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Contents

Introduction

This thesis is devoted to the study of the low-degree Poisson cohomology in the semilocal context, that is, in neighborhoods of symplectic leaves.

The cohomology theory of Poisson manifolds was introduced by Lichnerowicz, together with the description of Poisson brackets in terms of bivector fields [44]. Indeed, he observed that the Jacobi identity for a Poisson bracket can be translated to its associated bivector field in terms of the Schouten-Nijenhuis bracket for multivector fields. On the other hand, by the properties of the Schouten-Nijenhuis bracket, every bivector field induces a graded derivation of degree 1 on the algebra of multivector fields, namely, the adjoint operator with respect to this bracket. It turns out that such operator is a coboundary exactly when the given bivector field is associated with a Poisson bracket. This gives rise to the *Lichnerowicz-Poisson complex*, and its cohomology is known as the *Poisson cohomology* of the Poisson manifold.

From a geometric perspective, the Poisson cohomology in low degree has important interpretations. For example, the zeroth Poisson cohomology is isomorphic to the algebra of *Casimir functions*, being these functions the ones which are constant along the symplectic leaves of the Poisson manifold. Also, the first Poisson cohomology is the quotient of the Lie algebra of the *infinitesimal automorphisms* of the Poisson manifold by its ideal of *Hamiltonian vector fields* [44, 74].

On the other hand, the Poisson cohomology is a well-suited algebraic framework to express obstructions. For example, the *unimodularity* of an orientable Poisson manifold, which is the existence of a volume form invariant under every Hamiltonian flow, is controlled by the *modular class*, which lies in the first cohomology of the Poisson manifold [83]: the modular class vanishes if and only if such invariant volume form exists. The modular class is one of the most important Poisson cohomology classes, and is the first of the so-called *characteristic classes* [23]. On the other hand, the second Poisson cohomology is related to obstructions in quantization theory [63], [64, Chapter 6], and semilocal linearizability [73]. Second and higher-degree Poisson cohomology has applications, for instance, in deformation theory [21, Subsection 2.1.2].

As a cohomological theory, Poisson cohomology has many desirable properties. For example, there exists a natural morphism from the de Rham to the Poisson cohomology of the manifold. On the other hand, the Mayer-Vietoris sequence holds for Poisson cohomology, which allows to reduce the problem of global computation to smaller open sets. Moreover, the Lichnerowicz-Poisson complex admits a natural filtration whose associated spectral sequence is convergent to the Poisson cohomology [62, Section 1]. In the case of regular Poisson manifolds, there exists a recursive procedure for its computation, which was introduced first by Karasev and Vorobiev in the context of symplectic fibrations [74], and then adapted by Vaisman to any regular Poisson manifold [62, Section 2]. Also, Xu presented a method for the computation of Poisson cohomology in the regular case by means of symplectic groupoids [84].

In spite of its good properties, the computation of Poisson cohomology for general Poisson manifolds is not an easy task. In fact, the difficulties for its computation are mainly related to the singularities of the Poisson manifold. Several approaches have been developed for the computation of the Poisson cohomology in many particular cases of Poisson manifolds with singularities. For instance, Conn showed that if the isotropy algebra of a Poisson structure at a vanishing point is semisimple of compact type, then the Poisson structure is linearizable around that given point and its first Poisson cohomology group is trivial [12]. On the other hand, based on some classification results of Arnold [2], Monnier computed the cohomology of 2-dimensional Poisson manifolds around simple singularities [49]. These results may be interpreted as the computation of *germified Poisson cohomology.* Global computations of the cohomology of 3-dimensional linear Poisson structures can be found in works of Nakanishi [52, 53, 54]. In the context of 2-dimensional Poisson manifolds, a global computation was done by Radko, who obtained the cohomology of *topologically stable* Poisson structures on orientable closed surfaces [56, Section 4]. Furthermore, Lanius computed the Poisson cohomology of the so-called *partitionable* log *symplectic structures* [42], generalizing Radko's cohomological results.

The purpose of this work is to describe and compute Poisson cohomology in neighborhoods of symplectic leaves. As is well known, the semilocal model for Poisson manifolds is given by *coupling Poisson structures* [77]. Indeed, for each embedded symplectic leaf of a Poisson manifold, there exists a tubular neighborhood around it in which the Poisson structure is coupling. This means that the geometry around the symplectic leaf is controlled by some geometric data satisfying certain compatibility relations which are equivalent to the Jacobi identity of the Poisson structure [77, 78]. In fact, each coupling Poisson structure on a regular foliated manifold is equivalent to a triple consisting of a leaf-tangent Poisson structure equipped with an Ehresmann connection with Hamiltonian curvature, whose Hamiltonian is given by a covariantly constant 2-form vanishing along the leaves of the regular foliation [65, 77]. Such 2-form parameterizing the Hamiltonian of the curvature is called the *coupling form.*

In the semilocal context, the regular foliation which allows to apply the coupling method is provided by the fibers of a tubular neighborhood around the symplectic leaf [77]. The leaf-tangent (vertical) Poisson structure defines a locally trivial Poisson bundle [78, Prop. 3.3], and each fiber coincides with the transverse Poisson structure given by Weinstein Splitting Theorem [82, Theorem 2.1]. An adequate change of the tubular structure on a neighborhood of the symplectic leaf corresponds to the action of a group of an special class of diffeomorphisms around the zero section of its normal bundle. This corresponds to a gauge transformation of the Poisson structure, which allows to describe the change of the geometric data associated with each tubular structure [78, Sections 2 and 3]. In particular, the transverse Poisson bundles are isomorphic [78, Theorem 3.2].

Roughly speaking, the coupling method in Poisson geometry provides a geometric splitting into tangential and transversal components, which in the semilocal context is compatible with the singularities of the Poisson structure. Indeed, the singularities of a Poisson structure near an embedded symplectic leaf are encoded in the transverse Poisson bundle over the symplectic leaf. In this sense, the tangential component of the Poisson structure is regular, while the transversal part is singular. In the particular case when the symplectic leaf is regular, the tubular neighborhood can be chosen so that the transverse Poisson structure is trivial. This compatibility between the coupling method and the singularities of the Poisson structure is one of the key properties for which this method is a suitable tool for the study of Poisson cohomology in the semilocal context. Furthermore, since the semilocal model for Poisson manifolds is provided by coupling Poisson structures, we may restrict our attention in the computation of the Poisson cohomology of coupling Poisson structures.

A natural problem in semilocal Poisson geometry is to describe and compute the cohomology of coupling Poisson structures in terms of its associated geometric data. Some preliminary results in this direction were obtained by Itskov, Karasev, and Vorobiev in [75] and [33], where it is found a relation between the Poisson cohomology around symplectic leaves and the cohomology of some bigraded

($\mathbb{Z}\times\mathbb{Z}$-graded) complex. A further step in the description of the Poisson cohomology in the semilocal context was given by Crainic and Fernandes, who showed that the Lichnerowicz-Poisson complex of a coupling Poisson structure on a vector bundle is isomorphic to a bigraded cochain complex [15, Prop. 5.3]. In fact, they observed that the geometric data of a coupling Poisson structure induce differential operators on some bigraded Poisson algebra of degree $(0,-1)$: the vertical Poisson structure and the coupling form act via the adjoint of the bracket, and the Ehresmann connection induces the covariant exterior derivative. It turns out that this operators induce a coboundary if and only if they correspond to the geometric data of a coupling Poisson structure. In this case, the induced cochain complex is isomorphic to the Lichnerowicz-Poisson of the coupling structure. Finally, by considering a more general algebraic framework, Mărcuţ extended this result to the case of coupling Dirac structures on fibred manifolds [50, Subsection 4.2]. In this setting, the covariant exterior derivative is also presented as an adjoint operator (see also [16]).

In virtue of the above results, the approach adopted in this work for the description of the Poisson cohomology in the semilocal context is to compute the cohomology of a certain class of bigraded cochain complexes. More precisely, we consider a cochain complex $(\mathcal{C}^\bullet,\partial)$ with the following property: the $\mathcal{R}$-module $\mathcal{C}$ is endowed with a compatible bigrading $\mathcal{C}^{\bullet,\bullet}=\bigoplus_{p,q\in\mathbb{Z}}\mathcal{C}^{p,q}$ in the sense that the coboundary operator is the sum of three operators of given bigraded type,

$$\mathcal{C}^k=\bigoplus_{p+q=k}\mathcal{C}^{p,q},\qquad\text{and}\qquad \partial=\partial_{0,1}+\partial_{1,0}+\partial_{2,-1}. \tag{1}$$

It is important to remark that this class of cochain complexes appear in many different algebraic and geometric contexts. For instance, a particular case which is very well-studied in the literature is the *double complex* $(\mathcal{C}^{\bullet,\bullet},\partial=\partial_{0,1}+\partial_{1,0})$ (see, for example, [46, Chapter XI, Section 6], [6, Chapter II], [47, Section 2.4]), which is obtained from our general framework in the case when $\partial_{2,-1}=0$. Another example is provided by the de Rham complex of a fibred manifold. In fact, Vaisman observed that given a regular foliated manifold $(M,\mathcal{V})$, and fixing an Ehresmann connection γ on $(M,\mathcal{V})$, the de Rham differential has the following bigraded decomposition [61]:

$$\mathrm{d}=\mathrm{d}^\gamma_{0,1}+\mathrm{d}^\gamma_{1,0}+\mathrm{d}^\gamma_{2,-1}\,.$$

Here, $\mathrm{d}^\gamma_{0,1}$ is the γ-*covariant exterior derivative*, $\mathrm{d}^\gamma_{1,0}$ is the *foliated exterior derivative*, and $\mathrm{d}^\gamma_{2,-1}=-\,\mathrm{i}_{R^\gamma}$ is the negative of the insertion of the *curvature* R^γ of γ. These facts were also observed by Brahic as a particular case of cochain complexes arising in the context of extensions of Lie algebroids [7, Prop. 3.2], [8, Section 2]. Also, as a particular case of their constructions, García-Beltrán, Vallejo, and Vorobiev obtained some Lie algebroids on the tangent bundle whose de Rham complex is of the type in (1), [26, Section 4]. As a final example, we consider the case of a Poisson foliation $(M,\mathcal{F},P)$. For a fixed subbundle normal to the regular foliation $\mathcal{F}$, the Lichnerowicz-Poisson complex of (M,P) is of the form $(\Gamma(\wedge^{\bullet,\bullet}TM),\partial_P=(\partial_P)_{1,0}+(\partial_P)_{2,-1})$. This can be regarded as a particular case of our general framework, in which the operator of bigraded type $(0,1)$ vanishes. It is important to mention that the recursive procedures presented by Karasev and Vorobiev in [74], and by Vaisman in [62] for the computation of the cohomology of a regular Poisson manifold, are based on the given decomposition of the coboundary operator. In some sense, these results on the regular case motivate the approach and the results presented in this work.

Now we describe how a bigraded cochain complex of the form (1) arises in the context of coupling structures on foliated manifolds. Consider a regular foliation $\mathcal{V}$ on M, and a Poisson structure Π which is $\mathcal{V}$-coupling. Let us denote its associated geometric data by (P, γ, σ). This means that $(M, \mathcal{V}, P)$ is a Poisson foliation with Poisson connection γ, and σ is a coupling form, which is a γ-covariantly constant 2-form, vanishing along the $\mathcal{V}$-leaves, and which is the Hamiltonian of the curvature of γ. Let us denote by $H^\gamma := \ker(\gamma)$ the horizontal distribution of γ. Since (P, γ, σ) are associated with the bivector Π, the coupling form σ is horizontally non-degenerated, in the sense that the image of its flat mapping $\sigma^\flat : TM \to T^*M$ is the whole annihilator bundle of $T\mathcal{V}$, $\sigma^\flat(TM) = \mathrm{Ann}(T\mathcal{V})$. Therefore, we have an isomorphism from $TM = H^\gamma \oplus T\mathcal{V}$ to $\mathbb{V} := \mathrm{Ann}(T\mathcal{V}) \oplus T\mathcal{V}$, namely $X + Y \mapsto -\sigma^\flat(X) \oplus Y$, which can be extended to an isomorphism of bigraded exterior algebras $\flat_\sigma : \wedge^{\bullet,\bullet}TM \to \wedge^{\bullet,\bullet}\mathbb{V}$. It turns out that under this correspondence, the Lichnerowicz-Poisson complex $(\Gamma(\wedge^{\bullet,\bullet}TM), \partial_\Pi)$ is isomorphic to a bigraded cochain complex, $(\Gamma(\wedge^{\bullet,\bullet}\mathbb{V}), \partial^P_{0,1} + \partial^\gamma_{1,0} + \partial^\sigma_{2,-1})$, where the differential is the sum of three bigraded operators: the adjoint operator of P, $\partial^P_{0,1}$; the covariant exterior derivative of γ, $\partial^\gamma_{1,0}$; and the insertion of the differential of the coupling form σ, $\partial^\sigma_{2,-1}$. In this sense, the problem of the computation of the Poisson cohomology of (M, Π) turns into computing the cohomology of a bigraded cochain complex.

Based on the previous discussion, the structure of this work consists in three main parts: the computation of the cohomology of general bigraded complexes from an algebraic perspective (Chapter 3), the introduction of new geometric structures in which such algebraic formalism can be applied (Chapter 4), and the application of the general results to the first cohomology of coupling Poisson and Dirac structures on fibred and foliated manifolds (Chapters 5 and 6).

The cohomology of a bigraded complex

The most important contribution of this work is to provide a scheme for the computation of Poisson cohomology in the semilocal context. To this end, we have derived a general procedure which allows to compute the cohomology of a bigraded complex $(\mathcal{C}^{\bullet,\bullet}, \partial)$ with $\partial = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1}$, as described above. Here we present the results we have obtained in the cases of first, second, and third cohomology, but our procedure may be applied to derive similar results in any degree. In particular, we have recovered the results developed in [70, Chapter 1] for the cohomology of degree 1.

Among our main results, we mention the following.

Theorem (First Cohomology)**.** *We have the following short exact sequence*

$$0 \longrightarrow H^1(\mathcal{N}_0, \overline{\partial}) \longrightarrow H^1(\mathcal{C}, \partial) \longrightarrow \frac{\ker(\rho_1)}{B^1(\mathcal{C}^{0,\bullet}, \partial_{0,1})} \longrightarrow 0,$$

which describe the first cohomology of a bigraded cochain complex $(\mathcal{C}^{\bullet,\bullet}, \partial)$*.*

This result on the first cohomology of $(\mathcal{C}, \partial)$ is the bottom row of the diagram appearing in Theorem 3.11, and involves the map $\rho_1 : \mathcal{A}^1 \to H^2(\mathcal{N}_0, \overline{\partial})$, which is related to the second cohomology of $(\mathcal{N}_0, \overline{\partial})$.

Theorem (Second Cohomology)**.** *The following are short exact sequences which allow to describe the second cohomology of a bigraded cochain complex* $(\mathcal{C}^{\bullet,\bullet}, \partial)$*:*

$$0 \longrightarrow \frac{Z^2(\mathcal{N}_0, \overline{\partial})}{B^2(\mathcal{C}, \partial) \cap \mathcal{C}^{2,0}} \longrightarrow H^2(\mathcal{C}, \partial) \longrightarrow \frac{\ker(\rho_2)}{\mathcal{B}^2_1} \longrightarrow 0,$$

$$0 \longrightarrow \frac{\ker(\varrho_2)}{\mathcal{B}^2_1 \cap \mathcal{C}^{1,1}} \longrightarrow \frac{\ker(\rho_2)}{\mathcal{B}^2_1} \longrightarrow \frac{\mathcal{Z}^2_2}{B^2(\mathcal{C}^{0,\bullet}, \partial_{0,1})} \longrightarrow 0.$$

This result consists of the bottom rows of the diagrams appearing in Theorem 3.13. Moreover, it involves the maps $\rho_2 : \mathcal{A}^2 \to H^3(\mathcal{N}_0, \overline{\partial})$ and $\varrho_2 : \mathcal{J}^2 \to H^3(\mathcal{N}_0, \overline{\partial})$, related to the third cohomology of $(\mathcal{N}_0, \overline{\partial})$. Also, the subspace $\mathcal{Z}_2^2$ is related to the 3-coboundaries of $(\mathcal{N}_1, \overline{\partial})$.

Theorem (Third Cohomology)**.** *The third cohomology of* $(\mathcal{C}^{\bullet,\bullet}, \partial)$ *is described by the following short exact sequences:*

$$0 \longrightarrow \frac{Z^3(\mathcal{N}_0,\overline{\partial})}{B^3(\mathcal{C},\partial)\cap\mathcal{C}^{3,0}} \longrightarrow H^3(\mathcal{C},\partial) \longrightarrow \frac{\ker(\rho_3)}{\mathcal{B}_1^3} \longrightarrow 0,$$

$$0 \longrightarrow \frac{\ker(\varrho_3)}{\mathcal{B}_1^3\cap\mathcal{C}^{2,1}} \longrightarrow \frac{\ker(\rho_3)}{\mathcal{B}_1^3} \longrightarrow \frac{\mathcal{Z}_2^3}{\mathcal{B}_2^3} \longrightarrow 0,$$

$$0 \longrightarrow \frac{\mathcal{B}_2^3\cap\mathcal{C}^{1,2}}{\mathcal{Z}_2^3\cap\mathcal{C}^{1,2}} \longrightarrow \frac{\mathcal{Z}_2^3}{\mathcal{B}_2^3} \longrightarrow \frac{\mathcal{Z}_3^3}{B^3(\mathcal{C}^{0,\bullet},\partial_{0,1})} \longrightarrow 0.$$

These short exact sequences are the bottom rows appearing in the diagrams of Theorem 3.15. We note that this result on the third cohomology of $(\mathcal{C}^{\bullet,\bullet}, \partial)$ involves the maps $\rho_3 : \mathcal{A}^3 \to H^4(\mathcal{N}_0, \overline{\partial})$ and $\varrho_3 : \mathcal{J}^3 \to H^4(\mathcal{N}_0, \overline{\partial})$, related to the fourth cohomology of $(\mathcal{N}_0, \overline{\partial})$. Also, the subspace $\mathcal{Z}_3^3$ is related to the 4-coboundaries of $(\mathcal{N}_1, \overline{\partial})$.

Tools and the idea of the proof. The problem on the computation of the cohomology of a bigraded cochain complex $(\mathcal{C}^\bullet, \partial)$ with $\partial = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1}$ is addressed in Chapter 3. First of all, we tried to apply the spectral sequence technique. In fact, due to the particular decomposition of the coboundary operator, the filtration $F^p\mathcal{C} := \bigoplus_{i,j\in\mathbb{Z}, i\geq p} \mathcal{C}^{i,j}$ induces a first quadrant spectral sequence $E_r^{p,q}$, which converges to the cohomology $H^\bullet(\mathcal{C}, \partial)$. Since in the context of our applications we deal with cochain complexes over the field of real numbers, the convergence property implies $H^k(\mathcal{C}, \partial) \cong \bigoplus_{p+q=k} E_\infty^{p,q}$. In this sense, the problem of the computation of $H^k(\mathcal{C}, \partial)$ is just the computation of each $E_\infty^{p,q}$. Actually, the standard formula found in the modern literature to compute the spectral sequence of a graded filtered complex is $E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}}$ (see, for instance, [47, Theorem 2.6, Proposition 2.11]). However, the application of this formula to the case of our bigraded complex (1) did not produce something which could be adequately interpreted. Even in the case of the first cohomology, we did not recovered our previous results presented in [70, Chapter 1]. Fortunately, we were graced to found in [21, Eq. (2.46)] a different but equivalent formula for the computation of the spectral sequence: $E_r^{p,q} = \frac{Z_r^{p,q} + F^{p+1}\mathcal{C}^{p+q}}{B_{r-1}^{p,q} + F^{p+1}\mathcal{C}^{p+q}}$ (see also [46, p. 346]). this formula gives something manageable in the case of bigraded complexes, namely $E_\infty^{p,q} = \frac{\mathcal{Z}_q^k \cap \mathcal{C}^{p,q}}{\mathcal{B}_q^k \cap \mathcal{C}^{p,q}}$. Here, the elements of $\mathcal{Z}_q^k$ are called *pre-cocycles*, since are those which can be completed to a k-cocycle, in some sense. Similarly, the elements of $\mathcal{B}_q^k$ are called *pre-coboundaries*. In the case of the first cohomology, this formula coincides with some of the results in [70, Chapter 1] (see Lemma 5.14 and the discussion below).

As we have explained, the spectral sequence technique applied to a bigraded complex $(\mathcal{C}^\bullet, \partial)$ leads to splitting-type results which allow to compute the Poisson cohomology in the semilocal context. However, such results are weaker versions of the main results presented above, in the sense that, up to our knowledge, these splittings are not canonical. This fact is reflected, for instance, in the description of the modular class of coupling Poisson structures which we present below: the *generalized Reeb class* only exists when certain Poisson foliation is unimodular. If the splitting for the first cohomology were canonical, then such generalized Reeb class would always exist, regardless the unimodularity of the Poisson foliation. However, it is unlikely that this occurs, due to the way it is defined. For this reason, we tried to find a different approach that provides a broader view on the computation of the cohomology. We have found that, for each $k \in \mathbb{Z}$, the $\mathcal{R}$-modules of k-cocycles $Z^k(\mathcal{C}, \partial)$, k-coboundaries $B^k(\mathcal{C}, \partial)$, and k-cohomologies $H^k(\mathcal{C}, \partial)$ can be described by means of k commuting diagrams with exact rows and columns, given for each $q = 0, \ldots, k-1$, and $p := k - q$ as follows:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B}_q^k \cap \mathcal{C}^{p,q} \arrow[r, hook] \arrow[d, hook] & \mathcal{B}_q^k \arrow[r, "\pi_{q+1}"] \arrow[d, hook] & \mathcal{B}_{q+1}^k \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{Z}_q^k \cap \mathcal{C}^{p,q} \arrow[r, hook] \arrow[d] & \mathcal{Z}_q^k \arrow[r, "\pi_{q+1}"] \arrow[d] & \mathcal{Z}_{q+1}^k \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \frac{\mathcal{Z}_q^k \cap \mathcal{C}^{p,q}}{\mathcal{B}_q^k \cap \mathcal{C}^{p,q}} \arrow[r] \arrow[d] & \frac{\mathcal{Z}_q^k}{\mathcal{B}_q^k} \arrow[r] \arrow[d] & \frac{\mathcal{Z}_{q+1}^k}{\mathcal{B}_{q+1}^k} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

This result is presented in Theorem 3.7. The fact that this family of diagrams describes the coboundaries, cocycles, and cohomology of degree k readily follows from $\mathcal{Z}_0^k = Z^k(\mathcal{C}, \partial)$, and $\mathcal{B}_0^k = B^k(\mathcal{C}, \partial)$. In particular, the cohomology of degree k is described by means of the bottom row of each of the k diagrams. In the case of the cohomology in low degree, the module $\mathcal{Z}_1^k$ can be described in terms of the kernel of some natural mappings $\rho_k : \mathcal{A}^k \to H^{k+1}(\mathcal{N}_0, \overline{\partial})$ and $\varrho_k : \mathcal{J}^k \to H^{k+1}(\mathcal{N}_0, \overline{\partial})$. Combining this with some other calculations, we obtain an effective description of the cohomology in low degree. In particular, we get the main results presented above in the cases $k = 1, 2, 3$.

Generalized coupling structures

Another goal of this work is to enlarge the class of geometric structures for which its associated cochain complex naturally admits a bigrading of the type (1). As we explained above, there are many well-known examples of geometric structures with a cochain complexes of this kind. In fact, we observe that these examples can be presented as part of the following general framework: suppose that on the total space of a Lie algebroid $(E, q, [\cdot,\cdot]_E)$ we are given a pair of subbundles $H, V \subseteq E$ such that $H \oplus V = E$, and $\Gamma(V)$ is closed under the bracket $[\cdot,\cdot]_E$. In this case, the de Rham complex of the Lie algebroid E, $(\wedge^\bullet E^*, \partial)$, carries a natural bigrading such that $\partial = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1}$. In fact, consider the projection map $\mathrm{p} : E \to E$ such that $H = \ker(\mathrm{p})$ and $\mathrm{p}|_V = \mathrm{Id}_V$, and its curvature $R^{\mathrm{p}} \in \Gamma(\wedge^2 E^* \otimes E)$. If, for each $K \in \Gamma(\wedge^\bullet E^* \otimes E)$, i_K is the insertion on $\Gamma(\wedge^\bullet E^*)$, and $\mathrm{L}_K^\partial := [\mathrm{i}_K, \partial]$ is given by the graded commutator of endomorphisms, then we have:

Theorem (Bigrading on Lie algebroids)**.** *The bigraded components of the de Rham differential ∂ of $(E, q, [\cdot,\cdot]_E)$ satisfy*

$$\partial_{1,0} = \mathrm{L}^\partial_{\mathrm{Id}_E - \mathrm{p}} + 2\,\mathrm{i}_{R^\mathrm{p}}, \qquad \partial_{0,1} = \mathrm{L}^\partial_{\mathrm{p}} - \mathrm{i}_{R^\mathrm{p}}, \qquad \partial_{2,-1} = -\,\mathrm{i}_{R^\mathrm{p}}.$$

This result is presented in Theorem 4.1, but a detailed proof of it, given in a general algebraic context, can be found in the Appendix C. For the case $E = TM$, see [70, Corollary 2.5.7], [26, Section

5]. Under this general framework, we can present the cochain complex of some geometric structures, such as regular Poisson structures [74, 62], Poisson foliations [66, Lemma 4.1], the de Rham complex on a foliated manifold [61, 8], and the cohomology of an extension of a Lie algebroids [7, Prop. 3.2].

Now we observe how this theorem can be applied in the context of *coupling twisted Poisson and Dirac structures on foliated manifolds*. As we show in Section 4.3, each coupling twisted Dirac structure D with associated ψ-Dirac elements (P, γ, σ) admits the following splitting

$$D = \mathrm{Graph}(P^\sharp|_{A^\gamma}) \oplus \mathrm{Graph}(-\sigma^\flat|_{H^\gamma}),$$

in which $\Gamma(\mathrm{Graph}(P^\sharp|_{A^\gamma}))$ is closed under the ψ-Dorfman bracket. So, our previous theorem applies for this splitting, which means that the cochain complex of a coupling twisted Poisson or Dirac structure is of the kind described above, and our main results on the computation of its cohomology may be applied. Furthermore, this observation suggests that we may be able to find an adequate bigraded model for the associated cochain complexes of a coupling twisted structure on a foliated manifold, to effectively describe its cohomology.

Recall that a bigraded model for the cochain complex of coupling Poisson and Dirac structures was given in [15, Prop. 5.3] and [50, Prop. 4.2.8] in the category of fibred manifolds. However, as is known, the notion of coupling structures is not exclusive for fibred manifolds, and can be naturally extended to foliated manifolds [65, 67]. In this sense, Section 4.2 is devoted to adapt constructions presented in [50, Subsection 4.2] to the category of foliated manifolds by using Vinogradov calculus.

We believe that the construction of this generalization to the case of foliated manifolds gives more insight in the understanding of the algebraic aspects of the coupling method in Poisson geometry. In particular, we have found that it is possible to extend the notion of *coupling* to the case of Poisson and Dirac structures *with background* [79]. This straightforward generalization is presented in Section 4.3, where it is shown that the de Rham complex of the Lie algebroid of a coupling twisted Poisson and Dirac structure on a foliated manifolds is isomorphic to a bigraded cochain complex of the same type as before. More precisely, the results we have obtained are the following:

Theorem (Coupling Twisted Poisson structures). *Let $\psi \in \Gamma(\wedge^3 T^*M)$ be a closed 3-form on the foliated manifold $(M, \mathcal{V})$, and denote $V := T\mathcal{V}$, $V^\circ := \mathrm{Ann}(V)$. For each ψ-Poisson structure $\Pi \in \Gamma(\wedge^2 TM)$ such that $\Pi^\sharp(V^\circ) \oplus V = TM$, there exist unique $P \in \Gamma(\wedge^2 V)$, $\gamma \in \Gamma(T^*M \otimes V)$, and $\sigma \in \Gamma(\wedge^2 V^\circ)$ such that:*

1. *$(M, \mathcal{V}, P)$ is a ψ-Poisson foliation.*
2. *γ is a ψ-Poisson connection on $(M, \mathcal{V}, P)$, with horizontal distribution $H^\gamma := \ker\gamma$.*
3. *The curvature R^γ of γ is locally Hamiltonian, via the closed 3-form $(\mathrm{d}\,\sigma + \psi)$.*
4. *$\mathrm{d}\,\sigma + \psi$ vanishes on $\Gamma(\wedge^3 H^\gamma)$.*

In this case, we say that the ψ-Poisson structure Π is *coupling* on $(M, \mathcal{V})$, and (P, γ, σ) is called its associated *ψ-Dirac elements*. Of course, in the case $\psi = 0$ we recover the standard results for Poisson structures [77, 65]. We also note that this fact is proven in the more general context of Dirac structures with background (see Theorem 4.21). The first part of this result, relative to the existence of the geometric data, is proved by pure linear algebra, and then adapted to foliated manifolds in Proposition 2.23. The second part, which describes the structure equations of the geometric data, is obtained in Subsection 4.3.1 by straightforward computations with the Dorfman bracket on the Pontryagin bundle.

As mentioned above, there is a bigraded model for the cochain complex of a coupling twisted Poisson structure.

Theorem (Cohomological model for twisted structures)**.** *Let $\Pi \in \Gamma(\wedge^2 TM)$ be a coupling ψ-Poisson structure on the foliated manifold $(M, \mathcal{V})$ with geometric data (P, γ, σ). The cochain complex $(\Gamma(\wedge^\bullet TM), \partial_{\Pi,\psi})$ is isomorphic to a bigraded cochain complex $(\mathcal{C}^\bullet, \partial = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1})$, given by $\mathcal{C}^\bullet := \Gamma(\wedge^\bullet(V \oplus V^\circ))$, and $\partial_{0,1} = \partial_P + \mathrm{j}^{(0,1)}_{P,\psi}$, $\partial_{0,1} = \partial_\gamma + \mathrm{j}^{(1,0)}_{P,\psi}$, and $\partial_{2,-1} = \partial_\sigma + \mathrm{j}^{(2,-1)}_{\gamma,\psi}$.*

In the case $\psi = 0$, the bigraded components $\partial^P_{0,1}$, $\partial^\gamma_{1,0}$, and $\partial^\sigma_{2,-1}$ of the coboundary operator ∂ are in correspondence with the Dirac elements (P, γ, σ). However, in the general case, the presence of the background 3-form produces some intertwine of the Dirac elements in the description of the bigraded components.

Applications to cohomology in low degree in semilocal Poisson geometry

In the last part of this work, we present some applications of our general algebraic results, obtained the context of bigraded cochain complexes, to describe the infinitesimal Poisson automorphisms of coupling Poisson structures, the first cohomology, and the modular class.

Infinitesimal Poisson automorphisms around symplectic leaves. Given a coupling Poisson structure Π on a foliated manifold $(M, \mathcal{V})$ with associated geometric data (P, γ, σ), there are three geometric objects involved in the description of the first cohomology [70]: a cochain complex $(\mathcal{N}_0^\bullet, \overline{\partial}^\gamma)$, called the *de Rham-Casimir complex*, a Lie subalgebra $\mathcal{A}$ of $\mathcal{V}$-tangent infinitesimal automorphisms of P, and a mapping $\rho : \mathcal{A} \to H^2(\mathcal{N}_0^\bullet, \overline{\partial}^\gamma)$ from $\mathcal{A}$ to the second cohomology of $(\mathcal{N}_0^\bullet, \overline{\partial}^\gamma)$. In fact, the short exact sequence describing the first Poisson cohomology reads

$$0 \to H^1(\mathcal{N}_0, \overline{\partial}^\gamma) \xrightarrow{\sharp_H} H^1(M, \Pi) \xrightarrow{\gamma^*} \frac{\ker \rho^\gamma}{\mathrm{Ham}(M, P)} \to 0. \tag{2}$$

Here, $\sharp_H$ is the sharp map of the horizontal component of Π, and γ^* is the map induced in cohomology by the projection $\gamma : TM \to T\mathcal{V}$. We should note that the coboundary operator $\overline{\partial}^\gamma$ only depends on γ. Moreover, since $\mathcal{N}_0$ only depends on the Casimir functions of P, a Hamiltonian change of γ does not alter the coboundary operator. On the other hand, if the first cohomology of the associated Poisson foliation vanishes, $H^1(M, \mathcal{V}, P) = \{0\}$, then the right term in (2) is zero. Furthermore, one has $H^1(M, \Pi) \cong H^1(\mathcal{N}_0, \overline{\partial}^{\gamma_0})$ for any Poisson connection γ_0 on $(M, \mathcal{V}, P)$. Assuming that a flat Poisson connection exists, and under some projectability conditions, $H^1(M, \Pi)$ can be embedded in the first cohomology of a foliated de Rham complex. These conditions can be realized geometrically and, for instance, allows to present the following generalization of the cohomological part of Conn's results to neighborhoods of symplectic leaves:

Theorem (Triviality of the first cohomology)**.** *Let $S \subset M$ be an embedded symplectic leaf of the Poisson manifold (M, Ψ) such that the normal bundle of S (viewing as a Lie-Poisson bundle) is trivial. Assume that the isotropy algebra of the symplectic leaf S is a semisimple Lie algebra of compact type. If S is compact and simply connected, then there exists a tubular neighborhood N of S in M such that every Poisson vector field of Ψ is Hamiltonian on N.*

This fact readily follows from Proposition 5.21 in Chapter 5. We also give some other examples and classes of coupling Poisson and Dirac structures for which our general results can be effectively applied in the computation of the first Poisson cohomology.

Reeb and modular classes of Poisson structures on foliated manifolds

Poisson foliations and the modular class. As in the problem of the general computation of the cohomology, our results on the modular class are inspired in some sense by what occurs in the case of regular Poisson manifolds. Recall that the modular vector field of a regular Poisson manifold is always tangent to the symplectic foliation. So, the modular class of a regular Poisson manifold can be regarded as a foliated de Rham cohomology class of its characteristic foliation. More precisely, for a regular Poisson manifold (M, Π) with symplectic foliation $(\mathcal{S}, \omega)$ of rank $2k$, the modular class $\mathrm{Mod}(M, \Pi)$ of the Poisson manifold is equivalent to the *Reeb class* $\mathrm{Mod}(M, \mathcal{S})$ of its characteristic foliation, in the sense that

$$\mathrm{Mod}(M, \Pi) = -\Pi^\sharp \,\mathrm{Mod}(M, \mathcal{S}). \tag{3}$$

In particular, for regular Poisson manifolds, the unimodularity of (M, Π) only depends on its characteristic foliation $\mathcal{S}$, rather than in the leaf-wise symplectic form ω. The relationship between the Reeb and modular class was first observed by Weinstein [83, Section 5], and then formalized by Abouqateb and Boucetta [1] in the form (3). Geometrically, the Reeb class of a regular foliation is the obstruction to the existence of a transverse volume element invariant under the flow of every tangent vector field. On the other hand the *leaf-wise Liouville volume form* $\omega^k \in \Gamma(\wedge^{2k} T^*\mathcal{S})$ gives a correspondence between volume forms Ω on M, and transverse volume elements $\eta \in \Gamma(\wedge^{\mathrm{top}}(TM/T\mathcal{S})^*)$, namely, $\Omega = \omega^k \wedge \eta$. Since, tangentially, the leaf-wise Liouville volume form ω^k is already invariant under Hamiltonian flows, the problem of finding an invariant volume form Ω turns into finding an invariant transverse volume element η.

We first give a generalization of these results in the context of Poisson foliations. Recall that a regular Poisson manifold can be regarded as a Poisson foliation such that the leaf-tangent Poisson structure is symplectic at each leaf. As described in above, the fact that in this case the leaf-wise Liouville volume form is invariant under every Hamiltonian flow led to the cohomological relation of equation (3). However, for a general Poisson foliation $(M, \mathcal{V}, P)$, such an invariant leaf-wise volume form may not exists. Instead, we get a new cohomology class, called *the modular class of the Poisson foliation* $\mathrm{Mod}(M, \mathcal{V}, P)$. In fact, each leaf-wise volume form $\tau \in \Gamma(\wedge^{\mathrm{top}} T^*\mathcal{V})$ induces the so-called *modular vector field of the Poisson foliation* Z_P^τ. This vector field $Z_P^\tau \in \Gamma(T\mathcal{V})$ coincides on each leaf L of $\mathcal{V}$ with the modular vector field of the Poisson submanifold (L, P_L) induced by the volume form $\tau|_L$. It turns out that the family of such modular vector fields define a Poisson cohomology class, namely, $\mathrm{Mod}(M, \mathcal{V}, P)$, which is the obstruction to the existence of a leaf-wise volume form invariant under every Hamiltonian flow. As is shown in Proposition 6.3, the following more general cohomological relation is satisfied:

Theorem (Modular class of Poisson foliations)**.** *Let* $\mathrm{Mod}(M, P)$ *be the modular class of* (M, P) *as Poisson manifold,* $\mathrm{Mod}(M, \mathcal{V})$ *the Reeb class of the regular foliation* $\mathcal{V}$*, and* $\mathrm{Mod}(M, \mathcal{V}, P)$ *the modular class of the Poisson foliation* $(M, \mathcal{V}, P)$*. Then,*

$$\mathrm{Mod}(M, P) = -P^\sharp \,\mathrm{Mod}(M, \mathcal{V}) + \mathrm{Mod}(M, \mathcal{V}, P).$$

Semilocal unimodularity and the generalized Reeb class. We also present a description of the modular class in neighborhoods of symplectic leaves. Besides the results of [83, Section 5] and [1], which says that the unimodularity of regular Poisson manifolds is determined by the tangential geometry, it is convenient to recall what occurs in the local context. By Weinstein Splitting Theorem [82, Theorem 2.1], the local structure of a Poisson manifold at a given point is given as the product of a symplectic structure with the so called transverse Poisson structure. In this sense, the local unimodularity property is determined by the transverse Poisson structure, because symplectic manifolds are always unimodular. This means that, locally, the unimodularity is determined by the transversal geometry. Therefore, it is natural to expect that, in the semilocal context, the unimodularity is described by the interaction between the tangential and transversal geometry of the leaf.

This interaction can be effectively described in terms of coupling Poisson structures. Let Π be a coupling Poisson structure on the foliated manifold $(M, \mathcal{V})$ with associated geometric data (P, γ, σ). The first key observation in this context is that volume forms on M are in correspondence with leaf-wise volume forms on $\mathcal{V}$. In fact, due to the non-degeneracy of σ, every volume form Ω on M can be factored as $\Omega = \sigma^l \wedge \tau$. Here, $2l = \operatorname{rank} H^\gamma$, and τ is a leaf-wise volume form on $\mathcal{V}$. In terms of this correspondence, the modular vector field of (M, Π) induced by Ω splits into horizontal and vertical components depending on τ, namely,

$$Z_\Pi^\Omega = \Pi^\sharp \theta_\tau^\gamma + Z_P^\tau. \tag{4}$$

The vertical component Z_P^τ is the modular vector field of the Poisson foliation $(M, \mathcal{V}, P)$ induced by τ. The horizontal component is the image under $\Pi^\sharp$ of the so-called *divergence form* θ_τ^γ, which is the 1-form vanishing along the $\mathcal{V}$-leaves characterized by $\theta_\tau^\gamma(X) = \operatorname{div}^\tau(X)$ for all $X \in \Gamma(H^\gamma) \cap \operatorname{aut}(M, \mathcal{V})$. So, formula (4) says that the leaf-tangent component of the modular vector field Z_Π^Ω is modular vector field Z_P^τ of $(M, \mathcal{V}, P)$: $\gamma(Z_\Pi^\Omega) = Z_P^\tau$.

Our previous formula implies that at the level of cohomology one has

$$\gamma_* \operatorname{Mod}(M, \Pi) = \operatorname{Mod}(M, \mathcal{V}, P).$$

This means that an obstruction to the unimodularity of (M, Π) is the modular class of the Poisson foliation $(M, \mathcal{V}, P)$. However, the vanishing of $\operatorname{Mod}(M, \mathcal{V}, P)$ may be not sufficient for the vanishing of $\operatorname{Mod}(M, \Pi)$. In fact, due to the exactness of (2), we found that, under the unimodularity of $(M, \mathcal{V}, P)$, there exists a cohomology class in $H^1(\mathcal{N}_0, \overline{\partial}^\gamma)$ whose image under $\Pi^\sharp$ is the modular class. Furthermore, this cohomology class is the family of divergence 1-forms $\theta_{\tau_0}^\gamma \in \mathcal{N}_0^1$, where τ_0 varies over the leaf-wise volume forms invariant under every Hamiltonian flow of P, that is, $Z_P^{\tau_0} = 0$. More precisely, if $\operatorname{Mod}(M, \mathcal{V}, P) = 0$, then $[\theta_{\tau_0}^\gamma]$ exists, and we have

$$\operatorname{Mod}(M, \Pi) = -\Pi^\sharp [\theta_{\tau_0}^\gamma]. \tag{5}$$

If we take a look at the cohomological relation (3) for regular Poisson manifolds, we find that the cohomology class $[\theta_{\tau_0}^\gamma]$ in (5) plays an analogous role to the Reeb class, in the sense that it controls the unimodularity property. For this reason, we call it the *generalized Reeb class* of $(M, \mathcal{V}, \Pi)$, which only exists when the Poisson foliation $(M, \mathcal{V}, P)$ is unimodular. Furthermore, it is important to note that, given P and γ, the corresponding objects Z_P^τ and θ_τ^γ only depend on τ and, henceforth, their cohomology classes are well defined. This means that the unimodularity of (M, Π) is independent of the coupling form σ, which is an analog to the fact that, in the regular case, the unimodularity property is independent of the leaf-wise symplectic form.

These facts are translated to the semilocal context as follows:

Theorem (Generalized Reeb class)**.** *Let (M, Π) be an orientable Poisson manifold and S an embedded symplectic leaf. There exists a tubular neighborhood N of S such that*

$$\Pi = \Pi_H + \Pi_V \ \textit{on } N.$$

Here, Π_H is a bivector field on N with $\operatorname{rank}\Pi_H = \dim S$*, and Π_V is the* transverse Poisson structure *of S. Moreover, if* $\operatorname{Mod}(N, \Pi_V) = 0$*, then there exists a cohomology class* $\operatorname{Reeb}(S)$ *of the* de Rham - Casimir complex *of S, such that*

$$\operatorname{Mod}(N, \Pi) = -(\Pi_H^\sharp)^* \operatorname{Reeb}(S).$$

Observe that the question on the definition of the generalized Reeb class $\operatorname{Reeb}(S)$ of the symplectic leaf S is nontrivial. The main difficulty is to show that its construction is independent of the choice of a tubular neighborhood of the symplectic leaf (see Proposition 6.21).

Structure of the text

This thesis is organized as follows. The first part, which consists of the first two chapters, is devoted to review the preliminary notions which are needed throughout this work. In Chapter 1 we review the notions of Poisson manifolds, Dirac structures, Lie algebroids, and Poisson cohomology, while in Chapter 2 we present the coupling method on foliated manifolds. In the second part, we describe the cohomology of coupling Poisson structures. Chapter 3 presents an algebraic framework for the computation of the cohomology of a bigraded cochain complex. In Chapter 4, we describe some of the geometric structures which give rise to a bigraded cochain complex. In particular, we focus on the case of coupling twisted Poisson and Dirac structures. The third part of this work is devoted to the applications to the first cohomology of coupling Poisson and Dirac structures. In Chapter 5, the algebraic results on the computation of the first cohomology are reviewed from a geometric point of view. We also present different cases in which the first Poisson cohomology can be effectively described. Chapter 6 is devoted to the description of the modular class of Poisson structures on foliated manifolds, such as Poisson foliations, and coupling Poisson structures, with applications to the semilocal context. In particular, we discuss on the generalized Reeb class.

Part I

Preliminaries

Introduction to Part I

The main objective of this part is to present a brief summary of the basic notions in Poisson geometry and cohomology, as well as a description of the coupling method and semilocal Poisson geometry from an algebraic point of view.

Poisson geometry is a branch of mathematics which lies in the intersection of differential geometry and mathematical physics. For example, Poisson manifolds are the reduced phase space for Hamiltonian systems with symmetry. Furthermore, the formalism of Poisson manifolds arises in several contexts, such as Hamiltonian mechanics, integrable systems, topological field theories, deformation theory, representation theory and non-commutative geometry, to mention a few.

Given a point on a Poisson manifold, there exists a neighborhood of it which is Poisson isomorphic to the direct product of a symplectic manifold with a Poisson manifold vanishing at a point [82, Theorem 2.1]. The symplectic factor is just the symplectic leaf through the given point. The Poisson structure on the second factor is the so-called *transverse Poisson structure*, and encodes the singular behavior of the Poisson manifold at that given point. Thus, the problem of the local equivalence of Poisson manifolds reduces to the study of the transverse Poisson structure: two Poisson structures on a manifold are locally equivalent if and only if their transverse Poisson structures are isomorphic.

On the other hand, given two points on the same symplectic leaf, the corresponding transverse Poisson structures are isomorphic. In that sense, a natural problem is the existence of a way of gluing the transverse Poisson structures along the symplectic leaf to get a *transverse Poisson bundle*. Indeed, the answer to this question is affirmative: given an embedded symplectic leaf, there exists a tubular neighborhood of it which is a locally trivial Poisson bundle whose typical fiber is the transverse Poisson structure at some (any) point of the leaf. This transverse Poisson bundle is one of the invariants around the embedded symplectic leaf.

A second question, given in the context of the equivalence of Poisson structures around symplectic leaves, would be the following: if two Poisson structures share an embedded symplectic leaf with the same transverse Poisson bundle, do the Poisson structures are equivalent on a neighborhood of the symplectic leaf? Although the transverse Poisson structure at a given point is the only local invariant, the problem of equivalence of Poisson structures around symplectic leaves is more subtle, and the answer to this question is negative. This problem, as well as the existence of the transverse Poisson bundle, belong to the field of *semilocal Poisson geometry.*

In general, the term "semilocal Poisson geometry" refers to Poisson geometry in the context of neighborhoods of embedded submanifolds, such as symplectic, cosymplectic, or Poisson submanifolds.

The emergence of semilocal Poisson geometry is closely related with *the coupling method*. The origins of this method go back to Sternberg, who developed a covariant version of it to construct the symplectic form which gives the Hamiltonian structure to the Wong's equations describing the motion of a particle in a Yang-Mills field [59]. Later on, Vorobiev developed a contravariant version of it to describe the geometry of a Poisson manifold in neighborhoods of embedded symplectic leaves [77]. In fact, he showed that the class of *coupling Poisson structures* provides the semilocal model for Poisson manifolds around symplectic leaves. It is important to mention that the coupling method can be applied to any embedded symplectic leaf, regardless the possibly singular nature of it.

The coupling method allows to formulate some equivalence criteria between Poisson structures with a common symplectic leaf [78]. It also allows to define the linearized structure of a Poisson manifold around a symplectic leaf [77], and to formulate some linearizability criteria for Hamiltonian systems and Poisson structures [72, 73]. Also, coupling Poisson structures realize any transitive Lie algebroid with symplectic base as the restricted Lie algebroid of a coupling Poisson structure to a symplectic leaf [76].

Roughly speaking, the semilocal structure of a Poisson manifold around a symplectic leaf is described by the transverse Poisson bundle of the leaf, which is coupled with a horizontally non-degenerated 2-form, called the *coupling form*, via a possibly nonlinear Ehresmann connection on the tubular neighborhood. This triplet of geometric structures is said to be the *geometric data* of the Poisson manifold around the symplectic leaf [77, Theorem 2.1], also called *Dirac elements* [50, Definition 4.2.3]. In terms of the geometric data, the characteristic distribution is generated by the one of the transverse Poisson structure and the horizontal distribution of the connection. Similarly, the leaf-wise symplectic structure is the sum of the one of the transverse Poisson structure with the coupling form.

Later on, some generalizations of the coupling method were introduced. For example, Dufour and Wade applied it to describe the local and semilocal structure of Dirac manifolds [20]. They found out that, in neighborhoods of embedded presymplectic leaves, Dirac structures are described by a triple of geometric data for which the coupling form can be degenerated. In the same year, Vaisman introduced the notions of coupling Poisson, Dirac, and Jacobi structures on foliated manifolds [65, 67], giving an important step in the understanding the algebraic nature of the coupling method. In this sense, a coupling Dirac structure is Poisson if and only if its coupling form is horizontally non-degenerated. Moreover, Vaisman also introduced the more general notions of almost-coupling Poisson and Dirac structures, which appear in the context of slow-fast Hamiltonian systems and of the averaging method [4, 3, 68, 69].

There are some other important and recent contributions in the field of semilocal Poisson geometry. In [50, 51], some normal form theorems around Poisson submanifolds are presented. Furthermore, for several geometric structures in Poisson Geometry, splitting theorems around transversal submanifolds are given in [10] (for the case of Poisson structures and cosymplectic submanifolds, see also [24]). Since in this work, we are focused in the geometry and cohomology of Poisson manifolds around symplectic leaves, by the term "semilocal" we will refer to neighborhoods of symplectic leaves.

Chapter 1

Generalities on Poisson Geometry

Here we give a brief review of some structures in Poisson Geometry, which are used throughout this work. For a more detailed presentation of these notions we recommend [13, 18, 19, 21, 39, 79, 64, 82].

1.1 Poisson manifolds

Let M be a manifold, and $\Gamma(\wedge^\bullet TM)$ the algebra of multivector fields with the exterior product $\wedge$. Consider the Schouten-Nijenhuis bracket $[\cdot,\cdot]$ of multivector fields, defined on the in such a way that $(\Gamma(\wedge^\bullet TM), \wedge, [\cdot,\cdot])$ is a Poisson algebra of degree -1 (see, for instance, [21, Theorem 1.8.1]). A bivector field $\Pi \in \Gamma(\wedge^2 TM)$ on M is said to be a Poisson structure, or Poisson bivector field, if

$$[\Pi, \Pi] = 0.$$

This equation is called the Jacobi identity. The pair (M, Π) is said to be a Poisson manifold [44, Section 1], [27, Section 6], [82, Section 1]. The Poisson structure Π defines an $\mathbb{R}$-bilinear operation on $C^\infty(M)$ by

$$\{f, g\} := \Pi(\mathrm{d}\, f, \mathrm{d}\, g) \qquad \forall f, g \in C^\infty(M),$$

which is a Poisson bracket on $C^\infty(M)$ due to the Jacobi identity.

Consider the vector bundle map $\Pi^\sharp : T^*M \to TM$ given on $\alpha \in T_p^*M$ by $\beta(\Pi^\sharp\alpha) := \Pi_p(\alpha, \beta)$ for all $\beta \in T_p^*M$. This map can be extended to a morphism of exterior algebras $\Pi^\sharp : \wedge^k T^*M \to \wedge^k TM$ by setting $\Pi^\sharp\Theta(\alpha_1, \ldots, \alpha_k) := (-1)^k\Theta(\Pi^\sharp\alpha_1, \ldots, \Pi^\sharp\alpha_k)$. Given a smooth function $f \in C^\infty(M)$, then $\Pi^\sharp \mathrm{d}\, f \in \Gamma(TM)$ is said to be the Hamiltonian vector field of f. In this case, f is said to be the Hamiltonian function of $\Pi^\sharp \mathrm{d}\, f$. It follows from the Jacobi identity that

$$[\Pi^\sharp \mathrm{d}\, f, \Pi^\sharp \mathrm{d}\, g] = \Pi^\sharp \mathrm{d}\{f, g\} \qquad \forall f, g \in C^\infty(M),$$

which, together with the $\mathbb{R}$-linearity of d and $\Pi^\sharp$, implies that the Hamiltonian vector fields define an $\mathbb{R}$-Lie algebra, denoted by $\mathrm{Ham}(M, \Pi)$. Again, by the Jacobi identity, for all $Z \in \mathrm{Ham}(M, \Pi)$, we have

$$\mathrm{L}_Z \Pi = 0.$$

In general, whenever a vector field $Z \in \Gamma(TM)$ satisfies the previous relation, we say that Z is an infinitesimal Poisson automorphism of Π, or, shortly, a Poisson vector field. It follows from the graded Jacobi identity for the Schouten-Nijenhuis bracket that the Poisson vector fields also define a Lie algebra, denoted by $\mathrm{Poiss}(M, \Pi)$, for which $\mathrm{Ham}(M, \Pi)$ is an ideal. Moreover, $\mathrm{Poiss}(M, \Pi)$ is just the Lie algebra of derivations of the Poisson algebra $(C^\infty(M), \cdot, \{\cdot,\cdot\})$. Furthermore, Z is an infinitesimal Poisson automorphism of Π if and only if its flow Fl_Z^t satisfies the following relation for all $t \in \mathbb{R}$: $(\mathrm{Fl}_Z^t)^*\Pi = \Pi$.

Example 1.1. A symplectic manifold (M, ω) consists of a manifold M equipped with a closed and non-degenerated 2-form $\omega \in \Gamma(\wedge^2 T^*M)$: $\mathrm{d}\,\omega = 0$, and $\omega^\flat : TM \to T^*M$ is an isomorphism. Then,

$$\Pi_\omega(\alpha, \beta) := \omega\big((\omega^\flat)^{-1}\alpha, (\omega^\flat)^{-1}\beta\big), \qquad \forall \alpha, \beta \in \Gamma(T^*M),$$

defines a bivector field $\Pi_\omega \in \Gamma(\wedge^2 TM)$ which is Poisson, due to the closedness of ω. The Hamiltonian vector field X of f is given by $\mathrm{i}_X\,\omega = -\,\mathrm{d}\,f$, and the Poisson vector fields Z of Π_ω are those of the form $Z = \Pi_\omega^\sharp \alpha$, for some closed 1-form $\alpha \in \Gamma(T^*M)$. ▼

Example 1.2. Let $\mathfrak{g}^*$ be the dual of a finite-dimensional $\mathbb{R}$-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. A Poisson bracket on $\mathfrak{g}^*$ can be canonically defined as follows: for $f, g \in C^\infty(\mathfrak{g}^*)$, the Poisson bracket $\{f, g\} \in C^\infty(\mathfrak{g}^*)$ is given on $p \in \mathfrak{g}^*$ by

$$\{f, g\}(p) := p([\mathrm{d}_p\,f, \mathrm{d}_p\,g]_\mathfrak{g}).$$

Here we are using the identification $T_p^*(\mathfrak{g}^*) \cong \mathfrak{g}$. On the other hand, consider the identification $\mathfrak{g} \cong (\mathfrak{g}^*)^*$ in which each $\xi \in \mathfrak{g}$ is viewed as a linear function $\xi : \mathfrak{g}^* \to \mathbb{R}$. The Hamiltonian vector field of ξ is precisely the infinitesimal generator $\xi_{\mathfrak{g}^*} \in \Gamma(\mathfrak{g}^*)$ of the coadjoint action on $\mathfrak{g}^*$ of the connected and simply connected Lie group G integrating $\mathfrak{g}$. ▼

Example 1.3. On a two-dimensional manifold M, every bivector field Π is Poisson since $[\Pi, \Pi] \in \Gamma(\wedge^3 TM) = \{0\}$. On a 3-dimensional manifold, the set of Poisson vector fields in invariant under multiplication by smooth functions. Indeed, if $\Pi \in \Gamma(\wedge^2 TM)$ is a Poisson structure on M, and $f \in C^\infty(M)$, then

$$[f\Pi, f\Pi] = 2f[f, \Pi] \wedge \Pi + f^2[\Pi, \Pi] = -2f\Pi^\sharp\,\mathrm{d}\,f \wedge \Pi = -f\,\mathrm{i}_{\mathrm{d}\,f}(\Pi \wedge \Pi).$$

In the case when $\dim M = 3$, we have $\Pi \wedge \Pi \in \Gamma(\wedge^4 TM) = \{0\}$, and hence, $f\Pi$ is Poisson. ▼

Symplectic foliations. Given a Poisson manifold (M, Π), its characteristic distribution $C^\Pi \subseteq TM$ is the image of $\Pi^\sharp : T^*M \to TM$. It is clear from the definition of C^Π that for each $p \in M$, and $v \in C_p^\Pi$, there exists $f \in C^\infty(M)$ such that $v = \Pi_p^\sharp\,\mathrm{d}_p\,f$. In other words, C^Π is generated by $\mathrm{Ham}(M, \Pi)$. Now, let $\varphi := \mathrm{Fl}_X^t$ be the flow of $X := \Pi^\sharp\,\mathrm{d}\,f$ at some time $t \in \mathbb{R}$. Then,

$$(\varphi_*)_p(v) = (\varphi_*)_p(\Pi_p^\sharp\,\mathrm{d}_p\,f) = \Pi_{\varphi(p)}^\sharp(\varphi_{\varphi(p)}^*\,\mathrm{d}_p\,f) = \Pi_{\varphi(p)}^\sharp(\mathrm{d}_{\varphi(p)}\,\varphi^* f) \in C_{\varphi(p)}^\Pi.$$

The second equality uses the fact that $X \in \mathrm{Poiss}(M, \Pi)$, and the third equality is just the chain rule. Since $v \in C_p^\Pi$ is arbitrary, we get $((\mathrm{Fl}_X^t)_*)_p(C_p^\Pi) \subseteq C_{\mathrm{Fl}_X^t(p)}^\Pi$ for all $p \in M$, $X \in \mathrm{Ham}(M, \Pi)$, and $t \in \mathbb{R}$. This and the fact that $\mathrm{Ham}(M, \Pi)$ generates C^Π imply that C^Π is involutive and, therefore, integrable in the sense of Stefan-Sussmann [58, Corollary 1], [60, Theorem 4.2]. In other words, there exists a foliation $\mathcal{S}$ of M integrating the characteristic distribution, that is, $C^\Pi = T\mathcal{S}$.

On the other hand, there exists $\omega_\mathcal{S} \in \Gamma(\wedge^2 T^*\mathcal{S})$, well defined by $\omega_\mathcal{S}(\Pi^\sharp\,\mathrm{d}\,f, \Pi^\sharp\,\mathrm{d}\,g) = \Pi(\mathrm{d}\,f, \mathrm{d}\,g)$, which is closed and non-degenerated along the leaves of $\mathcal{S}$. Each leaf S of $\mathcal{S}$, together with the leaf-wise symplectic structure $\omega_S := \omega_\mathcal{S}|_S$, is said to be a *symplectic leaf* of (M, Π). The pair $(\mathcal{S}, \omega_\mathcal{S})$ is the *symplectic foliation* of (M, Π), which fully characterize the Poisson structure Π.

Example 1.4. For a symplectic manifold (M, ω), the symplectic foliation consists of the connected components of M, and the leaf-wise symplectic structure is just ω. ▼

Example 1.5. For the coalgebra $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ with integrating connected and simply connected Lie group G, the symplectic leaves of $\mathfrak{g}^*$ are the coadjoint G-orbits. This follows from the fact that the Hamiltonian vector fields, which generate the symplectic foliation, coincide with the infinitesimal generators of the coadjoint action, which generate the tangent space to the coadjoint orbits. The leaf-wise symplectic form is given by $\omega_{\mathcal{S}}(\xi_{\mathfrak{g}^*}, \eta_{\mathfrak{g}^*})(p) = p([\xi, \eta]_{\mathfrak{g}})$ for all $\xi, \eta \in \mathfrak{g}$. ▼

1.2 Lie algebroids

The idea of a Lie algebroid is to have a vector bundle endowed with a structure which is similar to the one of the tangent bundle of a manifold. Indeed, many geometric structures naturally induce a Lie algebroid, which encodes, for instance, the cohomological information of the geometric structure.

Definition 1.6. *A Lie algebroid is a triple $(E, q, [\cdot,\cdot]_E)$ which consists of a vector bundle $E \to M$ whose sections are endowed with a Lie bracket $[\cdot,\cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, and a vector bundle map $q : E \to TM$ satisfying*

$$[a, fb]_E = f[a, b]_E + \mathrm{L}_{q(a)} f \cdot b, \qquad \forall a, b \in \Gamma(E), f \in C^\infty(M). \tag{1.1}$$

The vector bundle map $q : E \to TM$ is called the *anchor* of the Lie algebroid, and the identity (1.1) is called the *Leibniz rule*. It follows from the definition of a Lie algebroid that the anchor map is a Lie algebra morphism [21, Lemma 8.1.4]:

$$q[a, b]_E = [q(a), q(b)] \qquad \forall a, b \in \Gamma(E). \tag{1.2}$$

Example 1.7 (Tangent Lie algebroid and regular foliations)**.** The simplest example of a Lie algebroid is the triple $(TM, \mathrm{Id}_{TM}, [\cdot,\cdot])$, consisting of the tangent bundle of M, the identity map, and the Lie bracket of vector fields. Similarly, on a foliated manifold $(M, \mathcal{F})$, the triple $(T\mathcal{F}, \iota, [\cdot,\cdot]_{\mathcal{F}})$ is a Lie algebroid, where $T\mathcal{F}$ is the tangent bundle to the $\mathcal{F}$-leaves, $\iota : T\mathcal{F} \hookrightarrow TM$ is the natural inclusion, and $[\cdot,\cdot]_{\mathcal{F}}$ is the restriction of the Lie bracket to the tangent vector fields $\Gamma(T\mathcal{F})$. ▼

Example 1.8 (Poisson manifolds)**.** Every Poisson structure on a manifold induces a Lie algebroid structure on the cotangent bundle. More precisely, if (M, Π) is a Poisson manifold, then the triple $(T^*M, \Pi^\sharp, \{\cdot,\cdot\}_\Pi)$, where

$$\{\alpha, \beta\}_\Pi := \mathrm{L}_{\Pi^\sharp \alpha} \beta - \mathrm{i}_{\Pi^\sharp \beta} \mathrm{d}\, \alpha, \qquad \forall \alpha, \beta \in \Gamma(T^*M), \tag{1.3}$$

is a Lie algebroid. In particular, the morphism property $\Pi^\sharp\{\alpha, \beta\}_\Pi = [\Pi^\sharp \alpha, \Pi^\sharp \beta]$ holds for all $\alpha, \beta \in \Gamma(T^*M)$. Conversely, if Π is a bivector field satisfying the morphism property, then Π is Poisson, and hence the triple $(T^*M, \Pi^\sharp, \{\cdot,\cdot\}_\Pi)$ is a Lie algebroid. ▼

Example 1.9 (Principal bundles)**.** Let $P \xrightarrow{p} M$ be a G-principal bundle, and consider the tangent lift of the action on P to the tangent bundle TP. The orbit space TP/G is a vector bundle over M whose sections naturally identify with the invariant vector fields on P,

$$\begin{tikzcd} TP \arrow[r, "\pi"] \arrow[d, "\tau"'] & TP/G \arrow[d, dotted, "\tilde{\tau}"] \\ P \arrow[r, "p"] & M \end{tikzcd} \qquad \begin{tikzcd} TP \arrow[rr, "\pi"] \arrow[dr, "p_*"'] & & TP/G \arrow[dl, dotted, "q"] \\ & TM & \end{tikzcd}$$

Hence, the Lie bracket of invariant vector fields on P induces a Lie bracket $[\cdot,\cdot]_{TP/G}$ on $\Gamma_M(TP/G)$. Furthermore, the differential $TP \xrightarrow{p_*} TM$ induces a vector bundle map $TP/G \xrightarrow{q} TM$ satisfying the Leibniz rule. Therefore $(TP/G, q, [\cdot,\cdot]_{TP/G})$ is a Lie algebroid, called the *Atiyah algebroid* of P. ▼

Example 1.10 (Restriction to a symplectic leaf)**.** Let $\iota : S \hookrightarrow M$ be a symplectic leaf of a Poisson manifold (M,Π). The triple $(T_SM := \iota^*(TM), \Pi_S^\sharp, \{\cdot,\cdot\}_S)$ is a Lie algebroid over S, where $\iota_* \circ \Pi_S^\sharp \circ \iota^* = \Pi^\sharp$, and $\{\iota^*\alpha, \iota^*\beta\}_S = \iota^*\{\alpha,\beta\}_\Pi$, holds for all $\alpha, \beta \in \Gamma(TM)$. ▼

The last two are examples of a *transitive* Lie algebroid, that is, the anchor map is surjective. In Example 1.10, S can be replaced by any submanifold N of M such that $\Pi^\sharp(T_N^*M) \subseteq TN$. Such submanifolds $\iota : N \hookrightarrow M$ are said to be *Poisson*, due to the fact that the Poisson structure Π can be naturally restricted to N, in the sense that there exists a Poisson structure $\Pi_N \in \Gamma(\wedge^2 TN)$ such that $\iota_* \circ \Pi_N^\sharp \circ \iota^* = \Pi^\sharp$.

Remark 1.11. *For $n \in \mathbb{N}$, consider the group of permutations S_n of $\{1,\ldots,n\}$. Given $\sigma \in S_n$, and $k \in \{1,\ldots,n\}$, we denote by $\sigma_k := \sigma(k)$ the value of σ on k. Explicitly,*

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{pmatrix}.$$

We will also denote by $(-1)^\sigma$ the sign of σ. On the other hand, for $i,j \in \mathbb{N}$, we denote by

$$S_{(i,j)} := \{\sigma \in S_{i+j} \mid \sigma_1 < \sigma_2 < \cdots < \sigma_i, \text{ and } \sigma_{i+1} < \sigma_{i+2} < \cdots < \sigma_{i+j}\}$$

the subset of shuffle permutations *of S_{i+j}. The subset $S_{(i,j,k)} \subseteq S_{i+j+k}$ is defined similarly.*

1.2.1 Equivalent definitions

There exist some alternative ways to define a Lie algebroid structure on a vector bundle $E \to M$ [21, Sections 8.2 and 8.6], [57, Sections 2.2 and 3.2]. For the purposes of this work, here we revise the following:

- A graded derivation $\partial_E \in \mathrm{Der}(\Gamma(\wedge^\bullet E^*))$ of degree 1 and of square zero, $\partial_E^2 = 0$.
- A graded Poisson bracket $[\cdot,\cdot]_E$ of degree -1 on $(\Gamma(\wedge^\bullet E), \wedge)$.
- A Poisson structure Π_E on E^* preserving the fiber-wise linear functions.

The de Rham differential. Let $(E, q, [\cdot,\cdot]_E)$ be a Lie algebroid. For each $\alpha \in \Gamma(\wedge^k E^*)$, $\partial_E\alpha \in \Gamma(\wedge^{k+1}E^*)$ is given on $a_1,\ldots,a_{k+1} \in \Gamma(E)$ by the Koszul's formula

$$\partial_E\alpha(a_1,\ldots,a_{k+1}) := \sum_{\sigma\in S_{(1,k)}} (-1)^\sigma \, \mathrm{L}_{q(a_{\sigma_1})}(\alpha(a_{\sigma_2},\ldots,a_{\sigma_{k+1}})) - \sum_{\sigma\in S_{(2,k-1)}} (-1)^\sigma \alpha([a_{\sigma_1},a_{\sigma_2}]_E, a_{\sigma_3},\ldots,a_{\sigma_{k+1}}).$$

This defines a graded derivation $\partial_E \in \mathrm{Der}^1(\Gamma(\wedge^\bullet E^*))$. The fact that $\partial_E^2 = 0$ follows from observing that $\partial_E^2 = \frac{1}{2}[\partial_E, \partial_E]$, where $[\cdot,\cdot]$ denotes the graded commutator of endomorphisms of $\Gamma(\wedge^\bullet E^*)$. Thus, ∂_E^2 is also a graded derivation of degree 2 vanishing on $C^\infty(M)$ and $\Gamma(E^*)$, due to (1.2) and the

Jacobi identity for $[\cdot,\cdot]_E$, respectively. Conversely, the anchor map q and the Lie bracket $[\cdot,\cdot]_E$ can be recovered from ∂_E by reversing the Koszul's formula for $k = 0, 1$, as follows:

$$\mathrm{L}_{q(a)} f := \partial_E f(a), \qquad \alpha([a,b]_E) := \mathrm{L}_{q(a)}(\alpha(b)) - \mathrm{L}_{q(b)}(\alpha(a)) - \partial_E \alpha(a,b),$$

where $a, b \in \Gamma(E)$, $f \in C^\infty(M)$, and $\alpha \in \Gamma(E)$. It is also important to note from (1.2) that the dual of the anchor map $q^* : \Gamma(\wedge^\bullet T^*M) \to \Gamma(\wedge^\bullet E^*)$ is a cochain complex morphism between the de Rham complexes of $(E, q, [\cdot,\cdot]_E)$ and $(TM, \mathrm{Id}_{TM}, [\cdot,\cdot])$, where the latter is the standard de Rham complex.

The Schouten bracket. Given a Lie algebroid $(E, q, [\cdot,\cdot]_E)$ and its differential ∂, we can associate to each $A \in \Gamma(\wedge^k E)$ the differential operator $\mathrm{i}_A \in \mathrm{End}^{-k}(\Gamma(\wedge^\bullet E^*))$ given as follows: if $A = a_1 \wedge \cdots \wedge a_k$ for some $a_1, \ldots, a_k \in \Gamma(E)$, then $\mathrm{i}_A := \mathrm{i}_{a_1} \circ \cdots \circ \mathrm{i}_{a_k}$, where i_{a_i} is the insertion operator. For a general $A \in \Gamma(\wedge^k E)$, i_A is defined by $\mathbb{R}$-linear extension. Moreover, we define $\mathrm{L}^\partial_A \in \mathrm{End}^{-k+1}(\Gamma(\wedge^\bullet E^*))$ by $\mathrm{L}^\partial_A := [\partial, \mathrm{i}_A]$, where $[\cdot,\cdot]$ is the graded commutator of graded endomorphisms. Explicitly, $\mathrm{L}^\partial_A = \partial \circ \mathrm{i}_A - (-1)^k \mathrm{i}_A \circ \partial$. Now, for $A \in \Gamma(\wedge^k E)$ and $B \in \Gamma(\wedge^l E)$, there exists a unique $[A,B]_E \in \Gamma(\wedge^{k+l-1} E)$ such that

$$\mathrm{i}_{[A,B]_E} = [\mathrm{L}^\partial_A, \mathrm{i}_B].$$

This gives an $\mathbb{R}$-bilinear operation $[\cdot,\cdot]_E : \Gamma(\wedge^\bullet E) \times \Gamma(\wedge^\bullet E) \to \Gamma(\wedge^\bullet E)$ such that $(\Gamma(\wedge^\bullet E), \wedge, [\cdot,\cdot]_E)$ is a graded Poisson algebra of degree -1. Indeed, the graded skew-symmetry, Leibniz rule, and Jacobi identity follow from the corresponding ones of the graded commutator on $\mathrm{End}^\bullet(\Gamma(\wedge^\bullet E^*))$. It is also important to mention that $[\cdot,\cdot]_E$ agrees on $\Gamma(E)$ with the original bracket of the Lie algebroid, so there is no ambiguity in the usage of this notation. Furthermore, $[a, f]_E = \mathrm{L}_{q(a)} f$ for all $a \in \Gamma(E)$ and $f \in C^\infty(M)$. These facts allow to recover the Lie algebroid structure on E from a graded Poisson bracket $[\cdot,\cdot]_E$ on $\Gamma(\wedge^\bullet E)$.

Definition 1.12. *The derivation* $\partial_E \in \mathrm{Der}^1(\Gamma(\wedge^\bullet E^*))$ *and the bracket* $[\cdot,\cdot]_E$ *described above are called the* de Rham differential *and the* Schouten bracket *of the Lie algebroid* $(E, q, [\cdot,\cdot]_E)$*, respectively.*

Fiber-wise linear Poisson structures. Let $E \xrightarrow{p} M$ be a vector bundle. For each $a \in \Gamma(E)$, let $\varphi_a \in C^\infty(E^*)$ be the corresponding fiber-wise linear function $\varphi_a : E^* \to \mathbb{R}$ defined by the natural pairing: $\varphi_a(\alpha) := \alpha(a_{p(\alpha)})$. Given a Lie algebroid $(E, q, [\cdot,\cdot]_E)$, there exists a unique Poisson structure $\Pi_E \in \Gamma(\wedge^2 T(E^*))$, defined on the total space of its dual bundle $E^* \xrightarrow{\pi} M$, by

$$\Pi_E(\mathrm{d}\,\varphi_a, \mathrm{d}\,\varphi_b) := \varphi_{[a,b]_E}. \tag{1.4}$$

The uniqueness of Π_E follows from the fact that, for all $\alpha \in E^*$, $T^*_\alpha(E^*) = \{\mathrm{d}_\alpha \varphi_a \mid a \in \Gamma(E)\}$, and the Jacobi identity for Π_E follows for the one for $[\cdot,\cdot]_E$. As a consequence of the Leibniz rule (1.1) and the fact that $\varphi_{fa} = \pi^* f \varphi_a$ for all $f \in C^\infty(M)$ and $a \in \Gamma(E)$, we get

$$\Pi_E(\mathrm{d}\,\varphi_a, \mathrm{d}\,\pi^* f) = \pi^* \mathrm{L}_{q(a)} f. \tag{1.5}$$

Relations (1.4) and (1.5) allow to recover a Lie algebroid structure $(E, q, [\cdot,\cdot])$ on E from a fiber-wise linear Poisson structure Π_E on E^*.

1.2.2 The characteristic foliation

It is well known that every Lie algebroid induces on the base manifold a generalized foliation.

Let $(E \to M, q, [\cdot,\cdot]_E)$ be a Lie algebroid. Its *characteristic distribution* is the image of the anchor map, $C^E := q(E)$. Now, consider its Poisson structure Π_E on the dual bundle $E^* \xrightarrow{\pi} M$. We claim that, for all $a \in \Gamma(E)$, the Hamiltonian vector field $X_a := \Pi_E^\sharp \operatorname{d}\varphi_a \in \Gamma(T(E^*))$ is π-related to $q(a) \in \Gamma(TM)$. This follows from (1.5),

$$\mathrm{L}_{X_a} \pi^* f = \Pi_E(\operatorname{d}\varphi_a, \operatorname{d}\pi^* f) = \pi^* \mathrm{L}_{q(a)} f, \qquad \forall f \in C^\infty(M).$$

Moreover, since the characteristic distribution C^{Π_E} of Π_E in E^* is generated by the Hamiltonian vector fields X_a of the fiber-wise linear functions φ_a, and the characteristic distribution of E is generated by the $q(a)$'s, our previous claim implies

$$\pi_{*_\alpha}(C^{\Pi_E}_\alpha) = C^E_{\pi(\alpha)} \qquad \forall \alpha \in E^*,$$

and $\pi \circ \mathrm{Fl}^t_{X_a} = \mathrm{Fl}^t_{q(a)} \circ \pi$, for all $t \in \mathbb{R}$ and $a \in \Gamma(E)$. Thus, denoting $\Phi := \mathrm{Fl}^t_{X_a}$, and $\phi := \mathrm{Fl}^t_{q(a)}$, and by applying the fact that C^{Π_E} is an involutive distribution on E^*, we get

$$\phi_{*_{\pi(\alpha)}}(C^E_{\pi(\alpha)}) = \phi_{*_{\pi(\alpha)}}(\pi_{*_\alpha}(C^{\Pi_E}_\alpha)) = \pi_{*_{\Phi(\alpha)}}(\Phi_{*_\alpha}(C^{\Pi_E}_\alpha)) = \pi_{*_{\Phi(\alpha)}}(C^{\Pi_E}_{\Phi(\alpha)}) = C^E_{\pi(\Phi(\alpha))} = C^E_{\phi(\pi(\alpha))}.$$

Therefore, the characteristic distribution C^E of the Lie algebroid E is also involutive in the sense of Stefan-Sussmann. Hence, it induces a foliation $\mathcal{L}$ of M such that $T\mathcal{L} = C^E$, which is also called the *characteristic foliation* of E.

1.3 Dirac structures

The formalism of Dirac structures gives a natural generalization of the notion of presymplectic and Poisson manifolds, unifying the covariant and contravariant approaches. This is achieved by endowing the Pontryagin bundle of a manifold, which is the Whitney sum of the tangent and cotangent bundles, with a suitable algebraic-geometric structure, called *Courant algebroid* [45, Definition 2.1]. In general, Courant algebroids are the natural framework to work with Dirac structures [13, Definition 2.3.1] and with some of its generalizations such as *twisted Dirac structures* [79].

1.3.1 Courant algebroids and Dirac structures

Let $\mathbb{E} \to M$ be a vector bundle, $p : \mathbb{E} \to TM$ a vector bundle map, $\langle\cdot,\cdot\rangle : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ a bilinear, symmetric, and non-degenerated product form on $\mathbb{E}$, and $[\![\cdot,\cdot]\!] : \Gamma(\mathbb{E}) \times \Gamma(\mathbb{E}) \to \Gamma(\mathbb{E})$ an $\mathbb{R}$-bilinear bracket.

Definition 1.13. *The tetrad $(\mathbb{E}, p, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!])$ is said to be a Courant algebroid whenever the following axioms are satisfied for all $f \in C^\infty(M)$ and $\eta, \eta_1, \eta_2 \in \Gamma(\mathbb{E})$ [37, Definition 2.1]:*

(CA1) $[\![\eta, [\![\eta_1, \eta_2]\!]]\!] = [\![[\![\eta, \eta_1]\!], \eta_2]\!] + [\![\eta_1, [\![\eta, \eta_2]\!]]\!]$,

(CA2) $\mathrm{L}_{p(\eta)}\langle\eta_1, \eta_2\rangle = \langle[\![\eta, \eta_1]\!], \eta_2\rangle + \langle\eta_1, [\![\eta, \eta_2]\!]\rangle$,

(CA3) $\langle\eta_1, [\![\eta_2, \eta_2]\!]\rangle = \frac{1}{2} \mathrm{L}_{p(\eta_1)}\langle\eta_2, \eta_2\rangle$.

Furthermore, on every Courant algebroid, the following properties also hold for all $\eta, \eta' \in \Gamma(\mathbb{E})$ and $f \in C^\infty(M)$ [37, Theorem 2.1]:

(CA4) $[\![\eta, f\eta']\!] = f[\![\eta, \eta']\!] + \mathrm{L}_{p(\eta)} f \cdot \eta'$,

(CA5) $p[\![\eta, \eta']\!] = [p(\eta), p(\eta')]$.

Axiom (CA1) is the Jacobi identity for the bracket $[\![\cdot, \cdot]\!]$, written in Leibniz form. Property (CA4) is called the Leibniz rule, and (CA5) says that $p : \Gamma(\mathbb{E}) \to \Gamma(TM)$ is a Loday algebra morphism. Note from (CA3) that $[\![\cdot, \cdot]\!]$ is not a Lie bracket in general.

Example 1.14. Let $(E, q, [\cdot, \cdot]_E)$ be a Lie algebroid, and ∂ its de Rham differential. The tetrad $(E \oplus E^*, q \circ \mathrm{pr}_E, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!])$ is a Courant algebroid, where $\mathrm{pr}_E : E \oplus E^* \to E$ is the natural projection,

$$\langle a \oplus \alpha, b \oplus \beta \rangle := \alpha(b) + \beta(a), \qquad \text{and} \qquad [\![a \oplus \alpha, b \oplus \beta]\!] := [a, b]_E \oplus (\mathrm{L}^{\partial}_a \beta - \mathrm{i}_b \partial \alpha)$$

for all $a, b \in \Gamma(E)$ and $\alpha, \beta \in \Gamma(E^*)$. In particular, when $E = TM$ we obtain the standard Courant algebroid on the Pontryagin bundle of M (see Subsection 1.3.2 below). ▼

As mentioned above, the failure of a Courant algebroid to be a Lie algebroid follows from the fact that the bracket is not always skew-symmetric, since the symmetric product gives an obstruction to this property. However, whenever we restrict the Courant algebroid structure to a subbundle in which the symmetric product vanishes, we indeed obtain a Lie algebroid.

Let $(\mathbb{E}, p, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!])$ be a Courant algebroid. Given subbundle $L \subseteq \mathbb{E}$, we denote by $L^\perp \subseteq \mathbb{E}$ its $\langle \cdot, \cdot \rangle$-orthogonal complement,

$$L^\perp := \{\eta \in \mathbb{E} \mid \langle \eta, \zeta \rangle = 0, \ \forall \zeta \in L\}.$$

We say that L is isotropic, coisotropic, or Lagrangian, if $L \subseteq L^\perp$, $L \supseteq L^\perp$, or $L = L^\perp$, respectively. Lagrangian subbundles are also called *maximally isotropic*, and, because of the non-degeneracy of $\langle \cdot, \cdot \rangle$, their rank must be equal to half of the rank of $\mathbb{E}$ (in particular, there are no maximally isotropic subbundles if the Courant algebroid has odd rank).

Suppose we are given an isotropic subbundle $L \subset \mathbb{E}$. By definition, the restriction of $\langle \cdot, \cdot \rangle$ to L is zero. If, in addition, the sections of L are closed under the bracket of $\mathbb{E}$,

$$[\![\Gamma(L), \Gamma(L)]\!] \subseteq \Gamma(L),$$

then the restriction $[\cdot, \cdot]_L := [\![\cdot, \cdot]\!]|_{\Gamma(L) \times \Gamma(L)}$ is a Lie bracket. Indeed, the Jacobi identity follows from (CA1), and the skew-symmetry is consequence of (CA3) and the isotropy of L. Furthermore, for the restriction $q := p|_L : L \to TM$, property (CA4) implies that the Leibniz rule for $[\cdot, \cdot]_L$ and q also holds. In other words, the triple $(L \to M, q, [\cdot, \cdot]_L)$ is a Lie algebroid.

Therefore, each isotropic subbundle $L \subset \mathbb{E}$ whose sections are closed under the bracket on $\mathbb{E}$ induces a foliation of the base manifold M, which integrates the characteristic distribution $q(L)$. We now restrict our attention to the case of maximally isotropic subbundles.

Definition 1.15. *Let $\mathbb{E} \to M$ be a Courant algebroid. A* Dirac subbundle *of $\mathbb{E}$ is a maximally isotropic subbundle $D \subset \mathbb{E}$ such that the sections of D are closed under the bracket of $\mathbb{E}$.*

As we have explained, Dirac structures are endowed with a Lie algebroid structure. On the other hand, for a maximally isotropic subbundle $D \subset \mathbb{E}$, the property of being Dirac is equivalent to

$$\langle \eta, [\![\xi, \zeta]\!] \rangle = 0 \qquad \forall \eta, \xi, \zeta \in \Gamma(D). \tag{1.6}$$

Indeed, observe that this property is equivalent to $[\![\xi, \zeta]\!] \in \Gamma(D^\perp)$ for all $\xi, \zeta \in \Gamma(D)$. Since, D is maximally isotropic, by definition we have $D^\perp = D$, so $[\![\xi, \zeta]\!] \in \Gamma(D)$ for all $\xi, \zeta \in \Gamma(D)$.

1.3.2 Twisted Dirac structures

Let TM and T^*M the tangent and cotangent bundles of a manifold M, respectively. Let us denote by $\mathbb{T}M := TM \oplus T^*M$ the Pontryagin bundle of M. Observe that $\mathbb{T}M$ is endowed with a canonical and non-trivial Courant algebroid structure, given as in Example 1.14. Consider the canonical projections $p_{TM} : \mathbb{T}M \to TM$ and $p_{T^*M} : \mathbb{T}M \to T^*M$, the canonical bilinear symmetric product $\langle \cdot, \cdot \rangle : \mathbb{T}M \times \mathbb{T}M \to \mathbb{R}$, given by

$$\langle X \oplus \alpha, Y \oplus \beta \rangle := \alpha(Y) + \beta(X), \qquad \forall\, X \oplus \alpha, Y \oplus \beta \in \mathbb{T}M, \tag{1.7}$$

and the *Dorfman bracket* $[\![\cdot, \cdot]\!] : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \to \Gamma(\mathbb{T}M)$ [18, eq. (2.15)], [19, p. 242],

$$[\![X \oplus \alpha, Y \oplus \beta]\!] := [X, Y] \oplus (\mathrm{L}_X \beta - \mathrm{i}_Y \mathrm{d}\, \alpha), \qquad \forall\, X \oplus \alpha, Y \oplus \beta \in \Gamma(\mathbb{T}M). \tag{1.8}$$

It is straightforward to verify that the tetrad $(\mathbb{T}M, p_{TM}, \langle \cdot, \cdot \rangle, [\![\cdot, \cdot]\!])$ is a Courant algebroid, that is, axioms (CA1)-(CA3) hold.

Dirac structures. Let M be a manifold, and $\mathbb{T}M$ its Pontryagin bundle. A *Dirac structure* on M is simply a Dirac subbundle $D \subset \mathbb{T}M$ of the Pontryagin bundle with respect its canonical Courant algebroid structure. Explicitly, a Dirac structure on M is a maximally isotropic subbundle $D \subset \mathbb{T}M$ whose sections are closed with respect to the Dorfman bracket (1.8). In this case, the pair (M, D) is sometimes called *Dirac manifold* [20].

Given a maximally isotropic subbundle $D \subset \mathbb{T}M$, the condition for D to be Dirac is equivalent to

$$\mathrm{i}_{X_3} \mathrm{L}_{X_1} \alpha_2 + \mathrm{i}_{X_1} \mathrm{L}_{X_2} \alpha_3 + \mathrm{i}_{X_2} \mathrm{L}_{X_3} \alpha_1 = 0 \qquad \forall X_i \oplus \alpha_i \in \Gamma(D).$$

This integrability condition is just (1.6) written in expanded form.

Observe that every Poisson manifold (M, Π) naturally induces a Dirac structure on M. Indeed,

$$D^\Pi := \{\Pi^\sharp \alpha \oplus \alpha \mid \alpha \in T^*M\}$$

is a maximally isotropic subbundle which is closed with respect to the Dorfman bracket due to the Jacobi identity of Π. In a similar fashion, the graph

$$D^\omega := \{X \oplus (-\,\mathrm{i}_X\, \omega) \mid X \in TM\}$$

of a 2-form $\omega \in \Gamma(\wedge^2 T^*M)$ is a maximally isotropic subbundle which is a Dirac structure if and only if $\mathrm{d}\,\omega = 0$. For a detailed explanation, we refer to Examples 1.17 and 1.18 below. Another family of Dirac structures on M are given by regular foliations. Indeed, as we explain in Subsection 4.2.1, if $\mathcal{F}$ is a regular foliation on M, then $D^{\mathcal{F}} := T\mathcal{F} \oplus \mathrm{Ann}(T\mathcal{F})$ is a Dirac structure, due to the involutivity of $T\mathcal{F}$.

Dirac structures with background. The most important example of a Courant algebroid is the Pontryagin bundle of a manifold endowed with the standard Dorfman bracket. However, we can slightly *twist the background* by means of a closed 3-form in order to generate new Courant algebroids.

Let $\psi \in \Gamma(\wedge^3 T^*M)$ be a closed 3-form on M. The *ψ-Dorfman bracket* $[\![\cdot,\cdot]\!]_\psi : \Gamma(\mathbb{T}M)\times\Gamma(\mathbb{T}M)\to\Gamma(\mathbb{T}M)$ is given by

$$[\![X\oplus\alpha, Y\oplus\beta]\!]_\psi := [X,Y]\oplus(\mathrm{L}_X\,\beta - \mathrm{i}_Y\,\mathrm{d}\,\alpha - \mathrm{i}_Y\,\mathrm{i}_X\,\psi), \qquad \forall\; X\oplus\alpha, Y\oplus\beta\in\Gamma(\mathbb{T}M). \tag{1.9}$$

We claim that the tetrad $\mathbb{T}M_\psi := (\mathbb{T}M, p_{TM}, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_\psi)$, where $p_{TM} : \mathbb{T}M\to TM$ is the canonical projection, and $\langle\cdot,\cdot\rangle$ is given as in (1.7), is a Courant algebroid. Indeed, axioms (CA2) and (CA3) hold because of the skew-symmetry of ψ, and the Jacobi identity (CA1) follows from the closedness of ψ. We call $\mathbb{T}M_\psi$ the Courant algebroid on $\mathbb{T}M$ with background ψ or, shortly, the ψ-Courant algebroid on $\mathbb{T}M$. For a detailed proof of the fact that $\mathbb{T}M_\psi$ is indeed a Courant algebroid, see Proposition B.5 of the Appendix.

We now recall the notion of Dirac and Poisson structures with background, which are also called *twisted structures.*

Definition 1.16. *A Dirac structure with background ψ or, shortly, ψ-Dirac structure on M, is a Lagrangian subbundle $D\subseteq\mathbb{T}M$ such that $[\![\Gamma(D),\Gamma(D)]\!]_\psi\subseteq\Gamma(D)$.*

In other words, a ψ-Dirac structure on M is a Dirac subbundle of the Courant algebroid $\mathbb{T}M_\psi$. For a maximally isotropic subbundle $D\subset\mathbb{T}M$, the integrability condition reads

$$\mathrm{i}_{X_3}\,\mathrm{L}_{X_1}\,\alpha_2 + \mathrm{i}_{X_1}\,\mathrm{L}_{X_2}\,\alpha_3 + \mathrm{i}_{X_2}\,\mathrm{L}_{X_3}\,\alpha_1 = \psi(X_1,X_2,X_3), \qquad \forall X_i\oplus\alpha_i\in\Gamma(D). \tag{1.10}$$

Example 1.17 (Presymplectic structures with background)**.** Let $\omega\in\Gamma(\wedge^2T^*M)$ be a 2-form on M. Then, the graph of $-\omega^\flat : TM\to T^*M$,

$$D^\omega := \{X\oplus(-\,\mathrm{i}_X\,\omega)\mid X\in\Gamma(TM)\}$$

is Lagrangian. Let us describe when D^ω is a ψ-Dirac structure. Fix $X_1,X_2,X_3\in\Gamma(TM)$, and set $\alpha_i := -\,\mathrm{i}_{X_i}\,\omega$, so that $X_i\oplus\alpha_i\in\Gamma(D^\omega)$ for all $i=1,2,3$. Then,

$$\begin{aligned}
\mathrm{i}_{X_3}\,\mathrm{L}_{X_1}\,\alpha_2 + \mathrm{i}_{X_1}\,\mathrm{L}_{X_2}\,\alpha_3 + \mathrm{i}_{X_2}\,\mathrm{L}_{X_3}\,\alpha_1 &= -\sum_{(1,2,3)} \mathrm{i}_{X_2}\,\mathrm{L}_{X_3}\,\mathrm{i}_{X_1}\,\omega\\
&= -\sum_{(1,2,3)} \mathrm{L}_{X_3}\,\mathrm{i}_{X_2}\,\mathrm{i}_{X_1}\,\omega + \mathrm{i}_{[X_2,X_3]}\,\mathrm{i}_{X_1}\,\omega\\
&= -\,\mathrm{d}\,\omega(X_1,X_2,X_3)
\end{aligned}$$

Because of (1.10), D^ω is a ψ-Dirac structure if and only if $\mathrm{d}\,\omega+\psi=0$ [79, p. 3]. ▼

Example 1.18 (Poisson structures with background)**.** Let $\Pi\in\Gamma(\wedge^2TM)$ be a 2-vector field on M. Then, the graph of $\Pi^\sharp : TM\to T^*M$,

$$D^\Pi := \{\Pi^\sharp\alpha\oplus\alpha\mid\alpha\in\Gamma(T^*M)\}$$

is Lagrangian. Let us describe when D^Π is a ψ-Dirac structure. Fix $\alpha_1, \alpha_2, \alpha_3 \in \Gamma(T^*M)$, and set $X_i = \Pi^\sharp \alpha_i$, so that $X_i \oplus \alpha_i \in \Gamma(D^\Pi)$ for all $i = 1, 2, 3$. By straightforward computations,

$$\begin{aligned} \mathrm{i}_{X_3} \mathrm{L}_{X_1} \alpha_2 + \mathrm{i}_{X_1} \mathrm{L}_{X_2} \alpha_3 + \mathrm{i}_{X_2} \mathrm{L}_{X_3} \alpha_1 &= - \sum_{(1,2,3)} \Pi(\alpha_1, \mathrm{d}\, \Pi(\alpha_2, \alpha_3)) - \alpha_3[\Pi^\sharp \alpha_1, \Pi^\sharp \alpha_2] \\ &= \tfrac{1}{2}[\Pi, \Pi](\alpha_1, \alpha_2, \alpha_3), \end{aligned}$$

where in the last equality we have applied Lemma B.1. Because of (1.10), D^Π is a ψ-Dirac structure if and only if $\frac{1}{2}[\Pi, \Pi] = -\Pi^\sharp \psi$ [79, p. 3]. ▼

Definition 1.19. *We say that $\omega \in \Gamma(\wedge^2 T^*M)$ is a presymplectic form with background ψ, or, shortly, ψ-presymplectic form, if $\mathrm{d}\,\omega + \psi = 0$. Similarly, $\Pi \in \Gamma(\wedge^2 TM)$ is a Poisson structure with background ψ, or, shortly, ψ-Poisson structure, if $\frac{1}{2}[\Pi, \Pi] = -\Pi^\sharp \psi$.*

As a consequence of Example 1.17, M admits ψ-presymplectic structures if and only if ψ is cohomologically trivial. On the other hand, given a ψ-Poisson structure Π on M, the projection $p_{T^*M} : D^\Pi \to T^*M$ is an isomorphism which induces the Lie algebroid $(T^*M, \Pi^\sharp, \{\cdot,\cdot\}_{\Pi,\psi})$, where

$$\{\alpha, \beta\}_{\Pi,\psi} := \mathrm{L}_{\Pi^\sharp \alpha} \beta - \mathrm{i}_{\Pi^\sharp \beta} \mathrm{d}\, \alpha - \mathrm{i}_{\Pi^\sharp \beta} \mathrm{i}_{\Pi^\sharp \alpha} \psi, \qquad \forall \alpha, \beta \in \Gamma(T^*M). \tag{1.11}$$

In terms of the closedness of the graph D^Π with respect to the ψ-Dorfman bracket, a bivector field $\Pi \in \Gamma(\wedge^2 TM)$ is ψ-Poisson if and only if

$$[\Pi^\sharp \alpha, \Pi^\sharp \beta] = \Pi^\sharp \{\alpha, \beta\}_{\Pi,\psi} \qquad \forall \alpha, \beta \in \Gamma(T^*M).$$

Remark 1.20. *In the case when $\psi = 0$, we recover the usual notions of presymplectic and Poisson structures as particular cases of Dirac structures. In particular, the bracket of 1-forms of Poisson structures is*

$$\{\alpha, \beta\}_{\Pi,\psi} := \mathrm{L}_{\Pi^\sharp \alpha} \beta - \mathrm{i}_{\Pi^\sharp \beta} \mathrm{d}\, \alpha, \qquad \forall \alpha, \beta \in \Gamma(T^*M). \tag{1.12}$$

Presymplectic foliations with background. Recall that every Dirac structure is equipped with a Lie algebroid structure, which induces a foliation on the base manifold. In the case of Dirac structures with background, the foliation is endowed with a leaf-wise presymplectic structure with background. The converse is also true: every presymplectic foliation with background is the characteristic foliation of the Lie algebroid of some Dirac structure with background. This generalizes the fact that Poisson structures are in correspondence with symplectic foliations.

Let $\psi \in \Gamma(\wedge^3 T^*M)$ be a background form, and consider the Pontryagin bundle $\mathbb{T}M_\psi$ equipped with the Courant algebroid structure given by the ψ-Dorfman bracket (1.9). Let also D be a Dirac structure. Let us denote by C^D its characteristic distribution, and by $\mathcal{S}$ its characteristic foliation, $T\mathcal{S} = C^D$. Let us define $\omega_\mathcal{S} \in \Gamma(\wedge^2 T^*\mathcal{S})$ by

$$\omega_\mathcal{S}(X, Y) := \beta(X),$$

where $X, Y \in \Gamma(T\mathcal{S})$ and $\beta \in \Gamma(T^*M)$ is such that $Y \oplus \beta \in \Gamma(D)$. We claim that $\omega_\mathcal{S}$ is well defined. Indeed, if $\beta' \in \Gamma(T^*M)$ is another such that $Y \oplus \beta' \in \Gamma(D)$, then $0 \oplus (\beta - \beta') \in \Gamma(D)$. So, for any $\alpha \in \Gamma(T^*M)$ such that $X \oplus \alpha \in \Gamma(D)$, we have by the isotropy of D,

$$0 = \langle 0 \oplus (\beta - \beta'), X \oplus \alpha \rangle = \beta(X) - \beta'(X).$$

This proves that the value $\omega_{\mathcal{S}}(X,Y)$ is independent of the choice of β. On the other hand, since D is isotropic, $0 = \langle X \oplus \alpha, Y \oplus \beta \rangle = \alpha(Y) + \beta(X)$, which implies

$$\omega_{\mathcal{S}}(Y,X) = \alpha(Y) = -\beta(X) = -\omega_{\mathcal{S}}(X,Y),$$

proving that $\omega_{\mathcal{S}}$ is skew-symmetric.

Finally, we will show that $\omega_{\mathcal{S}}$ is ψ-presymplectic along the $\mathcal{S}$-leaves. For a fixed leaf $\iota : S \hookrightarrow M$ of $\mathcal{S}$, let d_S be the de Rham differential of S, and $\omega_S \in \Gamma(\wedge^2 T^*S)$ the restriction of $\omega_{\mathcal{S}}$ to S. For $X_1, X_2, X_3 \in \Gamma(T\mathcal{S})$, and picking $\alpha_i \in \Gamma(T^*M)$ such that $X_i \oplus \alpha_i \in \Gamma(D)$, we get

$$\begin{aligned}
\mathrm{d}_S\, \omega_S(X_1|_S, X_2|_S, X_3|_S) &= \iota^* \sum_{(1,2,3)} \mathrm{L}_{X_3} \mathrm{i}_{X_2} \mathrm{i}_{X_1} \omega_{\mathcal{S}} - \mathrm{i}_{[X_3,X_2]} \mathrm{i}_{X_1} \omega_{\mathcal{S}} \\
&= \iota^* \sum_{(1,2,3)} \mathrm{i}_{X_2} \mathrm{L}_{X_3} \mathrm{i}_{X_1} \omega_{\mathcal{S}} = -\iota^* \sum_{(1,2,3)} \mathrm{i}_{X_2} \mathrm{L}_{X_3} \alpha_1 \\
&= -\iota^*(\psi(X_1, X_2, X_3)) = -\iota^*\psi(X_1|_S, X_2|_S, X_3|_S),
\end{aligned}$$

due to (1.10). In other words, the manifold (S, ω_S) is $\iota^*\psi$-presymplectic.

Conversely, if $(\mathcal{S}, \omega_{\mathcal{S}})$ is a foliation such that each leaf $\iota : S \hookrightarrow M$ is $\iota^*\psi$-presymplectic with the structure $\omega_S := \iota^*\omega_{\mathcal{S}}$, then

$$D^{\omega_{\mathcal{S}}} := \{X \oplus \alpha \in \mathbb{T}M \mid \alpha|_{T\mathcal{S}} = -\,\mathrm{i}_X\, \omega_{\mathcal{S}}\}$$

is a ψ-Dirac structure, provided that $D^{\omega_{\mathcal{S}}}$ is smooth. We have proved the following:

Theorem 1.21. *There exists a one-to-one correspondence between Dirac structures with background and presymplectic foliations with background.*

1.4 Poisson cohomology

Here, we review some basic notions in Poisson cohomology. For more details, see [21, Chapter 2], [62].

Let (M, Π) be a Poisson manifold. Because of the Leibniz rule of the Schouten-Nijenhuis bracket, the map

$$\partial_\Pi : \Gamma(\wedge^\bullet TM) \to \Gamma(\wedge^\bullet TM)$$

given by $\partial_\Pi(A) := [\Pi, A]$ is a graded derivation of degree 1. Furthermore, ∂_Π is a coboundary operator, $\partial_\Pi^2 = 0$, due to the Jacobi identities of Π and $[\cdot,\cdot]$. Thus, the pair $(\Gamma(\wedge^\bullet TM), \partial_\Pi)$ is a cochain complex, which is called the *Lichnerowicz-Poisson complex* of (M, Π) [44, Section 3]. The cohomology of this cochain complex is denoted by $H^\bullet_\Pi(M)$, and is called the *Poisson cohomology* of the Poisson manifold (M, Π). We also use the notation $Z^\bullet_\Pi(M)$ and $B^\bullet_\Pi(M)$ for the sets of cocycles and coboundaries.

The 0-cocycles are the *Casimir functions*, which are the smooth functions $f \in C^\infty(M)$ such that its Hamiltonian vector field vanishes, $\Pi^\sharp\, \mathrm{d} f = 0$, [44, Section 8], [74, p. 3]. Equivalently, a function is Casimir if and only if it is constant along the symplectic leaves. The algebra of Casimir functions is denoted by $\mathrm{Casim}(M, \Pi)$, and it is precisely the center of the Lie algebra $(C^\infty(M), \{\cdot,\cdot\})$.

The 1-coboundaries are just the Hamiltonian vector fields, $B^1_\Pi(M) = \mathrm{Ham}(M, \Pi)$, which are the *inner infinitesimal automorphisms* of $(C^\infty(M), \{\cdot,\cdot\})$, and the 1-cocycles are the infinitesimal Poisson automorphisms, $Z^1_\Pi(M) = \mathrm{Poiss}(M,\Pi)$. Thus, the Poisson cohomology of degree 1, which is given by [44, Section 8], [74, p. 3]

$$H^1(M,\Pi) = \tfrac{\mathrm{Poiss}(M,\Pi)}{\mathrm{Ham}(M,\Pi)},$$

may be interpreted as the space of *outer* infinitesimal automorphisms of Π [21, Subsection 2.1.2].

Now, suppose that $\Psi \in \Gamma(\wedge^2 TM)$ is a 2-cocycle of ∂_Π, $[\Pi,\Psi]=0$. Then, for $\varepsilon > 0$, we have

$$[\Pi + \varepsilon\Psi, \Pi + \varepsilon\Psi] = 0 \mod \varepsilon^2.$$

Thus, the 2-cocycles are the *infinitesimal deformations* of the Poisson structure Π [21, Subsection 2.1.2]. In this sense, the 2-coboundaries are the *trivial deformations.* Indeed, if $\Psi = [\Pi, X]$ for some $X \in \Gamma(TM)$, then

$$(\mathrm{Fl}^\varepsilon_X)^*(\Pi + \varepsilon\Psi) = \Pi \mod \varepsilon^2.$$

Since ∂_Π is a derivation of the Poisson algebra $(\Gamma(\wedge^\bullet TM), \wedge, [\cdot,\cdot])$, the space of coboundaries $B^\bullet_\Pi(M)$ is an ideal of the Poisson subalgebra of cocycles $Z^\bullet_\Pi(M)$. Hence, the exterior product and the Schouten-Nijenhuis bracket induce well-defined operations in cohomology, so that $(H^\bullet_\Pi(M), \wedge, [\cdot,\cdot])$ is also a graded Poisson algebra of degree -1 [21, Remark 2.1.5] [64, p. 64]. In particular, $H^\bullet_\Pi(M)$ is a module over the ring $H^0_\Pi(M) = \mathrm{Casim}(M,\Pi)$. Therefore, if there exists a non-constant Casimir function, and $H^k_\Pi(M) \neq \{0\}$, then $H^k_\Pi(M)$ is infinite-dimensional over $\mathbb{R}$ (regardless of the topological properties of the manifold).

By using the fact that Poisson manifolds induce a Lie algebroid, one can find a Koszul's type formula for the coboundary operator of the Lichnerowicz-Poisson complex. In fact, recall that the cotangent bundle of a Poisson manifold is a Lie algebroid, which induces a cochain complex structure on the exterior algebra of its dual bundle. It turns out that this cochain complex is precisely the Lichnerowicz-Poisson complex. More precisely, given a Poisson manifold (M,Π) and its cotangent Lie algebroid $(T^*M, \Pi^\sharp, \{\cdot,\cdot\}_\Pi)$, the coboundary operator ∂_Π of the Lichnerowicz-Poisson complex can be computed on $A \in \Gamma(\wedge^k TM)$ and $\alpha_1, \dots, \alpha_{k+1} \in \Gamma(T^*M)$ as follows [5]:

$$\partial_\Pi A(\alpha_1,\dots,\alpha_{k+1}) := \sum_{\sigma\in S_{(1,k)}} (-1)^\sigma \,\mathrm{L}_{\Pi^\sharp\alpha_{\sigma_1}}(A(\alpha_{\sigma_2},\dots,\alpha_{\sigma_{k+1}})) - \sum_{\sigma\in S_{(2,k-1)}} (-1)^\sigma A(\{\alpha_{\sigma_1},\alpha_{\sigma_2}\}_\Pi,\dots,\alpha_{\sigma_{k+1}}).$$

As a consequence of this fact, the map $\Pi^\sharp : \Gamma(\wedge^\bullet T^*M) \to \Gamma(\wedge^\bullet TM)$ is a cochain complex morphism between the Lichnerowicz-Poisson and the de Rham complexes. In particular, we have a well-defined map induced in cohomology,

$$(\Pi^\sharp)^* : H^\bullet_{\mathrm{dR}}(M) \to H^\bullet_\Pi(M).$$

Example 1.22 (Symplectic manifolds)**.** On a symplectic manifold (M,ω), the map $\Pi^\sharp_\omega : T^*M \to TM$ is an isomorphism, with inverse map $-\omega^\flat$. Therefore, the Lichnerowicz-Poisson complex is isomorphic to the de Rham complex of M. ▼

Example 1.23. Let N be any manifold endowed the zero Poisson bracket, (M,ω) a symplectic manifold, and $M \times N$ the product Poisson manifold. Then, (the lift of) every multivector field $A \in \Gamma(\wedge^\bullet TN)$ on N is a cocycle on $M \times N$. ▼

Example 1.24 (The Poisson manifold $\mathfrak{sl}^*(2)$)**.** Consider $M = \mathbb{R}^3$ endowed with the Poisson structure

$$\Pi := x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1} - x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}.$$

Then, $k(x) := (x^1)^2 + (x^2)^2 - (x^3)^2$ is a Casimir function. Now, let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(t) := \begin{cases} e^{-1/t^2} & \text{if } t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $K \in C^\infty(M)$ by $K := f \circ k$. Then, the vector field $Z \in \Gamma(TM)$ given by

$$Z := x^1 \frac{K(x)}{(x^1)^2+(x^2)^2} \frac{\partial}{\partial x^1} + x^2 \frac{K(x)}{(x^1)^2+(x^2)^2} \frac{\partial}{\partial x^2}$$

is an infinitesimal Poisson automorphism [52, p. 74]. Moreover, it is not Hamiltonian since it is not tangent to the symplectic foliation at each (x^1, x^2, x^3) with $(x^3)^2 < (x^1)^2 + (x^2)^2$. Therefore, $H^1_\Pi(M)$ is infinite dimensional. Furthermore,

$$H^1_\Pi(M)/\mathbb{R} \cong \{f \circ k \in C^\infty(M) \mid f \in C^\infty(\mathbb{R}) \text{ is flat at } 0\}.$$

The article [54] is devoted to the proof of this result. ▼

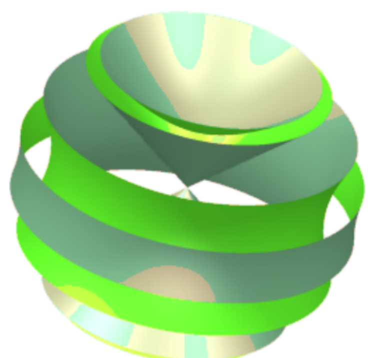

Figure 1.1: The characteristic foliation of $\mathfrak{sl}^*(2)$ consists of one- and two-sheeted hyperboloids, the upper and lower portions of the cone, and the origin as zero-dimensional leaf.

Example 1.25 (The Poisson manifold $\mathfrak{so}^*(3)$)**.** Let $M = \mathbb{R}^3$ be endowed with the Poisson structure

$$\Pi := x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}.$$

For $k(x) := (x^1)^2 + (x^2)^2 + (x^3)^2$, we have $\operatorname{Casim}(M, \Pi) = \{f \circ k \mid f \in C^\infty(\mathbb{R})\}$. Therefore, $H^0_\Pi(M)$ is infinite-dimensional. However, since $\mathfrak{so}^*(3)$ is a Lie algebra which is semi-simple and of compact type, one must have [12, Theroem 4.1]

$$H^1_\Pi(M) = \{0\}$$

(see also [53, Theorem 1.2]). To see this, observe that the symplectic foliation consists of spheres centered at the origin with different symplectic areas. Thus, infinitesimal Poisson automorphisms must be tangent to the symplectic foliation. Since the sphere is simply connected, such vector fields must be Hamiltonian. ▼

Figure 1.2: The characteristic foliation of $\mathfrak{so}^*(3)$ consists of concentric spheres at the origin, and the origin itself as zero-dimensional leaf.

Example 1.26. On $M = \mathbb{R}^3$, consider the Lie-Poisson structure

$$\Pi := \tfrac{\partial}{\partial x^3} \wedge \left(x^1 \tfrac{\partial}{\partial x^1} + x^2 \tfrac{\partial}{\partial x^2} \right).$$

The following vector fields

$$Z_1 = \frac{\partial}{\partial x_1}, \qquad Z_2 = x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3}, \qquad Z_3 = x_3 \frac{\partial}{\partial x_2}, \qquad Z_4 = x_2 \frac{\partial}{\partial x_3}$$

are generators of the first Poisson cohomology (see Section 5.8). Therefore,

$$H^1_\Pi(M) \cong \mathbb{R} \times \mathfrak{sl}(2, \mathbb{R})$$

as a Lie algebra [71, Example 7.1]. ▼

Figure 1.3: The Lie-Poisson structure described in Example 1.26 has characteristic foliation of open-book type. The two-dimensional leaves are half-planes whose boundary is the z-axis, which each of its points is a zero-dimensional leaf.

Cohomology of Poisson structures with background. Let $\psi \in \Gamma(\wedge^3 T^*M)$ be a closed 3-form on the manifold M, and $\Pi \in \Gamma(\wedge^2 TM)$ a ψ-Poisson structure, $\frac{1}{2}[\Pi,\Pi] = -\Pi^\sharp\psi$. The corresponding Lie algebroid is $(T^*M, \Pi^\sharp, \{\cdot,\cdot\}_{\Pi,\psi})$, where

$$\{\alpha,\beta\}_{\Pi,\psi} := \mathrm{L}_{\Pi^\sharp\alpha}\,\beta - \mathrm{i}_{\Pi^\sharp\beta}\,\mathrm{d}\,\alpha - \mathrm{i}_{\Pi^\sharp\beta}\,\mathrm{i}_{\Pi^\sharp\alpha}\,\psi, \qquad \forall \alpha,\beta \in \Gamma(T^*M).$$

The de Rham differential $\partial_{\Pi,\psi} : \Gamma(\wedge^\bullet TM) \to \Gamma(\wedge^\bullet TM)$ is $\partial_{\Pi,\psi} = \partial_\Pi + \mathrm{j}_{\Pi,\psi}$, where $\partial_\Pi := \mathrm{ad}_\Pi$ is the usual adjoint operator of Π with respect to the Schouten-Nijenhuis bracket of $\Gamma(\wedge^\bullet TM)$, and $\mathrm{j}_{\Pi,\psi} : \Gamma(\wedge^\bullet TM) \to \Gamma(\wedge^\bullet TM)$ is given on $A \in \Gamma(\wedge^k TM)$ and $\alpha_1,\dots,\alpha_{k+1} \in \Gamma(T^*M)$ by

$$\mathrm{j}_{\Pi,\psi}\,A(\alpha_1,\dots,\alpha_{k+1}) := -\sum_{\sigma\in S_{(2,k-1)}} (-1)^\sigma A(\mathrm{i}_{\Pi^\sharp\alpha_{\sigma_1}}\,\mathrm{i}_{\Pi^\sharp\alpha_{\sigma_2}}\,\psi, \alpha_{\sigma_3},\dots,\alpha_{\sigma_{k+1}}). \tag{1.13}$$

There is a nice description of $\mathrm{j}_{\Pi,\psi}$ as the insertion of a *1-form-valued bivector*. In fact, define $Q \equiv Q_{\Pi,\psi} \in \Gamma(\wedge^2 TM \otimes T^*M)$ by $Q(\alpha,\beta,X) := -\psi(\Pi^\sharp\alpha, \Pi^\sharp\beta, Y)$. Then, the right-hand side of (1.13) is simply the insertion $\mathrm{i}_Q\,A$ evaluated on $(\alpha_1,\dots,\alpha_{k+1})$. In other words, $\mathrm{j}_{\Pi,\psi} = \mathrm{i}_{Q_{\Pi,\psi}}$.

The 0-coboundaries are such $f \in C^\infty(M)$ satisfying $\Pi^\sharp\,\mathrm{d}\,f = 0$. In other words, $Z^0_{\Pi,\psi}(M) = H^0_{\Pi,\psi}(M)$ consists of the functions which are constant along the ψ-symplectic leaves. The 1-cocycles $X \in Z^1_{\Pi,\psi}(M)$ are those $X \in \Gamma(TM)$ such that

$$[\Pi, X](\alpha,\beta) - X(\mathrm{i}_{\Pi^\sharp\alpha}\,\mathrm{i}_{\Pi^\sharp\beta}\,\psi) = 0 \qquad \forall \alpha,\beta \in \Gamma(T^*M).$$

This is equivalent to $[X,\Pi] = \Pi^\sharp\,\mathrm{i}_X\,\psi$.

Chapter 2

The Coupling Method and Semilocal Poisson Geometry

In this chapter, we give a brief review of the coupling method for Poisson and Dirac structures on foliated manifolds. In Section 2.1, we describe the coupling method in the category of vector spaces. Sections 2.2-2.3 are devoted to the description of the geometric structures which appear in the context of the coupling method, such as Poisson foliations, connections, and curvature. The coupling method on foliated manifolds is described in Section 2.4. Finally, in Section 2.5 we apply the coupling method to describe Poisson geometry around symplectic leaves.

2.1 The coupling method in the vector space category

Let W be a finite dimensional vector space, and W^* its dual space. Denote $\mathbb{W} := W \oplus W^*$. Consider the canonical projections $p_W : \mathbb{W} \to W$, $p_{W^*} : \mathbb{W} \to W^*$, and the bilinear symmetric pairing $\langle \cdot, \cdot \rangle$ on $\mathbb{W}$ given by

$$\langle X \oplus \alpha, Y \oplus \beta \rangle := \alpha(Y) + \beta(X), \qquad \forall X, Y \in W, \alpha, \beta \in W^*.$$

Consider a subspace $L \subseteq \mathbb{W}$. The orthogonal complement $L^\perp$ of L with respect to $\langle \cdot, \cdot \rangle$ is

$$L^\perp := \{\eta \in \mathbb{W} \mid \langle \eta, \zeta \rangle = 0, \ \forall \zeta \in L\}.$$

We say that L is isotropic, coisotropic, or Lagrangian if $L \subseteq L^\perp$, $L \supseteq L^\perp$, or $L = L^\perp$, respectively. Lagrangian subspaces are also called *maximally isotropic*, and their dimension equals to the one of W. Given an isotropic subspace $L \subset \mathbb{W}$, the *characteristic subspace* $C^L \subseteq W$ is given by $C^L := p_W(L)$. An example of a Lagrangian subspace is the graph of a bivector $\Pi \in \wedge^2 W$, given by $\mathrm{Graph}(\Pi^\sharp) := \{\Pi^\sharp \alpha \oplus \alpha \mid \alpha \in W^*\}$. A maximally isotropic subspace D is the graph of a bivector if and only if $D \cap (W \oplus \{0\}) = \{0\}$.

Now, consider a fixed subspace $V \subseteq W$, and denote by

$$V^\circ := \mathrm{Ann}(V) = \{\alpha \in W^* \mid \alpha(v) = 0 \ \forall v \in V\}$$

its annihilator. Also, denote $\mathbb{V} := V \oplus V^\circ \subset \mathbb{W}$.

Definition 2.1. *Let $D \subset \mathbb{W}$ be a maximally isotropic subspace. We say that D is a coupling space on (W, V), or, shortly, a V-coupling subspace if*

$$D \oplus \mathbb{V} = \mathbb{W}.$$

If D is a maximally isotropic subspace, then $\dim D + \dim \mathbb{V} = \dim \mathbb{W}$ for any subspace $V \subset W$. Therefore, D is V-coupling if and only if $D \cap \mathbb{V} = \{0\}$.

Definition 2.2. *A triple (P, γ, σ) of geometric data on (W, V) consists of a bivector $P \in \wedge^2 V$; a linear map $\gamma : W \to W$ such that* $\operatorname{im}(\gamma) = V$*, and $\gamma^2 = \gamma$; and a 2-form $\sigma \in \wedge^2 V^\circ$.*

The goal of this part is to show the existence of a one-to-one correspondence between V-coupling subspaces of $\mathbb{W}$ and geometric data on (W, V). To do so, we need some preliminary constructions.

Given a Lagrangian subspace $D \subset \mathbb{W}$, let us define the subspaces $H \equiv H(D, V) \subset W$ and $A \equiv A(D, V) \subset W^*$ by

$$H(D, V) := \{X \in W \mid \exists \alpha \in V^\circ \ : \ X \oplus \alpha \in D\} = p_W(D \cap (W \oplus V^\circ)), \tag{2.1}$$
$$A(D, V) := \{\mu \in W^* \mid \exists v \in V \ : \ v \oplus \mu \in D\} = p_{W^*}(D \cap (V \oplus W^*)). \tag{2.2}$$

Observe that $A \subseteq \operatorname{Ann}(H)$. Indeed, for $\mu \in A$ and $X \in H$, there exist $v \in V$ and $\alpha \in V^\circ$ such that $v \oplus \mu, X \oplus \alpha \in D$. Taking into account that $\alpha(v) = 0$ and the isotropy of D, we get

$$0 = \langle v \oplus \mu, X \oplus \alpha \rangle = \mu(X).$$

Since $X \in H$ is arbitrary, $\mu \in \operatorname{Ann}(H)$, as claimed. This property is stronger in the coupling case.

Lemma 2.3. *For the Lagrangian subspace $D \subset \mathbb{W}$, the following assertions are equivalent:*

(a) D is V-coupling.

(b) $H \oplus V = W$.

(c) $V^\circ \oplus A = W^*$.

In this case, $\operatorname{Ann}(H) = A$.

See Lemma A.1 of the Appendix for a detailed proof.

Given a linear map $\gamma : W \to W$ such that $\gamma^2 = \gamma$ and $\operatorname{im}\gamma = V$, denote $H^\gamma := \ker \gamma \subset W$, and $A^\gamma := \operatorname{im}\gamma^* \subset W^*$. Then, it follows that $A^\gamma = \operatorname{Ann}(H^\gamma)$.

Proposition 2.4. *There exists a one-to-one correspondence between V-coupling Lagrangian subspaces $D \subset \mathbb{W}$ and geometric data (P, γ, σ) on (W, V).*

Proof. Given some geometric data (P, γ, σ), the subspace $D_{P,\gamma,\sigma} \subseteq \mathbb{W}$ given by

$$D_{P,\gamma,\sigma} := \{(P^\sharp \mu + X) \oplus (\mu - \mathrm{i}_X \sigma) \mid X \in H^\gamma, \mu \in A^\gamma\} = \operatorname{Graph}(P^\sharp|_{A^\gamma}) \oplus \operatorname{Graph}(-\sigma^\flat|_{H^\gamma})$$

is Lagrangian and V-coupling, due to Lemma A.2 of the Appendix. Conversely, each V-coupling Lagrangian subspace $D \subset \mathbb{W}$ induces a triple $(P_D, \gamma_D, \sigma_D)$ of geometric data given as follows: the linear map $\gamma_D : W \to W$ is the projection over V along the splitting $H \oplus V = W$ of Lemma 2.3 (b); the bivector $P_D \in \wedge^2 V$ and the 2-form $\sigma_D \in \wedge^2 V^\circ$ are defined by the following relations:

$$P_D^\sharp \mu \oplus \mu \in D, \quad \forall \mu \in A, \qquad\qquad X \oplus (-\mathrm{i}_X \sigma_D) \in D, \quad \forall X \in H.$$

The fact that P_D and σ_D are well defined follows from Lemma A.3. Finally, the proof of the fact that this correspondence is one to one is given in Proposition A.4 of the Appendix. ∎

Definition 2.5. *Given a V-coupling Lagrangian subspace $D \subset \mathbb{W}$, we say that (P, γ, σ) given by Proposition 2.4 is the geometric data associated with D.*

Let $\Pi \in \wedge^2 W$ be a bivector. Since the graph $\mathrm{Graph}(\Pi^\sharp)$ of Π is a maximally isotropic subspace of $\mathbb{W}$, it is natural to find conditions for which $\mathrm{Graph}(\Pi^\sharp)$ is V-coupling.

Proposition 2.6. *The graph* $\mathrm{Graph}(\Pi^\sharp)$ *of a bivector* $\Pi \in \wedge^2 W$ *is* V*-coupling if and only if*

$$\Pi^\sharp(V^\circ) \oplus V = W.$$

Moreover, if (P, γ, σ) *is the geometric data associated with a* V*-coupling Lagrangian subspace* $D \subseteq \mathbb{W}$*, then* D *is the graph of a bivector if and only if* $\ker \sigma^\flat = V$.

Proof. For $D^\Pi := \mathrm{Graph}(\Pi^\sharp)$, one has

$$H(D^\Pi, V) = \{X \in W \mid X \oplus \alpha \in D^\Pi, \alpha \in V^\circ\} = \{\Pi^\sharp \alpha \mid \alpha \in V^\circ\} = \Pi^\sharp(V^\circ).$$

By Lemma 2.3, we have that D^Π is V-coupling if and only if $\Pi^\sharp(V^\circ) \oplus V = W$. Now, if (P, γ, σ) is some geometric data and D is the corresponding V-coupling subspace, then

$$\begin{aligned} D \cap (W \oplus \{0\}) &= \{(X + P^\sharp \mu) \oplus (\mu - \mathrm{i}_X \sigma) \mid X \in H^\gamma, \mu \in A^\gamma, \mu - \mathrm{i}_X \sigma = 0\} \\ &= \{X \oplus 0 \mid X \in H^\gamma, \mathrm{i}_X \sigma = 0\} = \ker(\sigma^\flat|_{H^\gamma}) \oplus \{0\}. \end{aligned}$$

Thus, D is the graph of a bivector if and only if $\ker(\sigma^\flat|_{H^\gamma}) = \{0\}$. Since $H^\gamma \oplus V = TM$, and $V \subseteq \ker \sigma^\flat$, we conclude that the condition for D to be the graph of a bivector is equivalent to $\ker \sigma^\flat = V$. ■

2.2 Generalities on foliations and connections

For the purposes of this work, we are interested in the application of our previous description of coupling structures on vector spaces to the smooth category. More precisely, we apply our previous characterization to Dirac structures on foliated manifolds. In this sense, we need to review some basics on regular foliations, as well as some other structures that interrelate to describe the geometry of coupling structures, such as Ehresmann connections.

Regular foliations. Suppose we are given a regular foliation $\mathcal{V}$ on M, and let $V := T\mathcal{V}$ be the tangent bundle of $\mathcal{V}$. A multivector field $A \in \Gamma(\wedge^\bullet TM)$ on the foliated manifold $(M, \mathcal{V})$ is said to be *leaf-tangent* if $A \in \Gamma(\wedge^\bullet V)$. Since V is involutive, the Schouten-Nijenhuis bracket of leaf-tangent multivector fields is again leaf-tangent. This implies that $\Gamma(\wedge^\bullet V)$ is a graded Poisson subalgebra of the algebra of multivector fields.

A vector field $u \in \Gamma(TM)$ is said to be an *infinitesimal automorphism of* $(M, \mathcal{V})$, or, shortly, $\mathcal{V}$*-projectable*, if $[u, Y] \in \Gamma(V)$ for all $Y \in \Gamma(V)$. The set of $\mathcal{V}$-projectable vector fields will be denoted by $\mathrm{aut}(M, \mathcal{V})$. Algebraically, $\mathrm{aut}(M, \mathcal{V})$ is the normalizer of the Lie subalgebra of leaf-tangent vector fields $\Gamma(V)$, so $\mathrm{aut}(M, \mathcal{V})$ is an $\mathbb{R}$-Lie algebra containing $\Gamma(V)$. As a consequence, $[\mathrm{aut}(M, \mathcal{V}), \Gamma(\wedge^\bullet V)] \subseteq \Gamma(\wedge^\bullet V)$. It is important to remark that, for any regular foliation $\mathcal{V}$, the tangent bundle TM is locally finitely generated by $\mathcal{V}$-projectable vector fields.

Denote by $V^\circ := \mathrm{Ann}(V)$ the annihilator of the tangent bundle of $\mathcal{V}$. Then, $\Gamma(\wedge^\bullet V^\circ)$ is the exterior subalgebra of differential forms which vanish if any of its arguments is leaf-tangent. Moreover, the algebra of $\mathcal{V}$-projectable vector fields preserves $\Gamma(\wedge^\bullet V^\circ)$, in the sense that $\mathrm{L}_u \alpha \in \Gamma(\wedge^\bullet V^\circ)$ for all $u \in \mathrm{aut}(M, \mathcal{V})$ and $\alpha \in \Gamma(\wedge^\bullet V^\circ)$. Finally, note that $\Gamma(\wedge^\bullet V^\circ)$ is not a cochain subcomplex of the de Rham complex of M. In fact, the involutivity of V is equivalent to $\mathrm{i}_Y \mathrm{d}\,\alpha \in \Gamma(\wedge^\bullet V^\circ)$ for all $\alpha \in \Gamma(\wedge^\bullet V^\circ)$ and $Y \in \Gamma(V)$.

The contravariant analog of the $\mathcal{V}$-projectable vector fields are the basic differential forms. A k-form $\alpha \in \Gamma(\wedge^k V^\circ)$ is said to be *basic* if $\mathrm{d}\,\alpha \in \Gamma(\wedge^{k+1}V^\circ)$. The $\mathbb{R}$-space of basic k-forms is denoted by $\Gamma_{\mathrm{b}}(\wedge^k V^\circ)$. It is clear that $\Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ) := \bigoplus_{k\in\mathbb{Z}} \Gamma_{\mathrm{b}}(\wedge^k V^\circ)$ is a subalgebra of the differential forms with the exterior product, and a subcomplex of the de Rham complex $(\Gamma(\wedge^\bullet T^*M), \mathrm{d})$. Furthermore, for all $u \in \mathrm{aut}(M,\mathcal{V})$, and $\alpha \in \Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ)$, we have $\mathrm{i}_u\,\alpha \in \Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ)$. In particular, the algebra $\Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ)$ of basic forms is preserved by the $\mathcal{V}$-projectable vector fields,

$$\mathrm{L}_u\,\alpha \in \Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ), \qquad\qquad \forall u \in \mathrm{aut}(M,\mathcal{V}), \alpha \in \Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ).$$

For any regular foliation $\mathcal{V}$, the basic forms locally generate $\Gamma(\wedge^\bullet V^\circ)$.

Associated with a foliation $\mathcal{V}$ on M, the algebra of *leaf-wise differential forms* is $\Gamma(\wedge^\bullet V^*)$. The leaf-tangent multivector fields naturally act on $\Gamma(\wedge^\bullet V^*)$ by insertion, and for each $\mu \in \Gamma(\wedge^k V^*)$ and $u \in \mathrm{aut}(M,\mathcal{V})$, the Lie derivative $\mathrm{L}_u\,\mu$ is well defined by the standard formula

$$\mathrm{L}_u\,\mu(Y_1,\dots,Y_k) := \mathrm{L}_u(\mu(Y_1,\dots,Y_k)) - \sum_{i=1}^{k} \mu(Y_1,\dots,[u,Y_i],\dots,Y_k), \qquad \forall Y_i \in \Gamma(V).$$

Moreover, the *foliated exterior derivative* $\mathrm{d}_\mathcal{V} : \Gamma(\wedge^\bullet V^*) \to \Gamma(\wedge^\bullet V^*)$, is defined on $\mu \in \Gamma(\wedge^k V^*)$ by

$$\mathrm{d}_\mathcal{V}\,\mu(Y_1,\dots,Y_{k+1}) := \sum_{\sigma\in S_{(1,k)}} (-1)^\sigma\, \mathrm{L}_{Y_{\sigma_1}}(\mu(Y_{\sigma_2},\dots,Y_{\sigma_{k+1}})) - \sum_{\sigma\in S_{(2,k-1)}} (-1)^\sigma \mu([Y_{\sigma_1},Y_{\sigma_2}],Y_{\sigma_3},\dots,Y_{\sigma_{k+1}})$$

for all $Y_1,\dots,Y_{k+1} \in \Gamma(V)$. This is a graded derivation of degree 1 and a coboundary operator, $\mathrm{d}_\mathcal{V}^2 = 0$. The cochain complex $(\Gamma(\wedge^\bullet V^*), \mathrm{d}_\mathcal{V})$ is called the *foliated de Rham complex.* It is immediate to see that the foliated de Rham complex is precisely the de Rham complex of the tangent Lie algebroid $(V, \iota, [\cdot,\cdot]_V)$, where $\iota : V \to TM$ is the natural inclusion, and $[\cdot,\cdot]_V$ the restriction of the Lie bracket to the leaf-tangent vector fields (see Example 1.7). Its cohomology is called *foliated de Rham cohomology* and denoted by $H^\bullet_{\mathrm{dR}}(\mathcal{V})$.

The Frölicher-Nijenhuis bracket. Consider the Cartan's algebra $\Gamma(\wedge^\bullet T^*M)$ with the exterior product $\wedge$. Recall that, for each vector-valued form $K \in \Gamma(\wedge^k T^*M \otimes TM)$, one has the graded derivation i_K of $\Gamma(\wedge^\bullet T^*M)$ given on $\alpha \in \Gamma(\wedge^p T^*M)$ and $X_1,\dots,X_{p+k-1} \in \Gamma(TM)$ by

$$\mathrm{i}_K\,\alpha(X_1,\dots,X_{p+k-1}) := \sum_{\sigma\in S_{(k,p-1)}} (-1)^\sigma \alpha(K(X_{\sigma_1},\dots,X_{\sigma_k}),X_{\sigma_{k+1}},\dots,X_{\sigma_{p+k-1}}).$$

The graded derivation i_K has degree $k-1$ and is called the *insertion* of K. Similarly, if $\alpha \in \Gamma(\wedge^p T^*M \otimes TM)$, we define $\mathrm{i}_K\,\alpha \in \Gamma(\wedge^{p+k-1}T^*M \otimes TM)$ by the formula in above.

On the other hand the vector-valued form $K \in \Gamma(\wedge^k T^*M \otimes TM)$ induces the graded derivation L_K of degree k, defined in terms of the graded commutator with the exterior differential d,

$$\mathrm{L}_K := [\mathrm{i}_K, \mathrm{d}] = \mathrm{i}_K \circ \mathrm{d} + (-1)^k\, \mathrm{d} \circ \mathrm{i}_K\,.$$

The graded derivation L_K is called the *Lie derivative* along K. Moreover, because of the Jacobi identity of the graded commutator, it follows that $[\mathrm{L}_K, \mathrm{d}] = 0$. Also, $\mathrm{L}_K = [\mathrm{i}_K, \mathrm{d}] = 0$ if and only if $K = 0$.

It can be shown that every graded derivation of the Cartan's algebra is the sum of a Lie derivative and a insertion of vector-valued forms. More precisely,

Theorem 2.7 (Frölicher-Nijenhuis). *Let $D : \Gamma(\wedge^\bullet T^*M) \to \Gamma(\wedge^\bullet T^*M)$ be a derivation of degree k. There exist unique vector-valued forms $K \in \Gamma(\wedge^k T^*M \otimes TM)$, $L \in \Gamma(\wedge^{k+1} T^*M \otimes TM)$ such that*

$$D = \mathrm{L}_K + \mathrm{i}_L\,.$$

The proof of this fact can be found in [35, Subsection 8.3].

The Frölicher-Nijenhuis theorem, and the fact that the correspondence $K \mapsto \mathrm{L}_K$ is injective, implies that the only graded derivations commuting with the exterior differential are Lie derivatives,

$$[D, \mathrm{d}] = 0 \Leftrightarrow D = \mathrm{L}_K\,.$$

As a consequence, we have the following fact: given $K \in \Gamma(\wedge^k T^*M \otimes TM)$ and $L \in \Gamma(\wedge^l T^*M \otimes TM)$, the graded derivation $[\mathrm{L}_K, \mathrm{L}_L]$ of degree $k+l$ commutes with d, due to the Jacobi identity. Therefore, $[\mathrm{L}_K, \mathrm{L}_L]$ must be a Lie derivative. In other words, there exists a unique vector-valued form $[K, L]_{FN} \in \Gamma(\wedge^{k+l} T^*M \otimes TM)$ such that

$$\mathrm{L}_{[K,L]_{FN}} = [\mathrm{L}_K, \mathrm{L}_L].$$

This defines an $\mathbb{R}$-bilinear operation

$$[\cdot,\cdot]_{FN} : \Gamma(\wedge^\bullet T^*M \otimes TM) \times \Gamma(\wedge^\bullet T^*M \otimes TM) \to \Gamma(\wedge^\bullet T^*M \otimes TM),$$

called the *Frölicher-Nijenhuis bracket*. Because of the definition of the Frölicher-Nijenhuis bracket, the Lie derivative is a faithful representation of the vector-valued forms on the algebra of graded derivations with the graded commutator. In particular, the Frölicher-Nijenhuis bracket is a graded Lie bracket on $\Gamma(\wedge^\bullet T^*M \otimes TM)$ of degree zero.

We now present some properties regarding the Frölicher-Nijenhuis bracket. For more details, see [35, Section 8].

Lemma 2.8. *Let $K \in \Gamma(\wedge^k T^*M \otimes TM)$ and $L \in \Gamma(\wedge^l T^*M \otimes TM)$ be vector-valued forms. Then,*

$$[\mathrm{L}_K, \mathrm{i}_L] = \mathrm{i}_{[K,L]_{FN}} - (-1)^{kl}\, \mathrm{L}_{\mathrm{i}_L K}\,.$$

In the case when $K, L \in \Gamma(\wedge^1 T^*M \otimes TM)$, we have for $X, Y \in \Gamma(TM)$ that

$$\begin{aligned}[K, L]_{FN}(X, Y) = [KX, LY] - [KY, LX] - L([KX, Y] - [KY, X])\\ - K([LX, Y] - [LY, X]) + (LK + KL)[X, Y].\end{aligned}$$

Connections. Following [35, Section 8.13], recall that a connection form on M is a vector-valued 1-form $\gamma \in \Gamma(T^*M \otimes TM)$ corresponding to a vector bundle map $\gamma : TM \to TM$ such that $\gamma^2 = \gamma$. If we denote $H^\gamma := \ker(\gamma)$, and $V^\gamma := \mathrm{im}(\gamma)$, then we have

$$H^\gamma \oplus V^\gamma = TM. \tag{2.3}$$

In this case, H^γ and V^γ are said to be the *horizontal* and *vertical* distributions of γ, respectively. The *curvature* $R^\gamma \in \Gamma(\wedge^2 T^*M \otimes V^\gamma)$ and the *cocurvature* $S^\gamma \in \Gamma(\wedge^2 T^*M \otimes H^\gamma)$ of γ are

$$R^\gamma(X,Y) := \gamma[(\mathrm{Id}-\gamma)X, (\mathrm{Id}-\gamma)Y], \qquad S^\gamma(X,Y) = (\mathrm{Id}-\gamma)[\gamma X, \gamma Y], \qquad \forall X,Y \in \Gamma(TM),$$

which are the obstructions to the integrability of the distributions H^γ and V^γ, respectively.

Due to Lemma 2.8, the curvature and cocurvature of γ are related with the Frölicher-Nijenhuis bracket by

$$\tfrac{1}{2}[\gamma,\gamma]_{FN} = R^\gamma + S^\gamma.$$

Splitting (2.3) induces γ-dependent bigradings of the exterior algebras of multivector fields and differential forms on M. To see this, denote $\wedge^{p,q}TM := \wedge^p H^\gamma \otimes \wedge^q V^\gamma$ and $\wedge^{p,q}T^*M := \wedge^p \operatorname{Ann}(V^\gamma) \otimes \wedge^q \operatorname{Ann}(H^\gamma)$, where $\operatorname{Ann}(H^\gamma)$ and $\operatorname{Ann}(V^\gamma)$ are the annihilators of H^γ and V^γ, respectively. Then,

$$\Gamma(\wedge^\bullet TM) = \bigoplus_{p,q\in\mathbb{Z}} \Gamma(\wedge^{p,q}TM), \qquad \Gamma(\wedge^\bullet T^*M) = \bigoplus_{p,q\in\mathbb{Z}} \Gamma(\wedge^{p,q}T^*M). \tag{2.4}$$

For a multivector field A, the term of bidegree (p,q) in decomposition (2.4) is denoted by $A_{p,q}$. We follow same notation for differential forms. In a similar manner, we have a bigraded decomposition for any linear operator on these exterior algebras. For instance, and as a consequence of Proposition C.4 of the Appendix, the exterior differential splits as $\mathrm{d} = \mathrm{d}^\gamma_{2,-1} + \mathrm{d}^\gamma_{1,0} + \mathrm{d}^\gamma_{0,1} + \mathrm{d}^\gamma_{-1,2}$.

Connections on foliated manifolds. Suppose we are given a regular foliation $\mathcal{V}$ on M, and denote $V := T\mathcal{V}$. A connection on the foliated manifold $(M,\mathcal{V})$ is a connection γ on M such that $V^\gamma = V$. In this case, the horizontal distribution $H := \ker\gamma$ is a subbundle normal to $\mathcal{V}$, $H \oplus T\mathcal{V} = TM$. Furthermore, it follows from the involutivity of V that the cocurvature of γ is trivial, $S^\gamma = 0$. This implies the following relation between γ, its curvature R^γ, and the Frölicher-Nijenhuis bracket:

$$R^\gamma = \tfrac{1}{2}[\gamma,\gamma]_{FN}. \tag{2.5}$$

By Corollary C.6, $\mathrm{d} = \mathrm{d}^\gamma_{2,-1} + \mathrm{d}^\gamma_{1,0} + \mathrm{d}^\gamma_{0,1}$, and $(\mathrm{d}^\gamma_{0,1})^2 = 0$. Here, $\mathrm{d}^\gamma_{1,0}$ is the *covariant exterior derivative* of γ, and $\mathrm{d}^\gamma_{2,-1} = -\,\mathrm{i}_{R^\gamma}$. Furthermore, $\mathrm{d}^\gamma_{0,1}$ corresponds to the *foliated exterior derivative* introduced in Section 2.2. More precisely, the natural inclusion of the $\mathcal{V}$-leaves in M induces a cochain complex isomorphism, $(\Gamma(\wedge^\bullet V^*), \mathrm{d}_\mathcal{V}) \cong (\Gamma(\wedge^\bullet H^\circ), \mathrm{d}^\gamma_{0,1})$.

Now, observe that $(\mathrm{Id}-\gamma) \in \Gamma(V^\circ \otimes TM)$ can be viewed as a map $(\mathrm{Id}-\gamma) : TM \to TM$ such that

$$(\mathrm{Id}-\gamma)\,\mathrm{aut}(M,\mathcal{V}) = \Gamma(H^\gamma) \cap \mathrm{aut}(M,\mathcal{V}).$$

To see this, fix $u \in \mathrm{aut}(M,\mathcal{V})$. Since $\mathrm{im}(\gamma) = \Gamma(V) \subseteq \mathrm{aut}(M,\mathcal{V})$, one has $(\mathrm{Id}-\gamma)u = u - \gamma(u)$, which is the difference of elements in $\mathrm{aut}(M,\mathcal{V})$. Hence, $(\mathrm{Id}-\gamma)u \in \Gamma(H^\gamma) \cap \mathrm{aut}(M,\mathcal{V})$.

Remark 2.9. *For an arbitrary regular foliation $\mathcal{V}$, global $\mathcal{V}$-projectable vector fields transversal to $\mathcal{V}$ do not necessarily exist. Therefore, given a connection γ on $(M,\mathcal{V})$, the γ-horizontal $\mathcal{V}$-projectable vector fields* $\mathrm{aut}(M,\mathcal{V}) \cap \Gamma(H^\gamma)$ *are considered as local vector fields. Of course, if the foliation $\mathcal{V}$ is given by the fibers of a submersion $\pi : M \to B$, then the horizontal lifts* $\mathrm{hor}^\gamma(u)$ *of $u \in \Gamma(TB)$ are $\mathcal{V}$-projectable,* $\mathrm{hor}^\gamma(u) \in \mathrm{aut}(M,\mathcal{V}) \cap \Gamma(H^\gamma)$.

Finally, we say that a connection γ on $(M,\mathcal{V})$ is *flat* if the horizontal distribution H^γ is involutive, $R^\gamma = 0$. In this case, $\mathrm{d} = \mathrm{d}^\gamma_{0,1} + \mathrm{d}^\gamma_{1,0}$, where $\mathrm{d}^\gamma_{0,1}$ and $\mathrm{d}^\gamma_{1,0}$ are graded commutative coboundary operators corresponding to the foliated exterior derivatives of $\mathcal{V}$ and the foliation $\mathcal{H}$ integrating H^γ, $T\mathcal{H} = H^\gamma$.

2.3 Geometric data

Roughly speaking, a Poisson foliation is a regular foliation such that each leaf is endowed with a Poisson structure varying smoothly from leaf to leaf. Contrary to the case of leaf-wise differential forms, a leaf-wise multivector field defines a global leaf-tangent multivector field whose restriction to the leaves coincides with the leaf-wise one.

Definition 2.10. *A Poisson foliation is a triple $(M, \mathcal{V}, P)$, where $\mathcal{V}$ is a regular foliation on M, and P is a leaf-tangent Poisson structure: $[P, P] = 0$, and $P \in \Gamma(\wedge^2 T\mathcal{V})$.*

Given a Poisson foliation $(M, \mathcal{V}, P)$, the Hamiltonian vector fields of P are leaf-tangent, $P^\sharp \,\mathrm{d} f \in \Gamma(T\mathcal{V})$ for all $f \in C^\infty(M)$. Therefore, the symplectic leaves of P are contained in the leaves of $\mathcal{V}$. On the other hand, the infinitesimal Poisson automorphisms of P can be transversal to the foliation $\mathcal{V}$. In particular, this occurs whenever a Poisson foliation admits a *Poisson connection*.

Definition 2.11. *A Poisson connection γ on the Poisson foliation $(M, \mathcal{V}, P)$ is a connection on $(M, \mathcal{V})$ such that P is γ-parallel.*

The γ-parallel condition for P can be expressed in infinitesimal terms as $\Gamma(H^\gamma) \cap \mathrm{aut}(M, \mathcal{V}) \subseteq \mathrm{Poiss}(M, P)$. In other words, the γ-horizontal $\mathcal{V}$-projectable vector fields are Poisson,

$$\mathrm{L}_u P = 0 \qquad\qquad \forall u \in \Gamma(H^\gamma) \cap \mathrm{aut}(M, \mathcal{V}).$$

Denote by $A^\gamma := \mathrm{Ann}(H^\gamma)$ the annihilator of H^γ. Then, for arbitrary elements in $\Gamma(H^\gamma)$, the condition for γ to be Poisson on $(M, \mathcal{V}, P)$ becomes

$$\mathrm{L}_X P(\mu, \nu) = 0 \qquad\qquad \forall X \in \Gamma(H^\gamma), \mu, \nu \in \Gamma(A^\gamma).$$

Example 2.12 (Regular Poisson structures)**.** A Poisson structure Π such that the map $\Pi^\sharp : TM \to T^*M$ is of locally constant rank is said to be *regular*. In this case, the symplectic foliation $(\mathcal{S}, \omega_{\mathcal{S}})$ can be considered as a very special Poisson foliation $(M, \mathcal{S}, \Pi_{\omega_{\mathcal{S}}})$, such that $\Pi_{\omega_{\mathcal{S}}}$ is leaf-wise symplectic. If such Poisson foliation admits a Poisson connection, then Π is said to be transversally constant [62, pp. 959-960], [64, pp. 49-50]. ▼

An special class of Poisson foliations are, of course, Poisson fibrations.

Definition 2.13. *A* Poisson fibration *is a pair $(E \xrightarrow{\pi} M, P)$ in which π is a surjective submersion and $P \in \Gamma(\wedge^2 V)$ is a vertical Poisson structure. Given two Poisson fibrations $(E_1 \xrightarrow{\pi_1} M, P_1)$, and $(E_2 \xrightarrow{\pi_2} M, P_2)$ over M, a* morphism of Poisson fibrations *is a Poisson map $\phi : (E_1, P_1) \to (E_1, P_1)$ such that $\pi_2 \circ \phi = \pi_1$. A Poisson fibration $(E \xrightarrow{\pi} M, P)$ is said to be* locally trivial *if there exists a Poisson manifold (N, Υ), and an open cover $\mathcal{U}$ of M such that, for all $U \in \mathcal{U}$, $\pi^{-1}(U)$ and $U \times N$ are isomorphic as a Poisson foliations. If the open cover can be chosen as $\mathcal{U} = \{M\}$, then the Poisson fibration $(E \xrightarrow{\pi} M, P)$ is said to be* trivial.

Example 2.14 (Lie-Poisson bundles)**.** Let $(A \to M, q, [\cdot,\cdot]_A)$ be a transitive Lie algebroid. The isotropy bundle $I := \ker(q)$ has constant rank and hence is a vector subbundle of A. Furthermore, because of the Leibniz rule, $\Gamma(I)$ is closed with respect to $[\cdot,\cdot]_A$. Moreover, the restriction $[\cdot,\cdot]_I := [\cdot,\cdot]_A|_{\Gamma(I)\times\Gamma(I)}$ is $C^\infty(M)$-linear. So, one can think of $(I \to M, [\cdot,\cdot]_I)$ as a bundle of Lie algebras.

Therefore, the *coisotropy bundle* $E := I^*$ is a fiber-wise linear Poisson fibration over M. The fiber-wise linear Poisson structure P is defined as in (1.4), viewing I as a Lie algebroid. Furthermore, in this case $(E \xrightarrow{\pi} M, P)$ is a Poisson fibration. Indeed, since the anchor of I is zero, we get that $P^\sharp \, \mathrm{d}\, \pi^* f = 0$ for all $f \in C^\infty(M)$, which implies that $P \in \Gamma(\wedge^2 V)$. Since P is fiber-wise linear, $(E \xrightarrow{\pi} M, P)$ is in this case a Lie-Poisson bundle. Conversely, every Lie-Poisson bundle is the dual of a bundle of Lie algebras. ▼

Example 2.15. Let $P \xrightarrow{p} M$ be a principal G-bundle. Then, $A := TP/G$ is a transitive Lie algebroid (see Example 1.9). The isotropy bundle is $I = P \times_G \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G, together with the adjoint action. This is a vector bundle over M, which is locally trivial as a bundle of Lie algebras with typical fiber $\mathfrak{g}$. Therefore, the coisotropy bundle $E := I^*$ is locally trivial as a Lie-Poisson bundle. ▼

It is clear that trivial Poisson fibrations are endowed with a Poisson connection. Furthermore, this is also true for locally trivial Poisson fibrations.

Example 2.16 (Trivial Poisson fibrations)**.** Let M be a manifold, (N, Υ) a Poisson manifold, and consider the corresponding trivial Poisson fibration $(M \times N, P)$. Consider also the tangent bundle to the foliation $\mathcal{F} := \{M \times \{n\} \mid n \in N\}$ as the horizontal distribution of a connection γ on $(M \times N, P)$. It is clear that γ is a flat Poisson connection. ▼

Example 2.17 (Locally trivial Poisson fibrations)**.** Every locally trivial Poisson fibration admits a Poisson connection. In fact, let $(E \xrightarrow{\pi} M, P)$ be a locally trivial Poisson fibration, and $\mathcal{U}$ an open cover of M satisfying the triviality condition. For each $U \in \mathcal{U}$, denote by γ_U the Poisson connection defined on the trivial Poisson fibration $\pi^{-1}(U)$. Given a partition of unity $\{\rho_U : U \in \mathcal{U}\}$ on M subordinated to $\mathcal{U}$, it follows that $\gamma := \sum_{U \in \mathcal{U}} \pi^* \rho_U \gamma_U$ is a Poisson connection on E. This follows from the facts that $\sum_{U \in \mathcal{U}} \rho_U = 1$ and $\pi^* f \in \mathrm{Casim}(E, P)$ for all $f \in C^\infty(M)$. ▼

Example 2.18 (Coisotropy bundles)**.** Let $(A \to M, q, [\cdot,\cdot]_A)$ be a transitive Lie algebroid, and let $I := \ker(q)$ be the isotropy bundle. As explained above, the coisotropy bundle $E := I^*$ is a Lie-Poisson bundle, endowed with a fiber-wise linear and vertical Poisson structure P. Consider any subbundle $H \subset A$ complementary to I, $H \oplus I = A$. We claim that H induces a linear Poisson connection on $(E \xrightarrow{\pi} M, P)$. Since A is transitive, $\operatorname{rank} H = \operatorname{rank} A - \operatorname{rank} I = \operatorname{rank} = \operatorname{rank} TM$, which implies that $q|_H : H \to TM$ is an isomorphism of vector bundles. Let $h : TM \to H$ be the inverse of $q|_H$. Now, consider the map

$$\begin{aligned} \nabla : TM \times I &\to I, \\ (u, a) &\mapsto \nabla_u a := [h(u), a]_A. \end{aligned}$$

Since q is a Lie algebra morphism and $q(a) = 0$, it follows that $q([h(u), a]_A) = 0$, so ∇ is well defined. Furthermore, by the Leibniz rule,

$$\begin{aligned} \nabla_{fu} a &= [h(fu), a] = [f h(u), a] = f[h(u), a] - \mathrm{L}_{q(a)} f \cdot h(u) = f[h(u), a] = f \nabla_u a, \\ \nabla_u (fa) &= [h(u), fa] = f[h(u), a] + \mathrm{L}_{q(h(u))} f \cdot a = f \nabla_u a + \mathrm{L}_u f \cdot a, \end{aligned}$$

which means that ∇ is a linear connection on I. Furthermore, the dual connection $\nabla : TM \times E \to E$ is a Poisson connection on $(E \xrightarrow{\pi} M, P)$. In fact, for each $a \in \Gamma(I)$, consider the linear function

$\varphi_a : E \to \mathbb{R}$ given by $\varphi_a \alpha := \alpha(a_{\pi(\alpha)})$. Then, for each $u \in \Gamma(TM)$, the horizontal lift $\mathrm{hor}^{\nabla}(u)$ with respect to such dual connection is given by

$$\mathrm{L}_{\mathrm{hor}^{\nabla}(u)} \varphi_a = \varphi_{\nabla_u a}, \qquad \forall a \in \Gamma(I).$$

Now, fix $u \in TM$, $a, b \in I$. By the Jacobi identity of $[\cdot,\cdot]_A$,

$$\nabla_u [a,b]_A = [h(u), [a,b]_A]_A = [[h(u),a]_A, b]_A + [a, [h(u),b]_A]_A = [\nabla_u a, b]_A + [a, \nabla_u b]_A.$$

By applying φ on both sides and using the fact that $P(\mathrm{d}\,\varphi_a, \mathrm{d}\,\varphi_b) = \varphi_{[a,b]_A}$, we get

$$\mathrm{L}_{\mathrm{hor}^{\nabla}(u)}(P(\mathrm{d}\,\varphi_a, \mathrm{d}\,\varphi_b)) = P(\mathrm{d}\,\mathrm{L}_{\mathrm{hor}^{\nabla}(u)}\,\varphi_a, \mathrm{d}\,\varphi_b) + P(\mathrm{d}\,\varphi_a, \mathrm{d}\,\mathrm{L}_{\mathrm{hor}^{\nabla}(u)}\,\varphi_b).$$

Therefore, $\mathrm{hor}^{\nabla}(u) \in \mathrm{Poiss}(E,P)$ for all $u \in \Gamma(TM)$, proving that ∇ is a Poisson connection. ▼

Curvature. Given a Poisson connection γ on a Poisson foliation $(M, \mathcal{V}, P)$, the curvature $R^\gamma \in \Gamma(\wedge^2 T^*M \otimes V)$ takes values in infinitesimal Poisson automorphisms,

$$R^\gamma(X,Y) \in \Gamma(V) \cap \mathrm{Poiss}(M,P), \qquad \forall X, Y \in \mathrm{aut}(M,\mathcal{V}).$$

To see this, fix $X, Y \in \Gamma(H^\gamma) \cap \mathrm{aut}(M,\mathcal{V})$. Since $\mathrm{aut}(M,\mathcal{V}) \cap \mathrm{Poiss}(M,P)$ is a Lie algebra, one has $[X,Y] \in \mathrm{aut}(M,\mathcal{V}) \cap \mathrm{Poiss}(M,P)$. On the other hand, since $(\mathrm{Id} - \gamma)$ maps $\mathrm{aut}(M,\mathcal{V})$ to $\Gamma(H^\gamma) \cap \mathrm{aut}(M,\mathcal{V})$, one has $(\mathrm{Id} - \gamma)[X,Y] \in \Gamma(H^\gamma) \cap \mathrm{aut}(M,\mathcal{V}) \subseteq \mathrm{Poiss}(M,P)$. Therefore, $R^\gamma(X,Y)$ can be expressed as the difference of infinitesimal Poisson automorphisms, as follows:

$$R^\gamma(X,Y) = [X,Y] - (\mathrm{Id} - \gamma)[X,Y] \in \mathrm{Poiss}(M,P).$$

A more special situation is when the curvature of a Poisson connection takes values in Hamiltonian vector fields.

Definition 2.19. *Let γ be a Poisson connection on $(M,\mathcal{V},P)$. We say that the curvature R^γ of γ is* locally Hamiltonian *if there exists a closed 3-form $\Theta \in \Gamma(\wedge^3 T^*M)$ such that*

$$R^\gamma(X,Y) = -P^\sharp\, \mathrm{i}_Y\, \mathrm{i}_X\, \Theta, \qquad \forall X, Y \in \Gamma(TM).$$

In this case, we say that R^γ is locally Hamiltonian via Θ*. If, in addition, $\Theta = \mathrm{d}\,\sigma$ for some $\sigma \in \Gamma(\wedge^2 V^\circ)$, then R^γ is called* Hamiltonian*, and σ is said to be the* Hamiltonian form *of R^γ.*

Clearly, if the curvature R^γ of γ is Hamiltonian, then R^γ is locally Hamiltonian. In terms of $\mathcal{V}$-projectable vector fields, the Hamiltonian condition for R^γ reads

$$R^\gamma(X,Y) = -P^\sharp\, \mathrm{d}(\sigma(X,Y)), \qquad \forall X, Y \in \Gamma(H^\gamma) \cap \mathrm{aut}(M,\mathcal{V}),$$

which is indeed a Hamiltonian vector field.

Example 2.20. Suppose that $(A, q, [\cdot,\cdot]_A)$ is a transitive Lie algebroid, and let $E \xrightarrow{\pi} M$ be the coisotropy bundle, $E := I^*$, $I := \ker(q)$, endowed with the fiber-wise Lie-Poisson structure P and the linear Poisson connection $\nabla : TM \times E \to E$ induced by the choice of some $H \subset A$ complementary to

I. We refer to Examples 2.14 and 2.18 for details. Let us show that the curvature of ∇ is Hamiltonian. In fact, first consider $\sigma^H \in \Gamma(\wedge^2 TM \otimes I)$ given by

$$\sigma^H(u,v) := [h(u), h(v)]_A - h[u,v].$$

Since $q \circ h = \mathrm{Id}_{TM}$, it follows that $q(\sigma^H(u,v)) = 0$, so in fact $\sigma^H(u,v) \in I$. Now, consider the curvature $R^\nabla \in \Gamma(\wedge^2 TM \otimes V)$ of the linear connection ∇ on E. Observe that for $u, v \in \Gamma(TM)$ and $a \in \Gamma(I)$,

$$\begin{aligned}
\mathrm{L}_{R^\nabla(u,v)}\,\varphi_a &= \mathrm{L}_{[\mathrm{hor}^\nabla(u),\mathrm{hor}^\nabla(v)]}\,\varphi_a - \mathrm{L}_{\mathrm{hor}^\nabla[u,v]}\,\varphi_a \\
&= \varphi_{\nabla_u\nabla_v a - \nabla_v\nabla_u a} - \varphi_{\nabla_{[u,v]}a} = \varphi_{[h(u),[h(v),a]_A]_A - [h(v),[h(u),a]_A]_A} - \varphi_{[h[u,v],a]_A} \\
&= \varphi_{[[h(u),h(v)]_A - h[u,v],a]_A} = \varphi_{[\sigma^H(u,v),a]} \\
&= P(\mathrm{d}\,\varphi_{\sigma^H(u,v)}, \mathrm{d}\,\varphi_a) = \mathrm{L}_{P^\sharp\,\mathrm{d}\,\varphi_{\sigma^H(u,v)}}\,\varphi_a
\end{aligned}$$

This shows that $R^\nabla(u,v) = P^\sharp\,\mathrm{d}\,\varphi_{\sigma^H(u,v)}$. In other words, the curvature of ∇ is Hamiltonian. A similar computation shows that $\nabla\sigma^H = 0$, which means that σ^H is ∇-covariantly constant. ▼

Cohomology of Poisson foliations. Associated with a Poisson foliation $(M, \mathcal{V}, P)$, there exists an intrinsic Lie algebroid $(V^*, \iota \circ P^\sharp \circ \iota^*, \{\cdot,\cdot\}_P)$, where $V := T\mathcal{V}$, $\iota : V \hookrightarrow TM$ is the inclusion map, and

$$\{\mu,\nu\}_P = \mathrm{L}_{P^\sharp\mu}\,\nu - \mathrm{i}_{P^\sharp\nu}\,\mathrm{d}_{\mathcal{V}}\,\mu, \qquad \forall \mu,\nu \in \Gamma(V^*).$$

Here, $\mathrm{d}_{\mathcal{V}} \in \mathrm{Der}^1\,\Gamma(\wedge^\bullet V^*)$ is the foliated exterior derivative, and the Lie derivatives are defined by the standard formula (see Section 2.3). The de Rham differential $\partial_P : \Gamma(\wedge^\bullet V) \to \Gamma(\wedge^\bullet V)$ of $(V^*, \iota \circ P^\sharp \circ \iota^*, \{\cdot,\cdot\}_P)$ coincides with the Lichnerowicz-Poisson operator of the Poisson manifold (M, P) restricted to the algebra of leaf-tangent multivector fields, $\partial_P := [P, \,\cdot\,]$. The Lie algebroid cohomology is denoted by $H_P^\bullet(M, \mathcal{V})$. The map $P^\sharp : V^* \to V$ is a morphism from the Lie algebroid on V^* to $(V, \iota, [\cdot,\cdot]_V)$, which induces a linear mapping in cohomology $(P^\sharp)^* : H^\bullet_{\mathrm{dR}}(\mathcal{V}) \to H^\bullet_P(M, \mathcal{V})$. If we denote the space of cocycles of $(\Gamma(\wedge^\bullet V), \partial_P)$ by $Z_P^\bullet(M, \mathcal{V})$, then it is clear that $Z_P^\bullet(M, \mathcal{V}) = Z_P^\bullet(M) \cap \Gamma(\wedge^\bullet V)$, which are the leaf-tangent cocycles of the Poisson manifold (M, P). In particular,

$$H_P^1(M, \mathcal{V}) = \frac{\mathrm{Poiss}_{\mathcal{V}}(M, P)}{\mathrm{Ham}(M, P)}$$

is the quotient of the leaf-tangent infinitesimal Poisson automorphisms $\mathrm{Poiss}_{\mathcal{V}}(M, P) := \Gamma(V) \cap \mathrm{Poiss}(M, P)$ by the Hamiltonian vector fields of P.

2.4 The coupling method on foliated manifolds

Let M be a manifold and consider its Pontryagin bundle $\mathbb{T}M := TM \oplus T^*M$. Let also $V \subseteq TM$ be a regular distribution on M and $V^\circ := \mathrm{Ann}(V) \subseteq T^*M$ be its annihilator. Let us also denote $\mathbb{V} := V \oplus V^\circ \subset \mathbb{T}M$.

Definition 2.21. *A Lagrangian subbundle $D \subseteq \mathbb{T}M$ is said to be V-coupling if*

$$D \oplus \mathbb{V} = \mathbb{T}M. \tag{2.6}$$

Definition 2.22. *A triple (P, γ, σ) of geometric data on (M, V) consists of a bivector field $P \in \Gamma(\wedge^2 V)$, a connection form $\gamma \in \Gamma(T^*M \otimes V)$, and a 2-form $\sigma \in \Gamma(\wedge^2 V^\circ)$.*

The following result means that the correspondence given in Proposition 2.4 can be extended from vector spaces to the category of smooth manifolds.

Proposition 2.23. *There exists a correspondence between V-coupling Lagrangian subbundles and geometric data on (M, V).*

Proof. Because of Proposition 2.4, there exists a point-wise one-to-one correspondence between V-coupling Lagrangian subbundles and geometric data on (M, V). Furthermore, given geometric data (P, γ, σ), it is clear that $D_{P,\gamma,\sigma}$ is a (smooth) Lagrangian subbundle. Conversely, given a V-coupling subbundle $D \subseteq \mathbb{T}M$, observe that the distributions $H(D, V) \subseteq TM$ and $A(D, V) \subseteq T^*M$ defined as in (2.1) and (2.2) are smooth. Indeed, since projection $p_{TM} : D \cap (TM \oplus V^\circ) \to H(D, V)$ is a pointwise isomorphism, the distribution $D \cap (TM \oplus V^\circ)$ has constant rank. On the other hand, $D \cap (TM \oplus V^\circ)$ is the orthogonal complement of the smooth subbundle $(D + V \oplus \{0\}) \subseteq \mathbb{T}M$. Since $D \cap (TM \oplus V^\circ)$ has constant rank, so it has $(D + V \oplus \{0\})$. This and the smoothness of $(D + V \oplus \{0\})$ implies that $D \cap (TM \oplus V^\circ)$ is smooth. Hence, $H(D, V) = p_{TM}(D \cap (TM \oplus V^\circ))$ is smooth. In particular, its annihilator $A(D, V)$ is also smooth. Therefore,

$$\begin{gathered}
P_D^\sharp|_{A(D,V)} = p_{TM} \circ \big(p_{T^*M} : D \to A(D, V)\big)^{-1}, \\
\gamma_D = \mathrm{pr}_V : H(D, V) \oplus V \to V, \\
\sigma_D^\flat|_{H(D,V)} = p_{T^*M} \circ \big(p_{TM} : D \to H(D, V)\big)^{-1}
\end{gathered}$$

are smooth. ∎

Now, suppose we are given a foliated manifold $(M, \mathcal{V})$. We have the following result [77, 20, 65, 67]:

Theorem 2.24. *There is a one-to-one correspondence between coupling Dirac structures $D \subset \mathbb{T}M$ on $(M, \mathcal{V})$ and geometric data (P, γ, σ) satisfying*

$$[P, P] = 0, \qquad [X, P](\mu, \nu) = 0, \qquad R^\gamma(X, Y) = -P^\sharp \,\mathrm{i}_Y \,\mathrm{i}_X \,\mathrm{d}\,\sigma, \qquad \mathrm{d}\,\sigma(X, Y, Z) = 0,$$

for all $X, Y, Z \in \Gamma(H^\gamma)$ and $\mu, \nu \in \Gamma(A^\gamma)$. Under this correspondence, D is the graph of a Poisson structure if and only if $\ker \sigma^\flat = V$.

The proof of this fact is given in the more general context of Poisson and Dirac structures with background in Chapter 4 (see Theorem 4.21).

A natural example of coupling Poisson structures are the so-called *Vorobiev-Poisson structures* [65, Section 4], [66, Section 6]. These coupling Poisson structures, which arise in the total space of the coisotropy bundle of a transitive Lie algebroid over a symplectic base, were introduced by Vorobiev in [77, Section 4]. In fact, one of the applications of this class of Poisson structures is that they arise as the Hamiltonian setting of the Wong's equations for a colored particle in a Yang-Mills filed [77, Example 4.2]. Another application, which is given in the context of the linearization problem, is that they provide the linear model for Poisson structures around symplectic leaves [77, Section 5], [78, Section 4], [73]. Finally, it can be shown that every transitive Lie algebroid around the zero section is isomorphic to the restricted Lie algebroid of a Vorobiev-Poisson structure [76].

Example 2.25 (Vorobiev-Poisson structures)**.** Let $(A \overset{p}{\to} S, q, [\cdot,\cdot]_A)$ be a transitive Lie algebroid. Following Examples 2.14, 2.18, and 2.20, recall that on the coisotropy bundle $E := \ker(\rho)^*$ has a natural structure of a Lie-Poisson bundle $(E \overset{\pi}{\to} S, P)$. On the other hand, the choice of a subbundle $H \subset A$ such that $H \oplus \ker(\rho) = A$ gives a linear connection $\nabla \equiv \nabla^H : TM \times E \to E$ which is Poisson on $(E \overset{\pi}{\to} S, P)$. Furthermore, the curvature R^∇ of ∇ is Hamiltonian, with Hamiltonian form $-\varphi \circ \sigma^H$,

$$R^\nabla(u,v) = P^\sharp \,\mathrm{d}\, \varphi_{\sigma^H(u,v)} \forall u,v \in \Gamma(TM). \tag{2.7}$$

Because of Theorem 2.24, there exists a coupling Dirac structure D_H on E associated with the linear geometric data $(P, \nabla^H, -\varphi \circ \sigma^H)$, depending on the choice of the subbundle H. It is important to observe that since $\varphi \circ \sigma^H$ takes values on linear functions on E, it vanishes along the zero section. In particular D_H cannot be the graph of a Poisson structure. Now suppose that the base manifold S is symplectic, with a symplectic structure $\omega \in \Gamma(\wedge^2 T^*S)$. Defining $\sigma^{\omega,H} := -\varphi \circ \sigma^H + \pi^*\omega$, we get that $(P, \nabla^H, \sigma^{\omega,H})$ defines again some geometric data, and defines a Dirac structure $D_{H,\omega}$. Furthermore, since ω is non-degenerated, there exists a neighborhood U of the zero section in which $D_{H,\omega}$ is the graph of a coupling Poisson structure $\Pi_{H,\omega} \in \Gamma(\wedge^2 TU)$. ▼

As described in the previous example, Vorobiev-Poisson structures admit a natural generalization to Dirac structures. In fact, in the particular case of the Atiyah algebroid of a principal bundle, this setting is called *the classical Yang-Mills-Higgs setup* [81, Section 3].

2.5 Coupling neighborhoods of a symplectic leaf

One of the most relevant applications of the coupling method in Poisson geometry is that coupling Poisson structures serve as the semilocal model for Poisson manifolds around symplectic leaves. In other words, given an embedded symplectic leaf, there exists a tubular neighborhood of it in which the Poisson structure is coupling. Let us recall how this fact is achieved.

Let $S \hookrightarrow M$ be an embedded symplectic leaf of the Poisson manifold (M, Π). By the Tubular Neighborhood Theorem [43, Appendix 1.6], there exists a tubular neighborhood $N \overset{\pi}{\to} S$ of S in M. In particular, N is endowed with the foliation $\mathcal{V}$ provided by the fibers of the tubular structure of N. Let us show that the neighborhood N can be shrunk so that the Poisson structure Π is coupling on $(N, \mathcal{V})$.

We begin by showing that the coupling condition holds at S. Since $N \overset{\pi}{\to} S$ is a tubular neighborhood of S, we have the splitting $TS \oplus T_S\mathcal{V} = T_SM$. This induces the dual splitting $\mathrm{Ann}(TS) \oplus \mathrm{Ann}(T_S\mathcal{V}) = T^*_SM$, where $\mathrm{Ann}(TS) = \ker \Pi^\sharp_S$. Then,

$$TS = \Pi^\sharp(T^*_SM) = \Pi^\sharp(\mathrm{Ann}(TS) \oplus \mathrm{Ann}(T_S\mathcal{V})) = \Pi^\sharp(\mathrm{Ann}(T_S\mathcal{V})),$$

so $\Pi^\sharp(\mathrm{Ann}(T_S\mathcal{V})) \oplus T_S\mathcal{V} = TS \oplus T_S\mathcal{V} = T_SM$. This proves that the coupling condition holds at S. By continuity, it follows that the neighborhood N can be shrunk so that the coupling condition holds on all N.

Definition 2.26. *A tubular neighborhood N of the symplectic leaf S of (M, Π) is said to be a* coupling neighborhood *if the restriction $\Pi|_N$ is a coupling Poisson structure on the fiber bundle $\pi : N \to S$.*

The above discussion means that coupling Poisson structures provide a semilocal model for Poisson structures around symplectic leaves. Furthermore, we have [78, Prop. 3.1]

Proposition 2.27. *Let (S,ω) be an embedded symplectic leaf of a Poisson manifold (M,Π). Then, there exists a coupling neighborhood N of S. Moreover, if (P,γ,σ) are the associated geometric data of $\Pi|_N$, then*

$$\operatorname{rank} P_S = 0, \qquad H^\gamma_S = TS, \qquad \iota_S^*\sigma = \omega.$$

Parametrization via exponential maps. Observe that the associated geometric data given in the above proposition are not unique, since they depend on the choice of the exponential map. In fact, given a coupling neighborhood of an embedded symplectic leaf, its tubular structure depends on the choice of a exponential map. In particular, the geometric data, which is defined in terms of the tubular structure, varies if the exponential map is changed.

Let $S \hookrightarrow M$ be an embedding and let $E := T_SM/TS$ be the normal bundle of S. Let $\nu : T_SM \to E$ be the canonical quotient map, and let $\tau : T_SE \to E$ be the projection along the splitting $T_SE = TS \oplus E$.

Definition 2.28. *A diffeomorphism onto its image $\mathbf{f} : E \to M$ is called* exponential map over S *if $\mathbf{f}|_S = \mathrm{Id}_S$ and $\nu \circ \mathrm{d}_S \mathbf{f} = \tau$.*

We remark that the existence of tubular neighborhoods follows in fact from the existence of exponential maps. More precisely, the fiber bundle structure $\pi : N \to S$ of a tubular neighborhood is defined in such a way that $\mathbf{f} : E \to N$ is a vector bundle isomorphism. Therefore, it is clear that the foliation $\mathcal{V}$ given by the fibers of N depends on the choice of $\mathbf{f}$, as well as the induced geometric data of Proposition 2.27.

Let us describe the transformation of the geometric data under the change of the exponential map. Let $i : E \to T_SE$ be the natural inclusion. Consider the pseudogroup $\mathrm{Ex}_S(E) \subset \mathrm{Diff}(E)$ given by

$$\mathrm{Ex}_S(E) := \{(\phi, \mathrm{Dom}\,\phi) \mid \phi \in \mathrm{Diff}(E), \phi|_S = \mathrm{Id}_S, S\,\mathrm{Dom}\,\phi, \tau \circ \mathrm{d}_S\,\phi \circ i = \mathrm{Id}_E\}.$$

We first observe that $\mathrm{Ex}_S(E)$ is contained in the connected component of the identity in $\mathrm{Diff}(E)$. In fact, for each $t \in \mathbb{R}$, let us denote by $\rho_t : E \to E$ the fiberwise scalar multiplication by t. By fixing $\phi \in \mathrm{Ex}_S(E)$, and an open convex subset $U \subset \mathrm{Dom}\,\phi$ containing S, one can show that

$$\phi^t := \rho_{-t} \circ \phi \circ \varrho_t : N \to E$$

is a curve of diffeomorphisms connecting $\phi^1 = \phi$ with $\phi^0 = \mathrm{Id}_E$. Furthermore, one can show by straighforward computations that the exponential maps are parameterized by the pseudogroup $\mathrm{Ex}_S(E)$. More precisely,

Lemma 2.29. *If $\mathbf{f}, \widetilde{\mathbf{f}} : E \to M$ are exponential maps, then $\widetilde{\mathbf{f}}^{-1} \circ \mathbf{f} \in \mathrm{Ex}_S(E)$. Conversely, if $\phi \in \mathrm{Ex}_S(E)$, then $\mathbf{f} \circ \phi$ is an exponential map.*

In terms of this class of diffeomorphisms we can establish the uniqueness of the transverse Poisson structure of a symplectic leaf [78, Theorem 3.2, Theorem 3.4]:

Theorem 2.30 (Geometric Data)**.** *Let (S,ω) be a symplectic leaf of a Poisson manifold (M,Π), and let $\mathbf{f}, \widetilde{\mathbf{f}} : E \to M$ be exponential maps over S. Let (P,γ,σ) and $(\widetilde{P},\widetilde{\gamma},\widetilde{\sigma})$ be the geometric data associated with $\mathbf{f}^*\Pi$, and $\widetilde{\mathbf{f}}^*\Pi$ on E. Then, there exist $g \in \mathrm{Ex}_S(E)$ preserving the fibers of E, and $Q \in \Gamma(V^\circ)$ such that*

$$g^*\widetilde{P} = P, \qquad \gamma - g^*\widetilde{\gamma} = P^\sharp(\mathrm{d}\,Q)^\flat, \qquad g^*\widetilde{\sigma} = \sigma - (\mathrm{d}^\gamma_{1,0}\,Q + \tfrac{1}{2}\{Q \wedge Q\}_P).$$

Part II

The Cohomology of Coupling Structures

Introduction to Part II

This part is devoted to provide a general scheme for the computation of the cohomology of Poisson and Dirac structures near presymplectic leaves.

The relationship between bigraded cochain complexes and the computation of Poisson cohomology in neighborhoods of symplectic leaves has been observed since long ago. In fact, in [74], Karasev and Vorobiev gave the first steps in a semilocal description of Poisson cohomology in terms of bigraded operators. Later on, by means of the coupling method, Crainic and Fernandes showed the existence of a cochain complex isomorphism between the Lichnerowicz-Poisson complex near a symplectic leaf and a bigraded cochain complex [15, Prop. 5.3]. This fact also holds for Dirac structures, as shown by Mărcuţ in [50, Prop. 4.2.8]. In fact, recall form Chapter 2 that coupling structures on a fiber bundle $E \to S$ are described by triples (P, γ, σ) of geometric data, consisting of a vertical bivector field P, an Ehresmann connection γ, and a horizontal 2-form σ satisfying some *structure equations*. Then, one can associate to (P, γ, σ) some linear bigraded operators $\partial^P_{0,1}, \partial^\gamma_{1,0}, \partial^\sigma_{2,-1}$ on a certain bigraded space $\mathscr{V}^{\bullet,\bullet}$ defined in terms of the base S and the fibration on E. The coboundary condition for the total operator $\partial_{P,\gamma,\sigma} := \partial^P_{0,1} + \partial^\gamma_{1,0} + \partial^\sigma_{2,-1}$ turns out to be equivalent to the structure equations for (P, γ, σ). Furthermore, the resulting bigraded cochain complex is isomorphic to the cochain complex of the Lie algebroid of the coupling structure. In other words, $(\mathscr{V}^{\bullet,\bullet}, \partial_{P,\gamma,\sigma})$ is a bigraded model for the description of the cohomology of coupling Poisson and Dirac structures, in which the differential operator admits a bigraded decomposition of the form

$$\partial_{P,\gamma,\sigma} := \partial_{0,1} + \partial_{1,0} + \partial_{2,-1}.$$

The fact that the cohomology can be modelled by a bigraded complex of the above bigraded type is not exclusive of coupling Poisson and Dirac structures. In fact, one of the first examples in geometry leading to this class of complexes are foliated manifolds. In [61], Vaisman explains in detail that the de Rham complex $(\Gamma(\wedge^\bullet T^*M), \mathrm{d})$ of a foliated manifold $(M, \mathcal{F})$ endowed with a complementary distribution H is bigraded, and the de Rham differential is the sum of three operators of bigraded types (0,1), (1,0), and (2,-1), $\mathrm{d} = \mathrm{d}^\gamma_{0,1} + \mathrm{d}^\gamma_{1,0} + \mathrm{d}^\gamma_{2,-1}$. Here, $\gamma \equiv \gamma^H$ is the Ehresmann connection associated with H.

Another example in which a cochain complex with a bigrading arises is the case of regular Poisson manifolds. In [74], Karasev and Vorobiev provided a recursive procedure to compute the cohomology of a Poisson manifold (M, Π) in which its symplectic foliation is a fibration. This procedure is based in the fact that the choice of an Ehresmann connection on the fibration leads to a bigrading of the Lichnerowicz-Poisson complex $(\Gamma(\wedge^\bullet T^*M), \partial_\Pi)$ such that the differential is the sum of two operators of bigraded types $(1,0)$ and $(2,-1)$, $\partial_\Pi = (\partial_\Pi)_{1,0} + (\partial_\Pi)_{2,-1}$. Furthermore, Vaisman explains in [62] that this property holds for any regular Poisson manifold endowed with a complementary distribution to the symplectic foliation[1].

[1]The bigraded type of the operators in [62] are $(0,1)$ and $(-1,2)$, but this apparent discrepancy is simply due to a difference on the bidegree convention.

Transitive Lie algebroids also provide examples of cochain complexes endowed with a bigrading. In fact, given a transitive Lie algebroid $(A \to M, q, [\cdot,\cdot]_A)$ with isotropy bundle $I := \ker(q)$, there exists a short exact sequence of Lie algebroids over M, given by

$$0 \to I \hookrightarrow A \overset{q}{\twoheadrightarrow} TM \to 0.$$

The choice of some $h : TM \to A$ with $q \circ h = \mathrm{Id}_{TM}$ induces a bigrading of the de Rham complex $(\Gamma(\wedge^{\bullet} A^*), \partial_A)$ so that $\partial_A = (\partial_A)_{0,1} + (\partial_A)_{1,0} + (\partial_A)_{2,-1}$. In [33], Itskov, Karasev, and Vorobiev present several cohomological results on the cohomology of the transitive Lie algebroids obtained by the restriction of the Lie algebroid to a leaf of the characteristic foliation.

The case of a transitive Lie algebroid can be described as a particular case of an extension of Lie algebroids. Given two Lie algebroids $E \to M$ and $A \to N$, we say that E is an extension of A if we have a Lie algebroid epimorphism $\pi : E \to A$ covering a submersion base map $p : M \to N$. The kernel $V := \ker(\pi)$ is an involutive subbundle of E, so inherits a Lie algebroid structure such that

$$0 \to V \hookrightarrow E \overset{\pi}{\twoheadrightarrow} A \to 0$$

is a short exact sequence of Lie algebroids. This is described in detail by Brahic in [7]. Furthermore, it is shown that the choice of a map $h : A \to E$ such that $\pi \circ h = \mathrm{Id}_E$ also induces a bigrading of the de Rham complex of A, just as in the case of a transitive Lie algebroid. Furthermore, Brahic provides a bigraded model for the cochomology of a Lie algebroid extension.

These facts are indeed a motivation for the results presented in this part. Indeed, we consider an abstract bigraded cochain complex $(\mathcal{C}^{\bullet,\bullet}, \partial)$ such that $\partial = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1}$, and describe its cohomology in terms only of the bigrading. In particular, we present explicit computations for the cohomology in degree 1, 2 and 3 (see Section 3.4). We also provide the following general setting in which all of the previous geometric examples fit: Suppose we are given a Lie algebroid $(E, q, [\cdot,\cdot]_E)$ endowed with a fixed involutive subbundle $V \subset E$. Then, it is straightforward to verify that the choice of a complementary subbundle $H \subset E$, $H \oplus V = E$ gives a bigrading of the de Rham complex $(\Gamma(\wedge^{\bullet} E^*, \partial)$ such that $\partial = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1}$. Furthermore, the exterior algebra $\Gamma(\wedge^{\bullet,\bullet}(V^{\circ} \oplus V^*))$ is a bigraded model, in which the bigrading and the operator of bidegree (0,1) is independent of H, and is associated with the de Rham differential of the Lie subalgebroid $V \subset E$. The operator $\partial^{H}_{1,0}$ is a sort of covariant exterior derivative, and $\partial^{H}_{2,-1}$ only depends on the curvature of H.

Another of the results in this part is to show that the cohomology of coupling twisted Poisson and Dirac structures on foliated manifolds can be also described in terms of a bigraded cochain complex. To this end in Section 4.3 we extend the coupling method to twisted Poisson and Dirac structures on foliated manifolds. To this end, we based on the algebraic description of the coupling method given in Chapter 2, and some computations with the Dorfman bracket. Finally, by means of some Vinogadov calculus we propose a bigraded cohomological model which allows us to describe the cochain complex associated with coupling twisted Poisson and Dirac structures. This bigraded model is based on the one constructed in [50, Subsection 4.2] for the case of coupling structures on fiber bundles.

Chapter 3

The Cohomology of a Bigraded Cochain Complex

This chapter is devoted to the study of the cohomology of a special class of bigraded cochain complexes. For several examples of this clss of cochain complexes arising from geometric structures see [6, 7, 8, 26, 46, 47, 74, 75, 61, 62].

We begin in Section 3.1 by setting the algebraic framework in which this study is developed, and present a few steps in the direction of establish our main results. In Section 3.2, we introduce some of the objects which allows us to describe the cohomology of bigraded complexes. Section 3.3 is devoted to present our results in a recursive perspective, which is useful to describe particular cohomology classes. In Section 3.4 we give a more detailed description of the cohomology in degree 1, 2, and 3. Finally, Section 3.5 is devoted to the description of the cochomology of some special classes of bigraded complexes which are common in the literature.

3.1 Bigraded cochain complexes and its cohomology

Notations and conventions. Recall that a cochain complex is a pair $(\mathcal{C}^\bullet, \partial)$ consisting of a graded ($\mathbb{Z}$-graded) $\mathbb{R}$-linear space $\mathcal{C}^\bullet = \bigoplus_{k\in\mathbb{Z}} \mathcal{C}^k$ and an $\mathbb{R}$-linear operator $\partial \in \mathrm{End}^1_{\mathbb{R}}(\mathcal{C})$ on $\mathcal{C}$ of degree 1 such that $\partial^2 = 0$.

Suppose that, in addition, $\mathcal{C}$ is a bigraded ($\mathbb{Z}^2$-graded) linear space such that the bigrading is compatible with the original $\mathbb{Z}$-grading in the following sense:

$$\mathcal{C}^k = \bigoplus_{p+q=k} \mathcal{C}^{p,q} \qquad \forall k \in \mathbb{Z}. \tag{3.1}$$

We also assume that $\mathcal{C}^{p,q} = \{0\}$ whenever p or q is negative. Moreover, suppose that the coboundary operator ∂ splits in the sum of three bigraded operators with respect to the bigrading (3.1),

$$\partial = \partial_{2,-1} + \partial_{1,0} + \partial_{0,1}, \tag{3.2}$$

where $\partial_{i,j}(\mathcal{C}^{p,q}) \subseteq \mathcal{C}^{p+i,q+j}$ for $(i,j) \in \{(2,-1),(1,0),(0,1)\}$. The right-hand side of (3.2) is called the *bigraded decomposition* of ∂.

In terms of the decomposition (3.2), the coboundary condition $\partial^2 = 0$ reads

$$\partial^2_{2,-1} = 0, \tag{3.3}$$

$$\partial_{2,-1}\partial_{1,0} + \partial_{1,0}\partial_{2,-1} = 0, \tag{3.4}$$

$$\partial_{2,-1}\partial_{0,1} + \partial_{0,1}\partial_{2,-1} + \partial^2_{1,0} = 0, \tag{3.5}$$

$$\partial_{1,0}\partial_{0,1} + \partial_{0,1}\partial_{1,0} = 0, \tag{3.6}$$

$$\partial^2_{0,1} = 0. \tag{3.7}$$

Here, the left-hand sides of equations (3.3)-(3.7) are the bigraded components of ∂^2. In particular, (3.7) implies that $(\mathcal{C}^{p,\bullet}, \partial_{0,1})$ is a cochain complex for each $p \in \mathbb{Z}$. For any cochain complex, we use the notation $Z^\bullet$, $B^\bullet$, and $H^\bullet$ to indicate the linear spaces of cocycles, coboundaries, and cohomology, respectively.

Spectral sequence. Consider the decreasing filtration F of $\mathcal{C}$ given by

$$F^p\mathcal{C} := \bigoplus_{\substack{i,j\in\mathbb{Z}\\ i\geq p}} \mathcal{C}^{i,j}.$$

For every subspace $S \subseteq \mathcal{C}$, denote $F^pS := F^p\mathcal{C} \cap S$. In particular, $F^p\mathcal{C}^k = \bigoplus_{i\geq p} \mathcal{C}^{i,k-i}$. Moreover, since $\mathcal{C}^{\bullet,\bullet}$ lies in the first quadrant, we have $F^0\mathcal{C}^k = \mathcal{C}^k$ and $F^{k+1}\mathcal{C}^k = \{0\}$, so the filtration is bounded. Furthermore, it follows from the bigraded decomposition (3.2) that $\partial(F^p\mathcal{C}) \subseteq F^p\mathcal{C}$ for all $p \in \mathbb{Z}$. Hence, the triple $(\mathcal{C}^\bullet, \partial, F)$ is a graded filtered complex.

Let $(E_r^{\bullet,\bullet}, d_r)$ be the spectral sequence associated with $(\mathcal{C}^\bullet, \partial, F)$, that is, for each $p, q, r \in \mathbb{Z}$, $E_r^{p,q} := \frac{Z_r^{p,q}+F^{p+1}\mathcal{C}^{p+q}}{B_{r-1}^{p,q}+F^{p+1}\mathcal{C}^{p+q}}$ [21, Eq. (2.46)], where

$$Z_r^{p,q} := F^p\mathcal{C}^{p+q} \cap \partial^{-1}(F^{p+r}\mathcal{C}^{p+q}), \qquad B_{r-1}^{p,q} := F^p\mathcal{C}^{p+q} \cap \partial(F^{p-r+1}\mathcal{C}^{p+q}),$$

the sums $Z_r^{p,q} + F^{p+1}\mathcal{C}^{p+q}$, and $B_{r-1}^{p,q} + F^{p+1}\mathcal{C}^{p+q}$ are as $\mathbb{R}$-vector subspaces of $\mathcal{C}^\bullet$, and $d_r : E_r^{p,q} \to E_r^{p+r,q+1-r}$ is induced by the restriction of ∂ to $Z_r^{p,q}$. In particular, $E_0^{p,q} = \frac{F^p\mathcal{C}^{p+q}}{F^{p+1}\mathcal{C}^{p+q}} \cong \mathcal{C}^{p,q}$, so $(E_r^{\bullet,\bullet}, d_r)$ is a first quadrant spectral sequence. Therefore, $E_N^{p,q} = E_\infty^{p,q}$ for all $N \geq \max\{p+1, q+2\}$, where

$$E_\infty^{p,q} := \frac{Z^{p+q}(\mathcal{C},\partial) \cap F^p\mathcal{C}^{p+q} + F^{p+1}\mathcal{C}^{p+q}}{B^{p+q}(\mathcal{C},\partial) \cap F^p\mathcal{C}^{p+q} + F^{p+1}\mathcal{C}^{p+q}}.$$

Since the filtration F is bounded, the spectral sequence converges to the cohomology of $(\mathcal{C}^\bullet, \partial)$. Furthermore, taking into account that $(\mathcal{C}^\bullet, \partial)$ is an $\mathbb{R}$-vector space, we get the following splitting for the k-th cohomology of $(\mathcal{C}^\bullet, \partial)$:

$$H^k(\mathcal{C},\partial) \cong \bigoplus_{p+q=k} E_\infty^{p,q}. \tag{3.8}$$

In what follows, we give a more explicit description of the summands in the splitting (3.8). For each $q \in \mathbb{Z}$, define $G^q\mathcal{C} := \bigoplus_{\substack{i,j\in\mathbb{Z}\\ j\geq q}} \mathcal{C}^{i,j}$, and consider the projection

$$\pi_q : \mathcal{C}^\bullet \to G^q\mathcal{C}$$

along the splitting induced by the bigrading. In particular, $\pi_q = \mathrm{Id}_{\mathcal{C}}$ if $q \leq 0$. For simplicity, we use the same notation for the restriction of π_q to any subspace of $\mathcal{C}$.

Lemma 3.1. *For each $p, q \in \mathbb{Z}$ such that $p + q = k$, we have $E_\infty^{p,q} \cong \frac{\pi_q(Z^k(\mathcal{C},\partial))\cap\mathcal{C}^{p,q}}{\pi_q(B^k(\mathcal{C},\partial))\cap\mathcal{C}^{p,q}}$.*

Proof. Consider the projection $\mathrm{pr}_{p,q} : \mathcal{C} \to \mathcal{C}^{p,q}$ along the splitting (3.1). Observe that, for each subspace $S \subset \mathcal{C}^k$, we have

$$S \cap F^p\mathcal{C}^k + F^{p+1}\mathcal{C}^k = \mathrm{pr}_{p,q}(S \cap F^p\mathcal{C}^k) \oplus F^{p+1}\mathcal{C}^k.$$

On the other hand, it is straightforward to verify that every element of $\mathrm{pr}_{p,q}(S \cap F^p\mathcal{C}^k)$ is of the form $y_{p,q}$, for some $y \in S$ with bigraded decomposition $y = \sum_{i\geq p} y_{i,k-i}$. Thus,

$$\mathrm{pr}_{p,q}(S \cap F^p\mathcal{C}^k) = \pi_q(S) \cap \mathcal{C}^{p,q}.$$

Setting $S = Z^k(\mathcal{C},\partial)$ and $S = B^k(\mathcal{C},\partial)$, we get

$$\begin{aligned} E_\infty^{p,q} = \frac{Z^k(\mathcal{C},\partial) \cap F^p\mathcal{C}^k + F^{p+1}\mathcal{C}^k}{B^k(\mathcal{C},\partial) \cap F^p\mathcal{C}^k + F^{p+1}\mathcal{C}^k} &= \frac{\mathrm{pr}_{p,q}(Z^k(\mathcal{C},\partial) \cap F^p\mathcal{C}^k) \oplus F^{p+1}\mathcal{C}^k}{\mathrm{pr}_{p,q}(B^k(\mathcal{C},\partial) \cap F^p\mathcal{C}^k) \oplus F^{p+1}\mathcal{C}^k} \\ &= \frac{(\pi_q(Z^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}) \oplus F^{p+1}\mathcal{C}^k}{(\pi_q(B^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}) \oplus F^{p+1}\mathcal{C}^k} \cong \frac{\pi_q(Z^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}}{\pi_q(B^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}}. \end{aligned}$$

■

Splittings for cocycles and coboundaries. We now derive similar splittings for the spaces of k-cocycles and k-coboundaries. Observe that for each subspace $S \subset \mathcal{C}^k$, we have a family of short exact sequences, given by

$$0 \to \pi_q(S) \cap \mathcal{C}^{p,q} \hookrightarrow \pi_q(S) \overset{\pi_{q+1}}{\twoheadrightarrow} \pi_{q+1}(S) \to 0, \qquad 0 \leq q \leq k-1, \tag{3.9}$$

where $p := k - q$. In the case of cocycles and coboundaries, we get the following result.

Proposition 3.2. *For each $p, q \in \mathbb{Z}$, with $p + q = k$, we have the following commutative diagram with exact rows and columns which all describe the spaces of k-coboundaries $B^k(\mathcal{C},\partial)$, k-cocycles $Z^k(\mathcal{C},\partial)$, and k-cohomology $H^k(\mathcal{C},\partial)$:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi_q(B^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q} \arrow[r, hook] \arrow[d, hook] & \pi_q(B^k(\mathcal{C},\partial)) \arrow[r, two heads, "\pi_{q+1}"] \arrow[d, hook] & \pi_{q+1}(B^k(\mathcal{C},\partial)) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \pi_q(Z^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q} \arrow[r, hook] \arrow[d, two heads] & \pi_q(Z^k(\mathcal{C},\partial)) \arrow[r, two heads, "\pi_{q+1}"] \arrow[d, two heads] & \pi_{q+1}(Z^k(\mathcal{C},\partial)) \arrow[r] \arrow[d, two heads] & 0. \\
0 \arrow[r] & \frac{\pi_q(Z^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}}{\pi_q(B^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}} \arrow[r, tail] \arrow[d] & \frac{\pi_q(Z^k(\mathcal{C},\partial))}{\pi_q(B^k(\mathcal{C},\partial))} \arrow[r, two heads] \arrow[d] & \frac{\pi_{q+1}(Z^k(\mathcal{C},\partial))}{\pi_{q+1}(B^k(\mathcal{C},\partial))} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here, the mappings from the second to the third row are the canonical projections, and the maps $\frac{\pi_q(Z^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}}{\pi_q(B^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}} \to \frac{\pi_q(Z^k(\mathcal{C},\partial))}{\pi_q(B^k(\mathcal{C},\partial))}$ and $\frac{\pi_q(Z^k(\mathcal{C},\partial))}{\pi_q(B^k(\mathcal{C},\partial))} \to \frac{\pi_{q+1}(Z^k(\mathcal{C},\partial))}{\pi_{q+1}(B^k(\mathcal{C},\partial))}$ are defined in such a way that the lower 2×2 blocks commute.

Proof of Proposition 3.2. The upper 2×2 diagrams are clearly commutative because the arrows $\hookrightarrow$ are natural inclusions, and the arrows with a π_{q+1} are the restriction of the same mapping. On the other hand, the exactness of the first row is obtained from (3.9) by setting $S := B^k(\mathcal{C},\partial)$. Similarly, the exactness of the second row follows from setting $S := Z^k(\mathcal{C},\partial)$ in (3.9). Moreover, each column is exact by definition. Finally, the exactness of the last row follows from the commutativity and the exactness of the rest of the diagram. ■

Corollary 3.3. *The coboundary, cocycle, and cohomology spaces of degree k admit the following splittings:*

$$B^k(\mathcal{C},\partial) \cong \bigoplus_{p+q=k} \pi_q(B^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}, \qquad Z^k(\mathcal{C},\partial) \cong \bigoplus_{p+q=k} \pi_q(Z^k(\mathcal{C},\partial)) \cap \mathcal{C}^{p,q}, \qquad \textit{and}$$

$$H^k(\mathcal{C},\partial) \cong \bigoplus_{p+q=k} \tfrac{\pi_q(Z^k(\mathcal{C},\partial))\cap\mathcal{C}^{p,q}}{\pi_q(B^k(\mathcal{C},\partial))\cap\mathcal{C}^{p,q}}.$$

Note that the splitting for $H^k(\mathcal{C},\partial)$ in Corollary 3.3 coincides with (3.8) under Lemma 3.1

Remark 3.4. *Every result of this part is valid if the bigraded decomposition of ∂ has the more general form $\partial = \sum_{r\geq 0} \partial_{r,1-r}$. Moreover, Lemma 3.1 and Proposition 3.2 still hold if $(\mathcal{C}^\bullet,\partial)$ is a cochain complex over a ring, while the non-canonical splittings in* (3.8) *and in Corollary 3.3 only hold in the vector spaces category.*

Otherwise stated, we assume in what follows that $(\mathcal{C}^\bullet,\partial)$ is a cochain complex over a ring $\mathcal{R}$. In the cases when we require that the ring of scalars is a field, this condition will be explicitly indicated.

3.2 Describing the cohomology

In this Section, we introduce some useful objects which allow us to improve our description of the splittings in Corollary 3.3 and the diagram of Proposition 3.2.

The null subcomplexes. For simplicity, for each $p, q, k \in \mathbb{Z}$ we denote

$$\ker^k(\partial_{i,j}) := \ker(\partial_{i,j} : \mathcal{C}^k \to \mathcal{C}^{k+1}), \qquad \text{and} \qquad \ker^{p,q}(\partial_{i,j}) := \ker(\partial_{i,j} : \mathcal{C}^{p,q} \to \mathcal{C}^{p+i,q+j}),$$

for all $(i,j) \in \{(0,1),(1,0),(2,-1)\}$. Let us denote

$$\mathcal{N} := \ker(\partial_{0,1} : \mathcal{C} \to \mathcal{C}) \cap \ker(\partial_{2,-1} : \mathcal{C} \to \mathcal{C}),$$

$\mathcal{N}^k := \ker^k(\partial_{0,1}) \cap \ker^k(\partial_{2,-1})$, $\mathcal{N}^{p,q} := \ker^{p,q}(\partial_{0,1}) \cap \ker^{p,q}(\partial_{2,-1})$, and $\mathcal{N}_q := \bigoplus_{p\in\mathbb{Z}} \mathcal{N}^{p-q,q}$. Since $\partial_{0,1}$ and $\partial_{2,-1}$ are bigraded operators, we have $\mathcal{N} = \bigoplus_{k\in\mathbb{Z}} \mathcal{N}^k$ and $\mathcal{N}^k = \bigoplus_{p+q=k} \mathcal{N}^{p,q}$. Moreover,

Lemma 3.5. *The graded $\mathcal{R}$-module $\mathcal{N}$ is a cochain subcomplex of $(\mathcal{C},\partial)$. Moreover, for each $q \in \mathbb{Z}$, $\mathcal{N}_q$ is also a cochain subcomplex of $(\mathcal{C},\partial)$.*

Proof. Since $\mathcal{N} = \bigoplus_{q\in\mathbb{Z}} \mathcal{N}_q$, it suffices to show that each $\mathcal{N}_q$ is a cochain subcomplex of $(\mathcal{C},\partial)$. By definition, $\partial_{0,1}$ and $\partial_{2,-1}$ vanish on $\mathcal{N}_q$. Thus,

$$\partial(\mathcal{N}^{p-q,q}) = \partial_{1,0}(\mathcal{N}^{p-q,q}) \subseteq \partial_{1,0}(\mathcal{C}^{p-q,q}) \subseteq \mathcal{C}^{(p+1)-q,q}.$$

To complete the proof, we just need to verify that $\partial_{1,0}(\mathcal{N}) \subseteq \mathcal{N}$. Fix $\eta \in \mathcal{N}$. Then, $\partial_{2,-1}\eta = 0$ and $\partial_{0,1}\eta = 0$. By applying equations (3.4) and (3.6),

$$\partial_{2,-1}(\partial_{1,0}\eta) = -\partial_{1,0}\partial_{2,-1}\eta = 0, \qquad \text{and} \qquad \partial_{0,1}(\partial_{1,0}\eta) = -\partial_{1,0}\partial_{0,1}\eta = 0,$$

proving that $\partial_{1,0}\eta \in \mathcal{N}$. Thus, $\partial(\mathcal{N}_q) \subseteq \mathcal{N}_q$, as claimed. ∎

We denote by $\overline{\partial} := \partial|_{\mathcal{N}}$ the coboundary operator on $\mathcal{N}$. We use the same notation for any of the cochain subcomplexes $\mathcal{N}_q$. The cochain complexes $(\mathcal{N}, \overline{\partial})$ and $(\mathcal{N}_q, \overline{\partial})$ are called *the null subcomplexes* of $(\mathcal{C}, \partial)$. Finally, recall that $\mathcal{C}^{p,q} = \{0\}$ whenever p or q is negative. In particular, $\mathcal{C}^{\bullet,0} \subseteq \ker(\partial_{2,-1})$. Therefore, $\mathcal{N}_0 = \ker(\partial_{0,1} : \mathcal{C}^{\bullet,0} \to \mathcal{C}^{\bullet,1})$. In other words, for $q = 0$, the restriction of $\partial_{1,0}$ to the $\partial_{0,1}$-cocycles of bidegree $(p, 0)$ gives the null subcomplex $\mathcal{N}_0$.

Pre-coboundaries and pre-cocycles. Recall that the terms appearing in the upper row of the diagrams of Proposition 3.2 are of the form $\pi_q(B^k(\mathcal{C}, \partial))$ or $\pi_q(B^k(\mathcal{C}, \partial)) \cap \mathcal{C}^{p,q}$, whose elements are obtained by projecting a k-coboundary under $\pi_q : \mathcal{C} \to G^q\mathcal{C}$. We call the elements of $\pi_q(B^\bullet(\mathcal{C}, \partial))$ *pre-coboundaries*, and the elements of $\pi_q(B^\bullet(\mathcal{C}, \partial)) \cap \mathcal{C}^{p,q}$, *homogeneous pre-coboundaries*. In a similar fashion, we call the elements of $\pi_q(Z^\bullet(\mathcal{C}, \partial))$ *pre-cocycles*, and the elements of $\pi_q(Z^\bullet(\mathcal{C}, \partial)) \cap \mathcal{C}^{p,q}$, *homogeneous pre-cocycles.*

In this part, we give a detailed description of the $\mathcal{R}$-module of pre-cocycles. In fact, although a pre-cocycle is defined as the projection of a cocycle, we describe a bigger $\mathcal{R}$-module containing the cocycles such that the projection of its elements is again a pre-cocycle. In this sense, we have found some degrees of freedom in the construction of pre-cocycles.

Observe that $\eta \in \mathcal{C}^1$ is a 1-cocycle if and only if

$$\partial_{2,-1}\eta_{0,1} + \partial_{1,0}\eta_{1,0} = 0, \qquad \partial_{1,0}\eta_{0,1} + \partial_{0,1}\eta_{1,0} = 0, \qquad \partial_{0,1}\eta_{0,1} = 0.$$

The left-hand sides of each equation correspond to the bigraded components of $\partial\eta$. Similarly, $\eta \in \mathcal{C}^2$ is a 2-cocycle if and only if

$$\begin{aligned} \partial_{2,-1}\eta_{1,1} + \partial_{1,0}\eta_{2,0} &= 0, & \partial_{2,-1}\eta_{0,2} + \partial_{1,0}\eta_{1,1} + \partial_{0,1}\eta_{2,0} &= 0, \\ \partial_{1,0}\eta_{0,2} + \partial_{0,1}\eta_{1,1} &= 0, & \partial_{0,1}\eta_{0,2} &= 0. \end{aligned}$$

In general, for each $\eta \in \mathcal{C}^k$ with bigraded components $\eta_{p,q} \in \mathcal{C}^{p,q}$ $(p+q = k)$, the bigraded components of $\partial\eta$ are

$$(\partial\eta)_{i,j} = \partial_{0,1}\eta_{i,j-1} + \partial_{1,0}\eta_{i-1,j} + \partial_{2,-1}\eta_{i-2,j+1}, \qquad i + j = k + 1.$$

Let us consider the graded $\mathcal{R}$-modules

$$\mathcal{M} := \{\eta \in \mathcal{C} \mid \partial\eta \in B(\mathcal{N}, \overline{\partial})\}, \tag{3.10}$$

and

$$\mathcal{M}^k := \{\eta \in \mathcal{C}^k \mid (\partial\eta)_{i,j} \in B^{k+1}(\mathcal{N}_j, \overline{\partial}),\ i + j = k + 1\}.$$

Then, $\mathcal{M}^\bullet = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}^k$. Moreover, it is clear that $Z(\mathcal{C}, \partial) \subseteq \mathcal{M}$. Now, for each $q \in \mathbb{Z}$, define

$$\mathcal{Z}_q := \{\pi_q(\eta) \mid \eta \in \mathcal{M},\ \text{and } \pi_q(\partial\eta) = 0\}, \qquad \text{and} \qquad \mathcal{Z}_q^k := \{\pi_q(\eta) \mid \eta \in \mathcal{M}^k,\ \text{and } \pi_q(\partial\eta) = 0\}.$$

In other words, the elements of $\mathcal{Z}_q \subseteq G^q\mathcal{C}$ are of the form $\pi_q(\eta)$, for some $\eta \in \mathcal{C}$ satisfying

$$\partial_{0,1}\eta_{i,j-1} + \partial_{1,0}\eta_{i-1,j} + \partial_{2,-1}\eta_{i-2,j+1} \in \begin{cases} \{0\} & \text{if } j \geq q, \\ B(\mathcal{N}_j, \overline{\partial}) & \text{if } j < q. \end{cases}$$

Note that $\mathcal{Z}_q^\bullet = \bigoplus_{k\in\mathbb{Z}} \mathcal{Z}_q^k$. Furthermore, we claim that $\mathcal{Z}_q$ is precisely the $\mathcal{R}$-module of pre-cocycles in $G^q\mathcal{C}$.

Proposition 3.6. *For each $q \in \mathbb{Z}$, we have $\pi_q(Z(\mathcal{C},\partial)) = \mathcal{Z}_q$. In particular, $\xi \in \mathcal{C}^{p,q}$ is a pre-cocycle if and only if $\partial_{0,1}\xi = 0$ and there exist $\eta \in F^p\mathcal{M}^{p+q}$ such that $\eta_{p,q} = \xi$, and $\partial_{1,0}\eta_{p,q} + \partial_{0,1}\eta_{p+1,q-1} = 0$.*

Proof. The inclusion $\pi_q(Z(\mathcal{C},\partial)) \subseteq \mathcal{Z}_q$ simply follows from the already mentioned fact $Z(\mathcal{C},\partial) \subseteq \mathcal{M}$. Conversely, pick $\xi \in \mathcal{Z}_q^k$, of the form $\xi = \sum_{j\geq q} \xi_j$, where $\xi_j \in \mathcal{C}^{k-j,j}$. Then, there exists $\eta \in \mathcal{M}^k$ such that $\pi_q(\eta) = \xi$ and $\pi_q(\partial\eta) = 0$. Let $\eta_j \in \mathcal{C}^{k-j,j}$ be the bigraded components of η. The condition $\eta \in \mathcal{M}^k$ implies that for each $j < q$ there exists $\eta'_j \in \mathcal{N}^{k-j,j}$ such that

$$\partial_{0,1}\eta_{j-1} + \partial_{1,0}\eta_j + \partial_{2,-1}\eta_{j+1} = \partial_{1,0}\eta'_j,$$

Finally, set $\widetilde{\xi} := \eta - \sum_{j<q} \eta'_j$. Since $\partial_{0,1}\eta'_j = 0$ and $\partial_{2,-1}\eta'_j = 0$, it is straightforward to verify that $\partial\widetilde{\xi} = 0$. Furthermore, $\pi_q(\widetilde{\xi}) = \xi$, which proves that $\xi \in \pi_q(Z^k(\mathcal{C},\partial))$. ∎

For each $q, k \in \mathbb{Z}$ denote by $\mathcal{B}_q^k := \pi_q(B^k(\mathcal{C},\partial))$ the $\mathcal{R}$-module of pre-coboundaries. As a consequence, of Propositions 3.2 and 3.6, we have:

Theorem 3.7. *For each $p, q \in \mathbb{Z}$ with $p + q = k$, we have the following commutative diagrams with exact rows and columns describing the coboundary, cocycle, and cohomology of $(\mathcal{C}^{\bullet,\bullet}, \partial)$:*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B}_q^k \cap \mathcal{C}^{p,q} \arrow[r, hook] \arrow[d, hook] & \mathcal{B}_q^k \arrow[r, two heads, "\pi_{q+1}"] \arrow[d, hook] & \mathcal{B}_{q+1}^k \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{Z}_q^k \cap \mathcal{C}^{p,q} \arrow[r, hook] \arrow[d] & \mathcal{Z}_q^k \arrow[r, two heads, "\pi_{q+1}"] \arrow[d] & \mathcal{Z}_{q+1}^k \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \frac{\mathcal{Z}_q^k \cap \mathcal{C}^{p,q}}{\mathcal{B}_q^k \cap \mathcal{C}^{p,q}} \arrow[r, tail] \arrow[d] & \frac{\mathcal{Z}_q^k}{\mathcal{B}_q^k} \arrow[r, two heads, "\bar{\pi}_{q+1}"] \arrow[d] & \frac{\mathcal{Z}_{q+1}^k}{\mathcal{B}_{q+1}^k} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Corollary 3.8. *In the case when $\mathcal{R}$ is a field, we get the following splittings:*

$$B^k(\mathcal{C},\partial) \cong \bigoplus_{p+q=k} \mathcal{B}_q^k \cap \mathcal{C}^{p,q}, \qquad Z^k(\mathcal{C},\partial) \cong \bigoplus_{p+q=k} \mathcal{Z}_q^k \cap \mathcal{C}^{p,q}, \qquad H^k(\mathcal{C},\partial) \cong \bigoplus_{p+q=k} \frac{\mathcal{Z}_q^k \cap \mathcal{C}^{p,q}}{\mathcal{B}_q^k \cap \mathcal{C}^{p,q}}.$$

3.3 The recursive point of view

Before going further in the description of the diagrams appearing in Theorem 3.7 in the low-degree case, let us interpret this result from a recursive point of view. This perspective is particularly useful when we are interested in understanding an specific cohomology class of $(\mathcal{C},\partial)$.

Recall from Theorem 3.7 that the cocycles, coboundaries, and cohomology of $(\mathcal{C},\partial)$ is described by a family of diagrams, one for each $q = 0, 1, \ldots, k$. By denoting $\mathcal{H}_q^{p,q} := \frac{\mathcal{Z}_q^k \cap \mathcal{C}^{p,q}}{\mathcal{B}_q^k \cap \mathcal{C}^{p,q}}$, $\mathcal{H}_q^k := \frac{\mathcal{Z}_q^k}{\mathcal{B}_q^k}$, and $p = k - q$, the bottom row of the q-th diagram is

$$0 \longrightarrow \mathcal{H}_q^{p,q} \hookrightarrow \mathcal{H}_q^k \xrightarrow{\bar{\pi}_{q+1}} \mathcal{H}_{q+1}^k \longrightarrow 0. \tag{3.11}$$

Now, pick some $\eta \in Z^k(\mathcal{C}, \eth)$, with bigraded decomposition

$$\eta = \sum_{i+j=k} \eta_{i,j} = \eta_{0,k} + \eta_{1,k-1} + \cdots + \eta_{k,0}.$$

Since $H^k(\mathcal{C}, \eth) = \mathcal{H}^k_0$, we have $[\eta] \in \mathcal{H}^k_0$. Denote $[\eta]_0 := [\eta]$, and for each $q = 1, \dots, k$, recursively define $[\eta]_q \in \mathcal{H}^k_q$ by $[\eta]_q := \bar{\pi}_q [\eta]_{q-1}$. It is clear that $[\eta]_q$ is well defined for each q. Explicitly, we have $[\eta]_{k+1} = 0$, $[\eta]_k = \eta_{0,k} + \mathcal{B}^k_k$, and in general

$$[\eta]_q = \sum_{\substack{i+j=k \\ j \geq q}} \eta_{k-j,j} + \mathcal{B}^k_q.$$

In what follows, let us describe the obstructions for the vanishing of the cohomology class $[\eta] \in H^k(\mathcal{C}, \eth)$. By the relation $[\eta]_{q+1} = \bar{\pi}_{q+1}[\eta]_q$, a necessary condition for $[\eta]_q = 0$ is that $[\eta]_{q+1} = 0$. Conversely, if $[\eta]_{q+1} = 0$, then the exactness of (3.11) implies that $[\eta]_q = \eta_{p,q} + \mathcal{B}^k_q \cap \mathcal{C}^{p,q}$, where $\eta_{p,q} \in \mathcal{Z}^k_q \cap \mathcal{C}^{p,q}$. So, under the vanishing of $[\eta]_{q+1}$, the class $[\eta_{p,q}] \in \mathcal{H}^{p,q}_q$ is well defined, and the property $[\eta]_q = 0$ is equivalent to the vanishing of $[\eta_{p,q}]$.

To get more insight in the previous facts, let us describe them in an explicit fashion. Consider the bigraded decomposition $\eta = \sum_{i+j=k} \eta_{i,j}$. Clearly, $\pi_k \eta = \eta_{0,k}$, so $[\eta]_k = [\eta_{0,k}] \in \mathcal{H}^{0,k}_k$. Now, suppose that $[\eta_{0,k}] = 0$. Then, there exists $\eta' = \partial \xi \in B^k(\mathcal{C}, \partial)$ such that $\eta_{0,k} = \eta'_{0,k}$. Since $[\eta] = [\eta - \eta']$, the representative $\eta - \eta'$ is such that the component of bidegree $(0, k)$ vanishes. So, without loss of generality, we may assume that $\eta_{0,k} = 0$. Then, $[\eta]_{k-1} = [\eta_{1,k-1}] \in \mathcal{H}^{1,k-1}_{k-1}$. Assuming that $[\eta_{1,k-1}] = 0$, there exists $\eta'' = \partial \xi'$ such that $\pi_{k-1}\eta'' = \eta_{1,k-1}$ (so, in particular, $\eta''_{0,k} = 0$). The difference $\eta - \eta''$ is a representative of $[\eta]$ such that the components of bidegree $(0, k)$ and $(1, k-1)$ vanish, so we may assume that $\pi_{k-1}\eta = 0$. Thus, $[\eta]_{k-2} = [\eta_{2,k-2}] \in \mathcal{H}^{1,k-1}_{k-1}$, and so on.

In summary, the short exact sequences given by the bottom diagrams can be described in the following way. Given $[\eta] \in H^k(\mathcal{C}, \eth)$, an obstruction to $[\eta] = 0$ is the cohomology class $[\eta_{0,k}] \in \mathcal{H}^{0,k}_k$. If $[\eta_{0,k}] = 0$, then the class $[\eta_{1,k-1}] \in \mathcal{H}^{1,k-1}_{k-1}$ is well defined, and is a new obstruction to the vanishing of $[\eta]$. If in addition $[\eta_{1,k-1}] = 0$, then $[\eta_{2,k-2}] \in \mathcal{H}^{2,k-2}_{k-2}$ is well defined and is a new obstruction to the vanishing of $[\eta]$. On every stage, under the vanishing of $[\eta_{p,q}] \in \mathcal{H}^{p,q}_q$, the class $[\eta_{p+1,q-1}] \in \mathcal{H}^{p+1,q-1}_{q-1}$ is well defined, independent of the choice of the representative η, and is an obstruction to $[\eta] = 0$. In the last stage, our cohomology class is of the form $[\eta] = [\eta_{k,0}] \in \mathcal{H}^{k,0}_0$.

Low degree. Given $\eta \in Z^k(\mathcal{C}, \eth)$, it is important to remark that $[\eta_{p,q}] \in \mathcal{H}^{p,q}_q$ is well defined only in the case when $[\eta_{p-1,q+1}] \in \mathcal{H}^{p-1,q+1}_{q+1}$ also is well defined and vanishes, $[\eta_{p-1,q+1}] = 0$. As explained in the previous paragraphs, the vanishing of a cohomology class of degree k is controlled by a sequence of $(k+1)$ "simpler" cohomology classes, and each of them is obtained by projecting into the bigraded components of the original one, as long as the previous cohomology class vanishes. Let us illustrate this in some low-degree cases.

If $k = 1$, then the cohomology is described by only one short exact sequence,

$$0 \longrightarrow \mathcal{H}^{1,0}_0 \hookrightarrow H^1(\mathcal{C}, \eth) \xrightarrow{\bar{\pi}_1} \mathcal{H}^{0,1}_1 \longrightarrow 0.$$

In this case, the vanishing of a cohomology class $[\eta] \in H^1(\mathcal{C}, \eth)$ is controlled by at most two cohomology classes. In fact, a necessary condition for $[\eta] = 0$ is that $[\eta_{0,1}] \in \mathcal{H}^{0,1}_1$ vanishes. Conversely, under

$[\eta_{0,1}] = 0$, the cohomology class $[\eta_{1,0}] \in \mathcal{H}_0^{1,0}$ is well-defined and satisfies $[\eta] = [\eta_{1,0}]$, due to the exactness of the sequence. Thus, if $[\eta_{0,1}] = 0$, then the vanishing of $[\eta]$ is equivalent to $[\eta_{1,0}] = 0$.

For $k = 2$, the cohomology is described by means of two short exact sequences, namely

$$0 \longrightarrow \mathcal{H}_0^{2,0} \hookrightarrow H^2(\mathcal{C}, \partial) \xrightarrow{\bar{\pi}_1} \mathcal{H}_1^2 \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{H}_1^{1,1} \hookrightarrow \mathcal{H}_1^2 \xrightarrow{\bar{\pi}_2} \mathcal{H}_2^{0,2} \longrightarrow 0.$$

In this case, the vanishing of $[\eta] \in H^2(\mathcal{C}, \partial)$ is controlled by at most three cohomology classes. A necessary condition is $[\eta_{0,2}] = 0$. Under this condition, the cohomology class $[\eta_{1,1}] \in \mathcal{H}_1^{1,1}$ is well defined and satisfies $[\eta_{1,1}] = [\eta]_1$, due to the exactness of the second sequence. In this case, a necessary condition for the vanishing of $[\eta]$ is $[\eta_{1,1}] = 0$. Under this condition, the cohomology class $[\eta_{2,0}] \in \mathcal{H}_0^{2,0}$ is well defined and satisfies $[\eta_{2,0}] = [\eta]$, due to the exactness of the first sequence. Hence, if $[\eta_{0,2}] = 0$ and $[\eta_{1,1}] = 0$, then the vanishing of $[\eta]$ is equivalent to $[\eta_{2,0}] = 0$.

3.4 Cohomology in low degree

In this section, we describe the diagrams of Theorem 3.7 for the cases $k = 1, 2, 3$ in more detail.

Following the notation of Section 3.2, observe that the homogeneous pre-cocycles of bidegree $(k, 0)$ are just the k-cocycles in $(\mathcal{N}_0, \overline{\partial})$,

$$\mathcal{Z}_0^k \cap \mathcal{C}^{k,0} = \{\eta \in \mathcal{C}^{k,0} \mid \partial_{1,0}\eta = 0, \partial_{0,1}\eta = 0\} = Z^k(\mathcal{N}_0, \overline{\partial}). \tag{3.12}$$

On the other hand, the homogeneous pre-coboundaries of bidegree $(0, k)$ are precisely the k-coboundaries of the complex $(\mathcal{C}^{0,\bullet}, \partial_{0,1})$,

$$\mathcal{B}_k^k \cap \mathcal{C}^{0,k} = B^k(\mathcal{C}^{0,\bullet}, \partial_{0,1}). \tag{3.13}$$

We now refine our description of the $\mathcal{R}$-module $\mathcal{Z}_1^k$ of pre-cocycles for $q = 1$.

The mappings $\rho : \mathcal{A} \to H(\mathcal{N}, \overline{\partial})$ **and** $\varrho : \mathcal{J} \to H(\mathcal{N}, \overline{\partial})$**.** For each $k \in \mathbb{Z}$, consider the linear modules $\mathcal{A}^k$ and $\mathcal{J}^k$, where

$$\mathcal{A}^k := \{\pi_1(\eta) \mid \eta \in \mathcal{C}^k,\ \pi_1(\partial\eta) = 0\}, \text{ and } \mathcal{J}^k := \mathcal{A}^k \cap \mathcal{C}^{k-1,1}. \tag{3.14}$$

Explicitly, $\xi \in G^1\mathcal{C}^k$ lies in $\mathcal{A}^k$ if and only if $\pi_2(\partial\xi) = 0$ and there exists $\eta \in \mathcal{C}^{k,0}$ such that $\partial_{0,1}\eta + \partial_{1,0}\xi_{k-1,1} + \partial_{2,-1}\xi_{k-2,2} = 0$. In particular, $\xi \in \mathcal{C}^{k-1,1}$ lies in $\mathcal{J}^k$ if and only if $\partial_{0,1}\xi = 0$ and there exists $\eta \in \mathcal{C}^{k,0}$ such that $\partial_{0,1}\eta + \partial_{1,0}\xi = 0$.

Lemma 3.9. *For each $\eta \in \mathcal{C}^k$ such that $\pi_1(\partial\eta) = 0$, one has $\mathrm{pr}_{k+1,0}(\partial\eta) \in Z^{k+1}(\mathcal{N}_0, \overline{\partial})$.*

Proof. One must show that $\mathrm{pr}_{k+1,0}(\partial\eta) \in \ker \partial_{1,0} \cap \ker \partial_{0,1}$. First note that

$$\mathrm{pr}_{k+1,0}(\partial\eta) = \partial_{2,-1}\eta_{k-1,1} + \partial_{1,0}\eta_{k,0}.$$

By applying (3.5) and (3.6),

$$\begin{aligned} \partial_{0,1}(\mathrm{pr}_{k+1,0}(\partial\eta)) &= \partial_{0,1}\partial_{2,-1}\eta_{k-1,1} + \partial_{0,1}\partial_{1,0}\eta_{k,0} \\ &= -\partial_{1,0}^2\eta_{k-1,1} - \partial_{2,-1}\partial_{0,1}\eta_{k-1,1} - \partial_{1,0}\partial_{0,1}\eta_{k,0}. \end{aligned} \tag{3.15}$$

The condition $\pi_1(\partial\eta) = 0$ implies that $\mathrm{pr}_{k,1}(\partial\eta) = \partial_{0,1}\eta_{k,0} + \partial_{1,0}\eta_{k-1,1} + \partial_{2,-1}\eta_{k-2,2} = 0$ which, together with (3.15), leads to

$$\begin{aligned}\partial_{0,1}(\mathrm{pr}_{k+1,0}(\partial\eta)) &= \partial_{0,1}(\mathrm{pr}_{k+1,0}(\partial\eta)) + \partial_{1,0}(\mathrm{pr}_{k,1}(\partial\eta)) \\ &= -\partial_{2,-1}\partial_{0,1}\eta_{k-1,1} + \partial_{1,0}\partial_{2,-1}\eta_{k-2,2}.\end{aligned} \tag{3.16}$$

Now, from (3.4), we get

$$\partial_{0,1}(\mathrm{pr}_{k+1,0}(\partial\eta)) = -\partial_{2,-1}\partial_{0,1}\eta_{k-1,1} - \partial_{2,-1}\partial_{1,0}\eta_{k-2,2}. \tag{3.17}$$

Again, from $\pi_1(\partial\eta) = 0$ we get $\mathrm{pr}_{k-1,2}(\partial\eta) = \partial_{0,1}\eta_{k-1,1} + \partial_{1,0}\eta_{k-2,2} + \partial_{2,-1}\eta_{k-3,3} = 0$. By (3.17),

$$\partial_{0,1}(\mathrm{pr}_{k+1,0}(\partial\eta)) = \partial_{0,1}(\mathrm{pr}_{k+1,0}(\partial\eta)) + \partial_{2,-1}(\mathrm{pr}_{k-1,2}(\partial\eta)) = \partial_{2,-1}^2\eta_{k-3,3}.$$

Therefore, $\partial_{0,1}(\mathrm{pr}_{k+1,0}(\partial\eta)) = 0$, due to (3.3). In a similar fashion, by applying (3.4) and (3.5) we obtain

$$\partial_{1,0}(\mathrm{pr}_{k+1,0}(\partial\eta)) = -\partial_{2,-1}\partial_{1,0}\eta_{k-1,1} - \partial_{0,1}\partial_{2,-1}\eta_{k,0} - \partial_{2,-1}\partial_{0,1}\eta_{k,0}.$$

Note that $\partial_{2,-1}\eta_{k,0} = 0$, due to its negative bidegree. Taking into account that $\mathrm{pr}_{k,1}(\partial\eta) = \partial_{0,1}\eta_{k,0} + \partial_{1,0}\eta_{k-1,1} + \partial_{2,-1}\eta_{k-2,2} = 0$, we get

$$\partial_{1,0}(\mathrm{pr}_{k+1,0}(\partial\eta)) = \partial_{2,-1}^2\eta_{k-2,2},$$

which is zero because of (3.3). ■

Now, by definition, for each $\xi \in \mathcal{A}^k$ there exist $\eta \in \mathcal{C}^k$ such that $\pi_1\eta = \xi$ and $\pi_1(\partial\eta) = 0$. By Lemma 3.9, η induces a cohomology class

$$[\partial_{2,-1}\eta_{k-1,1} + \partial_{1,0}\eta_{k,0}] \in H^{k+1}(\mathcal{N}_0, \overline{\partial}).$$

We claim that the cohomology class only depends on ξ, that is, it is independent of the choice of η. Indeed, pick another $\widetilde{\eta} \in \mathcal{C}^k$ such that $\pi_1\widetilde{\eta} = \xi$ and $\pi_1(\partial\widetilde{\eta}) = 0$. Since $\eta_{k-1,1} = \widetilde{\eta}_{k-1,1} = \xi_{k-1,1}$, we get

$$(\partial_{2,-1}\widetilde{\eta}_{k-1,1} + \partial_{1,0}\widetilde{\eta}_{k,0}) - (\partial_{2,-1}\eta_{k-1,1} + \partial_{1,0}\eta_{k,0}) = \partial_{1,0}(\widetilde{\eta}_{k,0} - \eta_{k,0}).$$

To see that η and $\widetilde{\eta}$ induce the same cohomology class, we just need to check that $\widetilde{\eta}_{k,0} - \eta_{k,0} \in \mathcal{N}_0$. From $\pi_1(\partial\eta) = 0$ and $\pi_1(\partial\widetilde{\eta}) = 0$ we get that $\partial_{0,1}\eta + \partial_{0,1}\xi_{k,0} = 0$ and $\partial_{0,1}\widetilde{\eta} + \partial_{0,1}\xi_{k,0} = 0$. Therefore, $\widetilde{\eta}_{k,0} - \eta_{k,0} \in \ker^{k,0}\partial_{0,1} = \mathcal{N}^{k,0}$. Hence, the cohomology class is well defined.

This can be summarized in the following fact.

Lemma 3.10. *There exists a well-defined linear map $\rho_k : \mathcal{A}^k \to H^{k+1}(\mathcal{N}_0, \overline{\partial})$ given by*

$$\rho_k(\xi) := [\partial_{2,-1}\xi_{k-1,1} + \partial_{1,0}\eta_{k,0}],$$

where $\eta \in \mathcal{C}^k$ is such that $\pi_1\eta = \xi$ and $\pi_1(\partial\eta) = 0$. Moreover, we have the identity $\mathcal{Z}_1^k = \ker(\rho_k)$.

Proof. The fact that $\rho_k(\xi)$ is well defined follows from our previous discussion, in which we have explained that $[\partial_{2,-1}\xi_{k-1,1} + \partial_{1,0}\eta_{k,0}]$ only depends of ξ. Moreover, the linearity of ρ_k follows from the linearity of $\partial_{2,-1}$, $\partial_{1,0}$, and π_1. So, it is left to show that $\ker(\rho_k) = \mathcal{Z}_1^k$. Recall that, by definition, the elements of $\mathcal{Z}_1^k$ are of the form $\pi_1(\eta)$, where $\partial_{2,-1}\eta_{k-1,1} + \partial_{1,0}\eta_{k,0} \in B^{k+1}(\mathcal{N}_0, \overline{\partial})$, and $\pi_1(\partial\eta) = 0$. Therefore,

$$\ker(\rho_k) = \{\pi_1(\eta) \mid \eta \in \mathcal{C}^k,\ \pi_1(\partial\eta) = 0,\ \text{and } \partial_{2,-1}\eta_{k-1,1} + \partial_{1,0}\eta_{k,0} \in B^{k+1}(\mathcal{N}_0, \overline{\partial})\} = \mathcal{Z}_1^k.$$

■

For each $k \in \mathbb{Z}_{\geq 0}$, define $\varrho_k : \mathcal{J}^k \to H^{k+1}(\mathcal{N}_0, \overline{\partial})$ by the restriction $\varrho_k := \rho_k|_{\mathcal{J}^k}$. As a consequence of Lemma 3.10, we have

$$\ker(\varrho_k) = \mathcal{Z}_1^k \cap \mathcal{C}^{k-1,1}.$$

Refining the splittings for the low-degree cohomology. By applying Theorem 3.7, we describe the first, second, and third cohomology of the bigraded cochain complex $(\mathcal{C}, \partial)$ in terms of the cochain complexes $(\mathcal{C}^{p,\bullet}, \partial_{0,1})$, $(\mathcal{N}_q, \overline{\partial})$, and the mappings $\rho : \mathcal{A} \to H(\mathcal{N}, \overline{\partial})$ and $\varrho : \mathcal{J} \to H(\mathcal{N}, \overline{\partial})$ given in Lemmas 3.5 and 3.10.

First cohomology. Here we state our main result on the first cohomology of $(\mathcal{C}, \partial)$.

Theorem 3.11. *We have the following commutative diagram with exact rows and columns,*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^1(\mathcal{N}_0, \overline{\partial}) \arrow[r, hook] \arrow[d, hook] & B^1(\mathcal{C}, \partial) \arrow[r, "\pi_1"] \arrow[d, hook] & B^1(\mathcal{C}^{0,\bullet}, \partial_{0,1}) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & Z^1(\mathcal{N}_0, \overline{\partial}) \arrow[r, hook] \arrow[d] & Z^1(\mathcal{C}, \partial) \arrow[r, "\pi_1"] \arrow[d] & \ker(\rho_1) \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & H^1(\mathcal{N}_0, \overline{\partial}) \arrow[r] \arrow[d] & H^1(\mathcal{C}, \partial) \arrow[r] \arrow[d] & \frac{\ker(\rho_1)}{B^1(\mathcal{C}^{0,\bullet}, \partial_{0,1})} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Observe that this result on the first cohomology of $(\mathcal{C}, \partial)$ involves the map $\rho_1 : \mathcal{A}^1 \to H^2(\mathcal{N}_0, \overline{\partial})$, which is related to the second cohomology of $(\mathcal{N}_0, \overline{\partial})$.

Proof of Theorem 3.11. The fact that the diagram of Theorem 3.11 coincides with the one given in Theorem 3.7 for $k = 1$ follows from equations (3.12) and (3.13), the definition of ϱ_k, and from Lemma 3.10. We also need the following identity,

$$\mathcal{B}_0^1 \cap \mathcal{C}^{1,0} = \{\partial_{1,0} f \mid \partial_{0,1} f = 0, f \in \mathcal{C}^0\} = B^1(\mathcal{N}_0, \overline{\partial}).$$

■

Corollary 3.12. *In the case when $\mathcal{R}$ is a field, the coboundary, cocycle, and cohomology spaces of degree 1 admit the following splittings as vector spaces:*

$$B^1(\mathcal{C},\partial) \cong B^1(\mathcal{N}_0,\overline{\partial}) \oplus B^1(\mathcal{C}^{0,\bullet},\partial_{0,1}), \qquad Z^1(\mathcal{C},\partial) \cong Z^1(\mathcal{N}_0,\overline{\partial}) \oplus \ker(\rho_1),$$
$$H^1(\mathcal{C},\partial) \cong H^1(\mathcal{N}_0,\overline{\partial}) \oplus \frac{\ker(\rho_1)}{B^1(\mathcal{C}^{0,\bullet},\partial_{0,1})}.$$

Explicitly,

$$\begin{aligned}
B^1(\mathcal{N}_0,\overline{\partial}) &= \{\partial_{1,0}f \mid f \in \mathcal{C}^{0,0}, \partial_{0,1}f = 0\},\\
B^1(\mathcal{C}^{0,\bullet},\partial_{0,1}) &= \{\partial_{0,1}f \mid f \in \mathcal{C}^{0,0}\},\\
\mathcal{A}^1 &= \{Y \in \mathcal{C}^{0,1} \mid \partial_{0,1}Y = 0,\ \exists \alpha_Y \in \mathcal{C}^{1,0} : \partial_{0,1}\alpha_Y + \partial_{1,0}Y = 0\},\\
Z^1(\mathcal{N}_0,\overline{\partial}) &= \{\alpha \in \mathcal{C}^{1,0} \mid \partial_{0,1}\alpha = 0, \partial_{1,0}\alpha = 0\},\\
\ker(\rho_1) &= \{Y \in \mathcal{A}^1 \mid \partial_{2,-1}Y + \partial_{1,0}\alpha_Y \in B^1(\mathcal{N}_0,\overline{\partial})\}.
\end{aligned}$$

Second cohomology. Similarly, the $\mathcal{R}$-modules of cocycles, coboundaries, and cohomology of degree 2 of the bigraded cochain complex $(\mathcal{C},\partial)$ are described by the following more explicit diagrams.

Theorem 3.13. *We have the following commutative diagrams with exact rows and columns,*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^2(\mathcal{C},\partial)\cap\mathcal{C}^{2,0} \arrow[r, hook] \arrow[d, hook] & B^2(\mathcal{C},\partial) \arrow[r, "\pi_1"] \arrow[d, hook] & \mathcal{B}^2_1 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & Z^2(\mathcal{N}_0,\overline{\partial}) \arrow[r, hook] \arrow[d] & Z^2(\mathcal{C},\partial) \arrow[r, "\pi_1"] \arrow[d] & \ker(\rho_2) \arrow[r] \arrow[d] & 0, \\
0 \arrow[r] & \frac{Z^2(\mathcal{N}_0,\overline{\partial})}{B^2(\mathcal{C},\partial)\cap\mathcal{C}^{2,0}} \arrow[r] \arrow[d] & H^2(\mathcal{C},\partial) \arrow[r] \arrow[d] & \frac{\ker(\rho_2)}{\mathcal{B}^2_1} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B}^2_1\cap\mathcal{C}^{1,1} \arrow[r, hook] \arrow[d, hook] & \mathcal{B}^2_1 \arrow[r, "\pi_2"] \arrow[d, hook] & B^2(\mathcal{C}^{0,\bullet},\partial_{0,1}) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \ker(\varrho_2) \arrow[r, hook] \arrow[d] & \ker(\rho_2) \arrow[r, "\pi_2"] \arrow[d] & \mathcal{Z}^2_2 \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \frac{\ker(\varrho_2)}{\mathcal{B}^2_1\cap\mathcal{C}^{1,1}} \arrow[r] \arrow[d] & \frac{\ker(\rho_2)}{\mathcal{B}^2_1} \arrow[r] \arrow[d] & \frac{\mathcal{Z}^2_2}{B^2(\mathcal{C}^{0,\bullet},\partial_{0,1})} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Observe that this result on the second cohomology of $(\mathcal{C},\partial)$ involves the maps $\rho_2 : \mathcal{A}^2 \to H^3(\mathcal{N}_0,\overline{\partial})$ and $\varrho_2 : \mathcal{J}^2 \to H^3(\mathcal{N}_0,\overline{\partial})$, related to the third cohomology of $(\mathcal{N}_0,\overline{\partial})$. Also, the submodule $\mathcal{Z}^2_2$ is related to the 3-coboundaries of $(\mathcal{N}_1,\overline{\partial})$.

Corollary 3.14. *In the case when $\mathcal{R}$ is a field, the coboundary, cocycle, and cohomology spaces of degree 2 admit the following splittings as vector spaces:*

$$\begin{aligned}
&B^2(\mathcal{C},\partial) \cong (B^2(\mathcal{C},\partial)\cap\mathcal{C}^{2,0}) \oplus (\mathcal{B}^2_1\cap\mathcal{C}^{1,1}) \oplus B^2(\mathcal{C}^{0,\bullet},\partial_{0,1}),\\
&Z^2(\mathcal{C},\partial) \cong Z^2(\mathcal{N}_0,\overline{\partial}) \oplus \ker(\varrho_2) \oplus \mathcal{Z}^2_2,\\
&H^2(\mathcal{C},\partial) \cong \frac{Z^2(\mathcal{N}_0,\overline{\partial})}{B^2(\mathcal{C},\partial)\cap\mathcal{C}^{2,0}} \oplus \frac{\ker(\varrho_2)}{\mathcal{B}^2_1\cap\mathcal{C}^{1,1}} \oplus \frac{\mathcal{Z}^2_2}{B^2(\mathcal{C}^{0,\bullet},\partial_{0,1})}.
\end{aligned}$$

In a more explicit fashion, the modules appearing in Theorem 3.13 are

$$
\begin{aligned}
B^2(\mathcal{C},\partial)\cap\mathcal{C}^{2,0} &= \{\partial_{1,0}\alpha+\partial_{2,-1}Y \mid \alpha\in\mathcal{C}^{1,0}, Y\in\mathcal{C}^{0,1}, \partial_{0,1}\alpha+\partial_{1,0}Y=0, \partial_{0,1}Y=0\},\\
\mathcal{B}_1^2\cap\mathcal{C}^{1,1} &= \{\partial_{1,0}Y+\partial_{0,1}\alpha \mid \alpha\in\mathcal{C}^{1,0}, Y\in\mathcal{C}^{0,1}, \partial_{0,1}Y=0\},\\
B^2(\mathcal{C}^{0,\bullet},\partial_{0,1}) &= \{\partial_{0,1}Y \mid Y\in\mathcal{C}^{0,1}\},\\
\mathcal{J}^2 &= \{Q\in\mathcal{C}^{1,1} \mid \partial_{0,1}Q=0, \exists\beta_Q\in\mathcal{C}^{2,0} : \partial_{0,1}\beta_Q+\partial_{1,0}Q=0\},\\
Z^2(\mathcal{N}_0,\overline{\partial}) &= \{\beta\in\mathcal{C}^{2,0} \mid \partial_{1,0}\beta=0, \partial_{0,1}\beta=0\},\\
\ker(\varrho_2) &= \{Q\in\mathcal{J}^2 \mid \partial_{2,-1}Q+\partial_{1,0}\beta_Q\in B^3(\mathcal{N},\overline{\partial})\},\\
\mathcal{Z}_2^2 &= \left\{ V\in\ker^{0,2}\partial_{0,1} \;\middle|\; \exists Q\in\mathcal{C}^{1,1}, \beta\in\mathcal{C}^{2,0} : \begin{array}{c} \partial_{0,1}Q+\partial_{1,0}V=0,\\ \partial_{0,1}\beta+\partial_{1,0}Q+\partial_{2,-1}V\in B^3(\mathcal{N},\overline{\partial}),\\ \partial_{1,0}\beta+\partial_{2,-1}Q\in B^3(\mathcal{N},\overline{\partial}). \end{array} \right\}.
\end{aligned}
$$

Third cohomology. Finally, the following result gives a more explicit presentation of the $\mathcal{R}$-modules involved in the description of coboundaries, cocycles, and cohomology of degree 3.

Theorem 3.15. *We have the following commutative diagrams with exact rows and columns,*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B^3(\mathcal{C},\partial)\cap\mathcal{C}^{3,0} \arrow[r, hook] \arrow[d, hook] & B^3(\mathcal{C},\partial) \arrow[r, "\pi_1"] \arrow[d, hook] & \mathcal{B}_1^3 \arrow[r] \arrow[d, hook] & 0\\
0 \arrow[r] & Z^3(\mathcal{N}_0,\overline{\partial}) \arrow[r, hook] \arrow[d] & Z^3(\mathcal{C},\partial) \arrow[r, "\pi_1"] \arrow[d] & \ker(\rho_3) \arrow[r] \arrow[d] & 0,\\
0 \arrow[r] & \frac{Z^3(\mathcal{N}_0,\overline{\partial})}{B^3(\mathcal{C},\partial)\cap\mathcal{C}^{3,0}} \arrow[r] \arrow[d] & H^3(\mathcal{C},\partial) \arrow[r] \arrow[d] & \frac{\ker(\rho_3)}{\mathcal{B}_1^3} \arrow[r] \arrow[d] & 0\\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B}_1^3\cap\mathcal{C}^{2,1} \arrow[r, hook] \arrow[d, hook] & \mathcal{B}_1^3 \arrow[r, "\pi_2"] \arrow[d, hook] & \mathcal{B}_2^3 \arrow[r] \arrow[d, hook] & 0\\
0 \arrow[r] & \ker(\varrho_3) \arrow[r, hook] \arrow[d] & \ker(\rho_3) \arrow[r, "\pi_2"] \arrow[d] & \mathcal{Z}_2^3 \arrow[r] \arrow[d] & 0,\\
0 \arrow[r] & \frac{\ker(\varrho_3)}{\mathcal{B}_1^3\cap\mathcal{C}^{2,1}} \arrow[r] \arrow[d] & \frac{\ker(\rho_3)}{\mathcal{B}_1^3} \arrow[r] \arrow[d] & \frac{\mathcal{Z}_2^3}{\mathcal{B}_2^3} \arrow[r] \arrow[d] & 0\\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B}_2^3 \cap \mathcal{C}^{1,2} \arrow[r, hook] \arrow[d, hook] & \mathcal{B}_2^3 \arrow[r, "\pi_3"] \arrow[d, hook] & B^3(\mathcal{C}^{0,\bullet}, \partial_{0,1}) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{Z}_2^3 \cap \mathcal{C}^{1,2} \arrow[r, hook] \arrow[d] & \mathcal{Z}_2^3 \arrow[r, "\pi_3"] \arrow[d] & \mathcal{Z}_3^3 \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & \frac{\mathcal{B}_2^3 \cap \mathcal{C}^{1,2}}{\mathcal{Z}_2^3 \cap \mathcal{C}^{1,2}} \arrow[r] \arrow[d] & \frac{\mathcal{Z}_2^3}{\mathcal{B}_2^3} \arrow[r] \arrow[d] & \frac{\mathcal{Z}_3^3}{B^3(\mathcal{C}^{0,\bullet}, \partial_{0,1})} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

We note that this result on the third cohomology of $(\mathcal{C}, \partial)$ involves the maps $\rho_3 : \mathcal{A}^3 \to H^4(\mathcal{N}_0, \overline{\partial})$ and $\varrho_3 : \mathcal{J}^3 \to H^4(\mathcal{N}_0, \overline{\partial})$, related to the fourth cohomology of $(\mathcal{N}_0, \overline{\partial})$. Also, the submodule $\mathcal{Z}_3^3$ is related to the 4-coboundaries of $(\mathcal{N}_1, \overline{\partial})$.

Corollary 3.16. *In the case when $\mathcal{R}$ is a field, the coboundary, cocycle, and cohomology spaces of degree* 3 *admit the following splittings as vector spaces:*

$$
\begin{aligned}
B^3(\mathcal{C}, \partial) &\cong (B^3(\mathcal{C}, \partial) \cap \mathcal{C}^{3,0}) \oplus (\mathcal{B}_1^3 \cap \mathcal{C}^{2,1}) \oplus (\mathcal{B}_2^3 \cap \mathcal{C}^{1,2}) \oplus B^3(\mathcal{C}^{0,\bullet}, \partial_{0,1}), \\
Z^3(\mathcal{C}, \partial) &\cong Z^3(\mathcal{N}_0, \overline{\partial}) \oplus \ker(\varrho_3) \oplus (\mathcal{Z}_2^3 \cap \mathcal{C}^{1,2}) \oplus \mathcal{Z}_3^3, \\
H^3(\mathcal{C}, \partial) &\cong \frac{Z^3(\mathcal{N}_0, \overline{\partial})}{B^3(\mathcal{C}, \partial) \cap \mathcal{C}^{3,0}} \oplus \frac{\ker(\varrho_3)}{\mathcal{B}_1^3 \cap \mathcal{C}^{2,1}} \oplus \frac{\mathcal{Z}_2^3 \cap \mathcal{C}^{1,2}}{\mathcal{B}_2^3 \cap \mathcal{C}^{1,2}} \oplus \frac{\mathcal{Z}_3^3}{B^3(\mathcal{C}^{0,\bullet}, \partial_{0,1})}.
\end{aligned}
$$

Each of the terms appearing in the splittings of Corollary 3.16 are given as follows:

$$
B^3(\mathcal{C}, \partial) \cap \mathcal{C}^{3,0} = \left\{ \partial_{1,0}\beta + \partial_{2,-1}Q \;\middle|\; \beta \in \mathcal{C}^{2,0}, Q \in \mathcal{C}^{1,1}, \exists V \in \mathcal{C}^{0,2} : \begin{array}{l} \partial_{0,1}\beta + \partial_{1,0}Q + \partial_{2,-1}V = 0, \\ \partial_{0,1}Q + \partial_{1,0}V = 0, \\ \partial_{0,1}V = 0. \end{array} \right\},
$$

$$
\begin{aligned}
&\mathcal{B}_1^3 \cap \mathcal{C}^{2,1} = \{\partial_{0,1}\beta + \partial_{1,0}Q + \partial_{2,-1}V \mid \beta \in \mathcal{C}^{2,0}, Q \in \mathcal{C}^{1,1}, V \in \mathcal{C}^{0,2},\ \partial_{0,1}Q + \partial_{1,0}V = 0, \partial_{0,1}V = 0\}, \\
&\mathcal{B}_2^3 \cap \mathcal{C}^{1,2} = \{\partial_{0,1}Q + \partial_{1,0}V = 0 \mid Q \in \mathcal{C}^{1,1}, V \in \mathcal{C}^{0,2},\ \partial_{0,1}V = 0\}, \\
&B^3(\mathcal{C}^{0,\bullet}, \partial_{0,1}) = \{\partial_{0,1}V \mid V \in \mathcal{C}^{0,2}\}, \\
&\mathcal{J}^3 = \{R \in \mathcal{C}^{2,1} \mid \partial_{0,1}R = 0, \exists \varphi_R \in \mathcal{C}^{3,0} : \partial_{0,1}\varphi_R + \partial_{1,0}R = 0\}, \\
&Z^3(\mathcal{N}_0, \overline{\partial}) = \{\varphi \in \mathcal{C}^{3,0} \mid \partial_{1,0}\varphi = 0, \partial_{0,1}\varphi = 0\}, \\
&\ker(\varrho_3) = \{R \in \mathcal{J}^3 \mid \partial_{2,-1}R + \partial_{1,0}\varphi_R \in B^3(\mathcal{N}_0, \overline{\partial})\},
\end{aligned}
$$

$$
\mathcal{Z}_2^3 \cap \mathcal{C}^{1,2} = \left\{ S \in \mathcal{C}^{1,2} \;\middle|\; \exists R \in \mathcal{C}^{2,1}, \varphi \in \mathcal{C}^{3,0} : \begin{array}{l} \partial_{0,1}S = 0, \qquad \partial_{0,1}R + \partial_{1,0}S = 0, \\ \partial_{0,1}\varphi + \partial_{1,0}R + \partial_{2,-1}S \in B^4(\mathcal{N}_0, \overline{\partial}), \\ \partial_{1,0}\varphi + \partial_{2,-1}R \in B^4(\mathcal{N}_0, \overline{\partial}). \end{array} \right\},
$$

$$
\mathcal{Z}_3^3 = \left\{ W \in \ker^{0,3}(\partial_{0,1}) \;\middle|\; \exists S \in \mathcal{C}^{1,2}, R \in \mathcal{C}^{2,1}, \varphi \in \mathcal{C}^{3,0} : \begin{array}{l} \partial_{0,1}S + \partial_{1,0}W = 0, \\ \partial_{0,1}R + \partial_{1,0}S + \partial_{2,-1}W \in B^4(\mathcal{N}_0, \overline{\partial}), \\ \partial_{0,1}\varphi + \partial_{1,0}R + \partial_{2,-1}S \in B^4(\mathcal{N}_0, \overline{\partial}), \\ \partial_{1,0}\varphi + \partial_{2,-1}R \in B^4(\mathcal{N}_0, \overline{\partial}). \end{array} \right\}.
$$

The proofs of Theorems 3.13, and 3.15 are analogous to the proof of Theorem 3.11.

3.5 Particular cases

In this part we consider some particular cases regarding the bigraded cochain complex $(\mathcal{C}, \partial)$,

$$\partial = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1}.$$

The case $\partial_{2,-1} = 0$. In this part, we assume that $\partial_{2,-1} = 0$, which corresponds to the well-known case of a *double complex*. Namely, $(\mathcal{C}, \partial)$ is a bigraded cochain complex, such that the bigraded decomposition of the coboundary operator is

$$\partial = \partial_{0,1} + \partial_{1,0}.$$

The coboundary equations (3.3)-(3.7) read in this case

$$\partial_{0,1}^2 = 0, \qquad \partial_{0,1}\partial_{1,0} + \partial_{1,0}\partial_{0,1} = 0, \qquad \partial_{1,0}^2 = 0,$$

which means that the bigraded components $\partial_{0,1}$ and $\partial_{1,0}$ are coboundary operators which commute with each other in the graded sense.

The case of the double complex is a standard topic in the literature, since it naturally arises both from algebraic and geometric contexts [46, Chapter XI, Section 6], [47, Section 2.4], and has several applications [6, Chapter II]. However, the description of its cohomology is limited to explain that the natural filtration

$$F^p\mathcal{C}^\bullet := \bigoplus_{\substack{i,j\in\mathbb{Z}\\ i\geq p}} \mathcal{C}^{i,j}$$

induces a spectral sequence which converges to the cohomology, and whose second page is explicitly described in terms of the double complex, namely, $E_2^{p,q} = H^p(H^q(\mathcal{C}, \partial_{0,1}), \partial_{1,0})$. For several applications discussed in the literature, the computation of the second page of the spectral sequence is sufficient to describe the cohomology. In this sense, we have not found a general scheme for the computation of the cohomology of a double complex.

Theorems 3.11, 3.13, and 3.15 provide an explicit description of the low-degree cohomology of a bigraded cochain complex which, of course, also holds for the double complex. We remark that in this case, the null subcomplex is simply $\mathcal{N} = \ker \partial_{0,1}$, so the cocycles and coboundaries of $(\mathcal{N}, \overline{\partial})$ are

$$Z(\mathcal{N}, \overline{\partial}) = \ker \partial_{0,1} \cap \ker \partial_{1,0}, \qquad \text{and} \qquad B(\mathcal{N}, \overline{\partial}) = \partial_{1,0}(\ker \partial_{0,1}).$$

Furthermore, in the description of the cohomology of degree 1 provided by Theorem 3.11, we have

$$\ker(\rho_1) = \{Y \in \mathcal{C}^{0,1} \mid \partial_{0,1}Y = 0,\ \exists \alpha_Y \in \mathcal{C}^{1,0}\ :\ \partial_{1,0}Y + \partial_{0,1}\alpha_Y = 0, \partial_{1,0}\alpha_Y \in B^1(\mathcal{N}_0, \overline{\partial})\}.$$

On the other hand, the terms which simplify in the description of the cohomology of degree 2 of a double complex are

$$\begin{aligned} B^2(\mathcal{C},\partial)\cap\mathcal{C}^{2,0} &= \{\partial_{1,0}\alpha \mid \alpha\in\mathcal{C}^{1,0},\ \exists Y\in\mathcal{C}^{0,1}\ :\ \partial_{0,1}\alpha+\partial_{1,0}Y=0, \partial_{0,1}Y=0\},\\ \ker(\varrho_2) &= \{Q\in\mathcal{C}^{1,1}\mid \partial_{0,1}Q=0, \exists\beta_Q\in\mathcal{C}^{2,0}:\partial_{0,1}\beta_Q+\partial_{1,0}Q=0, \partial_{1,0}\beta_Q\in B^3(\mathcal{N},\overline{\partial})\},\\ \mathcal{Z}_2^2 &= \left\{ V\in\ker^{0,2}\partial_{0,1} \;\middle|\; \exists Q\in\mathcal{C}^{1,1},\beta\in\mathcal{C}^{2,0}: \begin{array}{c}\partial_{0,1}Q+\partial_{1,0}V=0,\\ \partial_{0,1}\beta+\partial_{1,0}Q\in B^3(\mathcal{N},\overline{\partial}),\\ \partial_{1,0}\beta\in B^3(\mathcal{N},\overline{\partial}).\end{array}\right\}. \end{aligned}$$

In a similar fashion, the terms appearing in the description of the cohomology of degree three that simplify in the case of the double complex are $\mathcal{B}_1^3\cap\mathcal{C}^{2,1}$, $\mathcal{B}_2^3\cap\mathcal{C}^{1,2}$, $\mathcal{Z}_2^3\cap\mathcal{C}^{1,2}$, and $\mathcal{Z}_3^3$.

The case $\partial_{0,1}=0$. We now consider a cochain complex $(\mathcal{C}^\bullet,\partial)$ endowed with a compatible bigrading such that the decomposition of the coboundary operator is of the form $\partial=\partial_{1,0}+\partial_{2,-1}$. This can be regarded as a particular case of our general scheme in which the operator of type $(0,1)$ vanishes, $\partial_{0,1}=0$. In this case, the coboundary property $\partial^2=0$ is equivalent to

$$\partial_{1,0}^2=0,\qquad \partial_{1,0}\partial_{2,-1}+\partial_{2,-1}\partial_{1,0}=0,\qquad \partial_{2,-1}^2=0,$$

which means that the bigraded components $\partial_{1,0}$ and $\partial_{2,-1}$ are graded commutative coboundary operators. Moreover, the null subcomplex is $\mathcal{N}=\ker\partial_{2,-1}$. In particular, $(\mathcal{N}_0,\overline{\partial})=(\mathcal{C}^{\bullet,0},\partial_{1,0})$,

$$Z^p(\mathcal{N}_0,\overline{\partial})=\ker(\partial_{1,0}:\mathcal{C}^{p,0}\to\mathcal{C}^{p+1,0}),\qquad\text{and}\qquad B^p(\mathcal{N}_0,\overline{\partial})=\partial_{1,0}(\mathcal{C}^{p-1,0}),\qquad \forall p\geq 0.$$

It is well known that this class of cochain complexes arise in the context of regular Poisson manifolds. In fact, the choice of a subbundle normal to the symplectic foliation of a regular Poisson manifold induces a bigrading of the Lichnerowicz-Poisson complex such that the coboundary operator is of this kind. Moreover, based on this fact, and motivated by the results in [74], a recursive scheme for the computation of the cohomology of regular Poisson manifolds is provided in [62, Section 2]. Such recursive scheme is similar to the one we have presented in Section 3.3, and leads to a description of the Poisson cohomology in terms of short exact sequences that coincide with the bottom rows of the diagrams of Proposition 3.2.

Finally, we remark that this class of cochain complexes also arise in the literature in the more general context of Poisson foliations [65, Proposition 2.2], [66, Lemma 4.1], which are Poisson structures such that the symplectic foliation admits an outer regularization.

The terms which appear in the description of the first cohomology which simplify in this case are

$$\begin{aligned} B^1(\mathcal{N}_0,\overline{\partial}) &= \partial_{1,0}(\mathcal{C}^{0,0}),\\ B^1(\mathcal{C}^{0,\bullet},\partial_{0,1}) &= \{0\},\\ \mathcal{A}^1 &= \{Y\in\mathcal{C}^{0,1}\mid\partial_{1,0}Y=0\},\\ Z^1(\mathcal{N}_0,\overline{\partial}) &= \{\alpha\in\mathcal{C}^{1,0}\mid\partial_{1,0}\alpha=0\},\\ \ker(\rho_1) &= \{Y\in\mathcal{A}^1\mid\partial_{2,-1}Y\in\partial_{1,0}(\mathcal{C}^{0,0})\}. \end{aligned}$$

In the description of the cohomology of degree two, the terms simplify to

$$\begin{aligned}
B^2(\mathcal{C},\partial)\cap\mathcal{C}^{2,0} &= \{\partial_{1,0}\alpha+\partial_{2,-1}Y \mid \alpha\in\mathcal{C}^{1,0}, Y\in\mathcal{C}^{0,1}, \partial_{1,0}Y=0\},\\
\mathcal{B}_1^2\cap\mathcal{C}^{1,1} &= \{\partial_{1,0}Y \mid Y\in\mathcal{C}^{0,1}\},\\
B^2(\mathcal{C}^{0,\bullet},\partial_{0,1}) &= \{0\},\\
\mathcal{J}^2 &= \{Q\in\mathcal{C}^{1,1} \mid \partial_{1,0}Q=0\},\\
Z^2(\mathcal{N}_0,\overline{\partial}) &= \{\beta\in\mathcal{C}^{2,0} \mid \partial_{1,0}\beta=0\},\\
\ker(\varrho_2) &= \{Q\in\mathcal{J}^2 \mid \partial_{2,-1}Q+\partial_{1,0}\beta_Q\in B^3(\mathcal{N},\overline{\partial})\},\\
\mathcal{Z}_2^2 &= \left\{ V\in\mathcal{C}^{0,2} \;\middle|\; \exists Q\in\mathcal{C}^{1,1},\beta\in\mathcal{C}^{2,0} : \begin{array}{c}\partial_{1,0}V=0,\\ \partial_{1,0}Q+\partial_{2,-1}V\in B^3(\mathcal{N},\overline{\partial}),\\ \partial_{1,0}\beta+\partial_{2,-1}Q\in B^3(\mathcal{N},\overline{\partial}).\end{array}\right\}.
\end{aligned}$$

In a similar fashion, most of the terms appearing in the description of the cohomology of degree three simplify.

Chapter 4

Bigraded Cohomological Models in Poisson Geometry

In this chapter, we review some of the geometric structures which induce a bigraded cochain complex such that the algebraic schemes developed in Chapter 3 can be applied. Furthermore, we show that the cochain complex of a coupling twisted Poisson or Dirac structure on a foliated manifold is endowed with a natural bigrading of this kind.

This chapter is organized as follows. Section 4.1 is devoted to describe the bigraded model on the general setting of Lie algebroids with fixed involutive subbundle. Also, we describe how the geometric structures described above fit in this general construction. In Section 4.2, we describe the algebraic structure of the bigraded model for coupling structures. Finally, in Section 4.3 we introduce the notions of coupling twisted Poisson and Dirac structures, show the correspondence with twisted Dirac elements, and describe the bigraded model for the cohomology.

4.1 Bigrading on Lie algebroids

Let $(E \to M, q, [\cdot,\cdot]_E)$ be a Lie algebroid, and let $\partial_E : \Gamma(\wedge^\bullet E^*) \to \Gamma(\wedge^\bullet E^*)$ be its de Rham differential, in the sense of Definition 1.12. Suppose we are given a regular distribution $V \subset E$ whose sections are closed under the bracket of the Lie algebroid, $[\Gamma(V), \Gamma(V)]_E \subseteq \Gamma(V)$. Such regular distributions will be called *involutive*. In addition, suppose we are given a vector bundle map $\mathrm{p} : E \to E$ such that $\mathrm{p}^2 = \mathrm{p}$ and $\mathrm{im}(\mathrm{p}) = V$. We say that p is a *connection* on E adapted to V. Denote $H \equiv H^{\mathrm{p}} := \ker(\mathrm{p})$, and let $V^\circ, H^\circ \subset E^*$ be the annihilators of V and H in E^*, respectively. Then, we have the splittings $H^{\mathrm{p}} \oplus V = E$, and $V^\circ \oplus H^\circ = E^*$. For simplicity, let us denote for each $p, q \in \mathbb{Z}$, $\wedge^{p,q} E^* := \wedge^p V^\circ \otimes \wedge^q H^\circ$. Then,

$$\wedge^\bullet E^* = \bigoplus_{p,q \in \mathbb{Z}} \wedge^{p,q} E^*, \tag{4.1}$$

Moreover, the *curvature* $R^{\mathrm{p}}(a,b) \in \Gamma(\wedge^2 V^\circ \otimes V)$ of p is given by

$$R^{\mathrm{p}}(a,b) := \mathrm{p}[(\mathrm{Id}_E - \mathrm{p})a, (\mathrm{Id}_E - \mathrm{p})b]_E, \qquad \forall a, b \in \Gamma(E),$$

and is the obstruction to the involutivity of H^{p}. On the other hand, for each $K \in \Gamma(\wedge^k E^* \otimes E)$, we consider the insertion operator $\mathrm{i}_K : \Gamma(\wedge^\bullet E^*) \to \Gamma(\wedge^\bullet E^*)$ of degree $k-1$ given on $\alpha \in \Gamma(\wedge^p E^*)$ by

$$\mathrm{i}_K \alpha(a_1, \ldots, a_{p+k-1}) = \sum_{\sigma \in S_{(k,p-1)}} (-1)^\sigma \alpha(K(a_{\sigma_1}, \ldots, a_{\sigma_k}), a_{\sigma_{k+1}}, \ldots, a_{\sigma_{p+k-1}}) \quad \forall a_1, \ldots, a_{p+k-1} \in \Gamma(E).$$

Here, $S_{(k,p-1)}$ is the set of *shuffle permutations* (see Remark 1.11). Moreover, the ∂_E-Lie derivative $\mathrm{L}_K^{\partial_E} : \Gamma(\wedge^\bullet E^*) \to \Gamma(\wedge^\bullet E^*)$ of degree k is defined in terms of the commutator of graded endomorphisms by $\mathrm{L}_K^{\partial_E} := [\mathrm{i}_K, \partial_E] = \mathrm{i}_K \circ \partial_E + (-1)^k \partial_E \circ \mathrm{i}_K$.

Following the Appendix C, we get the following result on the bigraded decomposition of the de Rham differential.

Theorem 4.1. *Let* $\mathrm{p} \in \Gamma(\wedge^2 E^* \otimes V)$ *be a connection on the Lie algebroid* $(E, q, [\cdot,\cdot]_E)$ *adapted to the involutive subbundle* $V \subseteq E$*. The bigraded decomposition of the de Rham differential is*

$$\partial_E = (\partial_E^{\mathrm{p}})_{0,1} + (\partial_E^{\mathrm{p}})_{1,0} + (\partial_E^{\mathrm{p}})_{2,-1},$$

where, for each $i, j \in \mathbb{Z}$*, one has the property* $(\partial_E^{\mathrm{p}})_{i,j}(\Gamma(\wedge^{p,q}E^*)) \subseteq \Gamma(\wedge^{p+i,q+j}E^*)$ *for all* $p, q \in \mathbb{Z}$*. In terms of the connection* p*, the bigraded components of* ∂_E *are given by*

$$(\partial_E^{\mathrm{p}})_{0,1} = \mathrm{L}_{\mathrm{p}}^{\partial_E} - \mathrm{i}_{R^{\mathrm{p}}}, \qquad (\partial_E^{\mathrm{p}})_{1,0} = \mathrm{L}_{\mathrm{Id}_E - \mathrm{p}}^{\partial_E} + 2\,\mathrm{i}_{R^{\mathrm{p}}}, \qquad (\partial_E^{\mathrm{p}})_{2,-1} = -\,\mathrm{i}_{R^{\mathrm{p}}}\,.$$

This result implies that, whenever a Lie algebroid $(E, q, [\cdot,\cdot]_E)$ is endowed with a projection map $\mathrm{p} : E \to E$, $\mathrm{p}^2 = \mathrm{p}$, with involutive image $V = \mathrm{im}(\mathrm{p})$, its de Rham complex is bigraded in the sense of the complexes studied in Chapter 3. In other words, the results of Chapter 3 may be applied for the computation of the Lie algebroid cohomology.

Let us propose an adequate bigraded model for the description of the cohomology of a Lie algebroid $(E, q, [\cdot,\cdot]_E)$ in which an involutive subbundle $V \subseteq E$ is fixed. To this end, recall that the closedness of $\Gamma(V)$ under the bracket of $\Gamma(E)$ implies that, the triple $(V, q|_V, [\cdot,\cdot]_V)$ is a Lie algebroid such that $[\cdot,\cdot]_V := [\cdot,\cdot]_E|_{\Gamma(V)\times\Gamma(V)}$ is the restriction, and the natural inclusion $\iota : V \hookrightarrow E$ is a morphism. In particular, V has its own de Rham complex $(\Gamma(\wedge^\bullet V^*), \partial_V)$.

Note that each complementary subbundle $H \subset E$, $H \oplus V = E$ corresponds to a unique connection $\mathrm{p} \in \Gamma(E^* \otimes V)$ on the Lie algebroid adapted to V, via the relation $\ker(\mathrm{p}) = H$. We also observe that the splitting $V^\circ \oplus H^\circ = E^*$ induces an isomorphism $H^\circ \cong V^*$, given by the dual of the inclusion map $\iota^* : E^* \twoheadrightarrow V^*$ restricted to H°. For each $p, q \in \mathbb{Z}$, this induces an isomorphism $\wedge^{p,q}E^* \cong \wedge^p V^\circ \otimes \wedge^q V^*$. Let us denote $C^{p,q} := \wedge^p V^\circ \otimes \wedge^q V^*$, $\mathcal{C}^{p,q} := \Gamma(C^{p,q})$, and $\mathcal{C}^{\bullet,\bullet} := \bigoplus_{p,q\in\mathbb{Z}} \mathcal{C}^{p,q}$. We then have a bigraded exterior algebra isomorphism $\Gamma(\wedge^{\bullet,\bullet}E^*) \cong \mathcal{C}^{\bullet,\bullet}$. Moreover, under this correspondence, the de Rham complex $(\Gamma(\wedge^\bullet E^*), \partial_E)$ induces a cochain complex structure on $\mathcal{C}^\bullet$.

Observe that the bigraded model $\mathcal{C}^{\bullet,\bullet}$ only depends of V, but the bigraded exterior algebra isomorphism between $\Gamma(\wedge^\bullet E^*)$ and $\mathcal{C}^\bullet$ do depend of the choice of p. So, the coboundary operator induced on $\mathcal{C}^\bullet$ also depends on p. Since the de Rham complex $(\Gamma(\wedge^\bullet E^*), \partial_E)$ is bigraded and such that $\partial_E = (\partial_E^{\mathrm{p}})_{0,1} + (\partial_E^{\mathrm{p}})_{1,0} + (\partial_E^{\mathrm{p}})_{2,-1}$, it follows that the coboundary operator ∂^{p} on $\mathcal{C}^\bullet$ has a bigraded decomposition of the same type, $(\mathcal{C}^{\bullet,\bullet}, \partial^{\mathrm{p}} = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1})$.

We now describe each of the bigraded components $\partial_{0,1}$, $\partial_{1,0}$, and $\partial_{2,-1}$ of the coboundary operator $\partial^{\mathrm{p}} : \mathcal{C}^\bullet \to \mathcal{C}^\bullet$.

- **The operator** $\partial_{0,1}^V$**.** Observe that the coboundary operator $\partial_{0,1}$ is independent of the choice of the connection p. In fact, the bigraded operator $\partial_{0,1}$ is an extension of the de Rham differential ∂_V of $\mathcal{C}^{0,\bullet} = \Gamma(\wedge^\bullet V^*)$ to the whole bigraded space $\mathcal{C}^{\bullet,\bullet}$, in the sense that $\partial_{0,1}|_{\Gamma(\wedge^\bullet V^*)} = \partial_V$. In fact, the value of $\partial_{0,1}\alpha$, with $\alpha \in \Gamma(\wedge^p V^\circ)$, on $v \in \Gamma(V)$ and $a_1, \dots, a_p \in \Gamma(E)$ is

$$\partial_{0,1}\alpha(v; a_1, \dots, a_p) = \mathrm{L}_v^{\partial}\,\alpha(a_1, \dots, a_p) = \mathrm{L}_{q(v)}(\alpha(a_1, \dots, a_p)) - \sum_{i=1}^{p} \alpha(a_1, \dots, [v, a_i]_E, \dots, a_p).$$

  Since $\partial_{0,1}$ is a derivation, these relations determine its value on the whole $\mathcal{C}^{\bullet,\bullet}$. It now becomes clear that $\partial_{0,1}$ is in fact independent of the choice of p.

- **The operator $\partial^{\mathrm{p}}_{1,0}$.** Proceeding as before, the differential $\partial^{\mathrm{p}}_{1,0}$ on $\alpha \in \Gamma(\wedge^p V^\circ)$ is given on $a_1, \dots, a_{p+1}$ by

$$\partial^{\mathrm{p}}_{1,0}\alpha(a_1,\dots,a_{p+1}) = \sum_{\sigma \in S_{(1,p)}} (-1)^\sigma \, \mathrm{L}_{q((\mathrm{Id}-\mathrm{p})a_{\sigma_1})}(\alpha(a_{\sigma_2},\dots,a_{\sigma_{p+1}})) - \sum_{\sigma \in S_{(2,p-1)}} (-1)^\sigma \alpha((\mathrm{Id}-\mathrm{p})[a_{\sigma_1},a_{\sigma_2}]_E, a_{\sigma_3},\dots,a_{\sigma_{p+1}}).$$

Similarly, if $\mu \in \Gamma(\wedge^q V^*)$, then for $v_1,\dots,v_q \in \Gamma(V)$, and $a \in \Gamma(E)$,

$$\partial^{\mathrm{p}}_{1,0}\mu(a;v_1,\dots,v_q) = \mathrm{L}_{q((\mathrm{Id}-\mathrm{p})a)}(\mu(v_1,\dots,v_q)) - \sum_{i=1}^{q} \mu(v_1,\dots,\mathrm{p}[(\mathrm{Id}-\mathrm{p})a, v_i]_E,\dots,v_q).$$

- **The operator $\partial^{\mathrm{p}}_{2,-1}$.** This operator is given by the curvature $R^{\mathrm{p}} \in \Gamma(\wedge^2 V^\circ \otimes V)$ of p. Viewing each $\eta \in \mathcal{C}^{p,q}$ as a p-linear skew-symmetric map $\eta : \Gamma(E) \times \cdots \times \Gamma(E) \to \Gamma(\wedge^q V^*)$ vanishing if any argument lies in $\Gamma(V)$, the operator $\partial^{\mathrm{p}}_{2,-1}$ is readily described by the curvature, as follows:

$$\partial^{\mathrm{p}}_{2,-1}\eta(a_1,\dots,a_{p+2}) = -(-1)^p \sum_{\sigma \in S_{(2,p)}} (-1)^\sigma \, \mathrm{i}_{R^{\mathrm{p}}(a_{\sigma_1},a_{\sigma_2})}(\eta(a_{\sigma_3},\dots,a_{\sigma_p+2})).$$

The operator $\partial^V_{0,1}$ is the extension of the de Rham differential of V, and $\partial^{\mathrm{p}}_{1,0}$ is the *covariant exterior derivative* of p. The operator $\partial^{\mathrm{p}}_{2,-1}$ is the negative of the insertion of the curvature.

In some particular contexts, the formulas presented in the previous general context of a Lie algebroid with an involutive subbundle can be simplified. This is in fact the case when $V \subset E$ can be realized as the kernel of a surjective Lie algebroid morphism $\pi : E \to A$ onto $(A \to N, q_A, [\cdot,\cdot]_A)$ covering some map $M \to N$ [7, Section 3]. In this case, E is called an *extension* of A via V, and the above formulas get a simpler formula when applied to the H^{p}-lifting of sections of A. Another case in which such formulas simplify are $E = TM$ is the tangent Lie algebroid. In such context, Frobenius Theorem implies that V is the tangent bundle of some foliation $\mathcal{V}$, $V = T\mathcal{V}$. By evaluating in local $\mathcal{V}$-projectable vector fields, the above formulas also simplify.

4.1.1 Geometric applications

We now briefly describe some well-known geometric settings in which the corresponding Lie algebroid is naturally equipped with a connection.

Example 4.2 (Regular Poisson structures)**.** Let (M,Π) be a regular Poisson manifold, with symplectic foliation $(\mathcal{S},\omega_{\mathcal{S}})$, and consider its cotangent Lie algebroid $(T^*M, \Pi^\sharp, \{\cdot,\cdot\}_\Pi)$. The annihilator of the characteristic distribution $V := \mathrm{Ann}(T\mathcal{S})$ is a subbundle of T^*M which coincides with the kernel of the anchor map, $V = \ker(\Pi^\sharp)$. Then, it is clear from (1.12) that $\{\Gamma(V),\Gamma(V)\}_\Pi = \{0\}$. Furthermore, the involutivity of $T\mathcal{S}$ implies $\{\Gamma(V),\Gamma(T^*M)\}_\Pi \subseteq \Gamma(V)$. Therefore, by taking any vector subbundle $H \subset T^*M$ such that $H \oplus V = T^*M$, it follows from Corollary C.7 that the corresponding bigrading of the Lichnerowicz-Poisson operator is

$$\partial_\Pi = (\partial_\Pi)_{1,0} + (\partial_\Pi)_{2,-1}.$$

Furthermore, Corollary C.6 implies that $(\partial_\Pi)_{2,-1}$ is the zero operator if and only if the subbundle $H \subseteq T^*M$ is involutive, $\{\Gamma(H), \Gamma(H)\}_\Pi \subseteq \Gamma(H)$. Because of Lemma B.3, this is equivalent to the following geometric condition: if $N \subseteq T^*M$ is the annihilator of H, then every $X \in \Gamma(N) \cap \mathrm{aut}(M, \mathcal{S})$ is Poisson, $X \in \mathrm{Poiss}(M, \Pi)$. In other words, N is a Poisson connection if and only if $(\partial_\Pi)_{2,-1} = 0$ (see Example 2.12). In particular, this implies that Π must be transversally constant [62, pp. 959-960], [64, pp. 49-50]. ▼

Example 4.3 (Poisson foliations)**.** All of the arguments in the previous example are valid for the case of regular Poisson foliations. Given a Poisson foliation $(M, \mathcal{F}, P)$, one can take $V := \mathrm{Ann}(T\mathcal{F})$. Then, it is clear that V is an Abelian ideal of $(T^*M, \{\cdot,\cdot\}_P)$ such that $V \subseteq \ker(P^\sharp)$. By fixing H such that $H \oplus V = T^*M$, the Lichnerowicz-Poisson operator has the form $\partial_P = (\partial_P)_{1,0} + (\partial_P)_{2,-1}$ [65, Proposition 2.2], [66, Lemma 4.1], and the term $(\partial_P)_{2,-1}$ vanishes if and only if V is the annihilator of a Poisson connection. ▼

Example 4.4 (Extensions of Lie algebroids)**.** Let $\pi : E \to A$ be a Lie algebroid epimorphism covering the base map $p : M \to N$, and let $V := \ker(\pi)$. Observe that for each $m \in M$, the image of dual map $\pi_m^* : A_{p(m)}^* \to E_m^*$ is V_m°. This implies that the image of $\pi^* : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^\bullet E^*)$ generates $\Gamma(\wedge^\bullet V^\circ)$. Moreover, for each $v, w \in \Gamma(V)$, and $\alpha \in \Gamma(A^*)$, we have

$$\pi^*\alpha[v,w]_E = \mathrm{L}_{q(v)}(\pi^*\alpha(w)) - \mathrm{L}_{q(w)}(\pi^*\alpha(v)) - \pi^*\,\mathrm{d}_A\,\alpha(v,w) = 0.$$

Since $\alpha \in \Gamma(A^*)$ is arbitrary, this implies that $[v,w]_E \in \Gamma(\mathrm{Ann}(V^\circ)) = \Gamma(V)$. Therefore, V is involutive. Now, consider the bigraded model $\mathcal{C}^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{Z}} \mathcal{C}^{p,q}$ described above, where $\mathcal{C}^{p,q} = \Gamma(\wedge^p V^\circ \otimes \wedge^q V^*)$. Then, the $C^\infty(N)$-module isomorphism $\pi^* : \Gamma(\wedge^\bullet A^*) \to \Gamma(\wedge^\bullet V^\circ)$ extends to

$$\pi^* : \Gamma(\wedge^\bullet A^*) \otimes_{C^\infty(N)} \Gamma(\wedge^\bullet V^*) \to \mathcal{C}^{\bullet,\bullet}.$$

Now, consider a connection p adapted to $V \subset E$. For each $a \in \Gamma(A)$, define $h(a) \in \Gamma(H)$ by the relation, and $\pi^*\alpha(h(a)) = p^*(\alpha(a))$, for all $\alpha \in \Gamma(A)$. This induces a linear A-connection $\nabla \equiv \nabla^h$ on V, given by

$$\nabla^h : \Gamma(A) \times \Gamma(V) \to \Gamma(V), \qquad \nabla_a^h v := [h(a), v]_E, \quad \forall a \in \Gamma(A), v \in \Gamma(V).$$

Consider also the curvature 2-form $\sigma^{\mathrm{p}} \in \Gamma(\wedge^2 A^*) \otimes_{C^\infty(N)} \Gamma(V)$, given on $a, b \in \Gamma(A)$ by $\sigma^{\mathrm{p}}(a,b) := [h(a), h(b)] - h[a,b]$. Our last isomorphism induces a coboundary operator ∂^{p} on $\Gamma(\wedge^\bullet A^*) \otimes_{C^\infty(N)} \Gamma(\wedge^\bullet V^*)$, and the formulas for $\partial_{1,0}^{\mathrm{p}}$, $\partial_{0,1}^{V}$, $\partial_{2,-1}^{\mathrm{p}}$ simplify to [7, Prop. 3.2]

$$\begin{aligned}
\partial_{1,0}^{V}\eta(a_1,\dots,a_p) &= (-1)^p\,\mathrm{d}_V(\eta(a_1,\dots,a_p)),\\
\partial_{0,1}^{\mathrm{p}}\eta(a_1,\dots,a_{p+1}) &= \sum_{(1,p)}(-1)^\sigma \nabla_{a_{\sigma_1}}^*(\eta(a_{\sigma_2},\dots,a_{\sigma_{p+1}})) - \sum_{(2,p-1)}(-1)^\sigma \eta([a_{\sigma_1},a_{\sigma_2}]_A, a_{\sigma_3},\dots,a_{\sigma_{p+1}}),\\
\partial_{2,-1}^{\mathrm{p}}\eta(a_1,\dots,a_{p+2}) &= -(-1)^p \sum_{(2,p)}(-1)^\tau\, \mathrm{i}_{\sigma^{\mathrm{p}}(a_{\tau_1},a_{\tau_2})}\,\eta(a_{\tau_3},\dots,a_{\tau_{p+1}}),
\end{aligned}$$

for all $\eta \in \Gamma(\wedge^p A^*) \otimes_{C^\infty(N)} \Gamma(\wedge^q V^*)$, and $a_1, \dots, a_{p+2} \in \Gamma(A)$. ▼

Example 4.5 (De Rham complex on foliated manifolds)**.** Let $(M, \mathcal{V})$ be a foliated manifold, and let $V := T\mathcal{V}$. As a consequence of Theorem 4.1, for any connection γ on $(M, \mathcal{V})$, the bigraded decomposition of the exterior derivative is of the form $\mathrm{d} = \mathrm{d}^{\gamma}_{0,1} + \mathrm{d}^{\gamma}_{1,0} + \mathrm{d}^{\gamma}_{2,-1}$ [61]. Observe that in the particular case when the leaves of $\mathcal{V}$ are given by the fibers of a surjective submersion $p : M \to N$, V is the kernel of the Lie algebroid map $\pi = p_* : TM \to TN$, so this example fits in the previous case of extensions of Lie algebroids. However, even though the leaves $\mathcal{V}$ of M are not the fibers of a submersion, locally they are. So, the formulas of the previous example still hold for the bigraded model $\mathcal{C}^{\bullet,\bullet} = \Gamma(\wedge^{\bullet} V^{\circ} \otimes \wedge^{\bullet} V^*)$ when we evaluate on $\mathcal{V}$-projectable vector fields. In fact, for $u_1, \ldots, u_{p+2} \in \mathrm{aut}(M, \mathcal{V})$, and $\eta \in \mathcal{C}^{p,q}$ we have

$$\begin{aligned}
\eth^{V}_{1,0}\eta(u_1, \ldots, u_p) &= (-1)^p \, \mathrm{d}_V(\eta(u_1, \ldots, u_p)), \\
\eth^{\mathrm{p}}_{0,1}\eta(u_1, \ldots, u_{p+1}) &= \sum_{(1,p)} (-1)^{\sigma} \, \mathrm{L}_{u_{\sigma_1}}(\eta(u_{\sigma_2}, \ldots, u_{\sigma_{p+1}})) - \sum_{(2,p-1)} (-1)^{\sigma} \eta([u_{\sigma_1}, u_{\sigma_2}], u_{\sigma_3}, \ldots, u_{\sigma_{p+1}}), \\
\eth^{\mathrm{p}}_{2,-1}\eta(u_1, \ldots, u_{p+2}) &= -(-1)^p \sum_{(2,p)} (-1)^{\sigma} \, \mathrm{i}_{R^{\mathrm{p}}(u_{\sigma_1}, u_{\sigma_2})} \, \eta(u_{\sigma_3}, \ldots, u_{\sigma_{p+1}}).
\end{aligned}$$

▼

4.2 The bigraded model for coupling structures

4.2.1 The Dirac structure of a foliation

There are two Lie algebroids associated with a regular foliation. One of them is the Lie algebroid of the tangent bundle to the foliation. The other one is the Lie algebroid of the Dirac structure of the regular foliation.

Let $\mathcal{V}$ be a regular foliation on the manifold M, with tangent bundle $V := T\mathcal{V}$. Let us denote by $V^{\circ} := \mathrm{Ann}(V)$ its annihilator. We also denote the Whitney sum of V and V° by $\mathbb{V} := V \oplus V^{\circ}$. It is clear that $\mathbb{V} \subset \mathbb{T}M$ is a maximally isotropic subbundle of $\mathbb{T}M$ with respect to the canonical pairing. Furthermore, $\mathbb{V}$ is a Dirac structure (with the standard Dorfman bracket (1.8)). Indeed, given $X, Y \in \Gamma(V)$, and $\alpha, \beta \in \Gamma(V^{\circ})$, we have

$$[\![X \oplus \alpha, Y \oplus \beta]\!] = [X, Y] \oplus (\mathrm{L}_X \beta - \mathrm{i}_Y \mathrm{d}\, \alpha) = [X, Y] \oplus (\mathrm{L}_X \beta - \mathrm{L}_Y \alpha).$$

Here, we have applied the fact that $\mathrm{i}_Y \alpha = 0$ and Cartan's formula. Since V is involutive, the vectorial component is again leaf-tangent, $[X, Y] \in \Gamma(V)$. Furthermore, the fact that each of $\mathrm{L}_X \beta$ and $\mathrm{L}_Y \alpha$ are sections of V° follow from standard computations and the involutivity of V. Therefore, $[\![X \oplus \alpha, Y \oplus \beta]\!] \in \Gamma(\mathbb{V})$, and hence $\mathbb{V}$ is a Dirac structure. As explained in Subsection 1.3.1, this means that $\mathbb{V}$ has a natural structure of Lie algebroid, where the Lie bracket $[\cdot, \cdot]_{\mathbb{V}}$ is just the restriction of the Dorfman bracket.

The bigrading of the Lie algebroid $\mathbb{V}$. Consider the Lie algebroid $(\mathbb{V}, q, [\cdot, \cdot]_{\mathbb{V}})$, where the anchor map and the bracket are the restrictions of the projection $p_{TM} : \mathbb{T}M \to TM$ and of the Dorfman bracket $[\![\cdot, \cdot]\!]$, respectively. Let us describe the bigraded properties of its de Rham differential $\eth_{\mathbb{V}}$,

$$\eth_{\mathbb{V}} : \Gamma(\wedge^{\bullet} \mathbb{V}^*) \to \Gamma(\wedge^{\bullet} \mathbb{V}^*),$$

with respect to the canonical splitting $\mathbb{V} = V \oplus V^{\circ}$. Denote by $\mathrm{p} : \mathbb{V} \to \mathbb{V}$ the projection over V along the splitting $\mathbb{V} = V \oplus V^{\circ}$. The horizontal and vertical distributions of p are $\{0\} \oplus V^{\circ}$ and

$V \oplus \{0\}$, respectively. Then, the vector bundle $\wedge^\bullet \mathbb{V}$ acquires structure of bigraded ($\mathbb{Z} \times \mathbb{Z}$-graded) vector bundle. Denoting, for each $p, q \in \mathbb{Z}$, $\wedge^{p,q}\mathbb{V} := \wedge^p V^\circ \otimes \wedge^q V$, we get

$$\wedge^{\bullet,\bullet}\mathbb{V} := \bigoplus_{p,q\in\mathbb{Z}} \wedge^{p,q}\mathbb{V}.$$

Following the Appendix C, by the definition of the anchor and Dorfman bracket, we observe that the p-vertical distribution $V \oplus \{0\}$ is involutive, and the p-horizontal distribution satisfies $\{0\} \oplus V^\circ = \ker(q)$. Since each of the hypotheses of Corollary C.8 hold, we conclude that the de Rham differential of the Lie algebroid $\mathbb{V}$ has bidegree $(0,1)$, $\eth_{\mathbb{V}} = (\eth^{\mathrm{p}}_{\mathbb{V}})_{0,1}$.

Let us use this fact to describe also the bigraded properties of the Schouten bracket of the Lie algebroid on $\mathbb{V}$,

$$[\cdot,\cdot]_{\mathbb{V}} : \Gamma(\wedge^\bullet\mathbb{V}) \times \Gamma(\wedge^*\mathbb{V}) \to \Gamma(\wedge^{\bullet+*-1}\mathbb{V})$$

(see Section 1.2). Recall that the Schouten bracket is characterized in terms of the graded commutator of endomorphisms on $\Gamma(\wedge^\bullet\mathbb{V}^*)$ by

$$\mathrm{i}_{[\eta,\eta']_{\mathbb{V}}} = [[\mathrm{i}_\eta, \eth_{\mathbb{V}}], \mathrm{i}_{\eta'}] \qquad\qquad \forall \eta, \eta' \in \Gamma(\wedge^\bullet\mathbb{V}). \tag{4.2}$$

We claim that the Schouten bracket has bidegree $(0,-1)$. Indeed, suppose that $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$ and $\eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$. Then, i_η and $\mathrm{i}_{\eta'}$ have bidegree $(-p,-q)$ and $(-p',-q')$, respectively. Since the bidegree of $\eth_{\mathbb{V}}$ is $(0,1)$, it follows from (4.2) that $\mathrm{i}_{[\eta,\eta']_{\mathbb{V}}}$ has bidegree $(-p-p', -q-q'+1)$ (the sum of the bidegrees of $\eth_{\mathbb{V}}$, i_η, and $\mathrm{i}_{\eta'}$). In consequence, $[\eta,\eta']_{\mathbb{V}}$ must have bidegree $(p+p', q+q'-1)$, which is the sum of (p,q) (the bidegree of η), (p',q') (the bidegree of η') and $(0,-1)$. Since $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$ and $\eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$ are arbitrary, we conclude that the Schouten bracket of $(\mathbb{V}, q, [\cdot,\cdot]_{\mathbb{V}})$ has bidegree $(0,-1)$. In other words, $(\Gamma(\wedge^{\bullet,\bullet}\mathbb{V}), [\cdot,\cdot]_{\mathbb{V}})$ is a bigraded $\mathbb{R}$-Lie algebra of bidegree $(0,-1)$.

Viewing $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$ and $\eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$ as $C^\infty(M)$-multilinear maps $\Gamma(TM) \times \cdots \times \Gamma(TM) \to \Gamma(\wedge^\bullet V)$, their exterior product $\eta \wedge \eta'$ and the Schouten bracket $[\eta,\eta']_{\mathbb{V}}$ evaluated on $\mathcal{V}$-projectable vector fields $u_1, \ldots, u_{p+p'} \in \mathrm{aut}(M,\mathcal{V})$ are given by

$$(\eta \wedge \eta')(u_1, \ldots, u_{p+p'}) = (-1)^{p'q} \sum_{\sigma \in S_{(p,p')}} (-1)^\sigma \eta(u_{\sigma_1}, \ldots, u_{\sigma_p}) \wedge \eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p+p'}}),$$

$$[\eta,\eta']_{\mathbb{V}}(u_1, \ldots, u_{p+p'}) = (-1)^{p'(q-1)} \sum_{\sigma \in S_{(p,p')}} (-1)^\sigma [\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}), \eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p+p'}})].$$

Here, the wedge products and brackets on each summand in the right-hand sides are the standard exterior product and Schouten-Nijenhuis bracket of multivector fields. If $\eta = \alpha \otimes A$, and $\eta' = \beta \otimes B$, where $\alpha, \beta \in \Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ)$, and $A, B \in \Gamma(\wedge^\bullet V)$, then

$$(\alpha \otimes A) \wedge (\beta \otimes B) = (-1)^{p'q}(\alpha \wedge \beta) \otimes (A \wedge B), \tag{4.3}$$

$$[\alpha \otimes A, \beta \otimes B]_{\mathbb{V}} = (-1)^{p'(q-1)}(\alpha \wedge \beta) \otimes [A, B]. \tag{4.4}$$

In particular, for leaf-tangent multivector fields $A, B \in \Gamma(\wedge^\bullet V) = \Gamma(\wedge^{0,\bullet}\mathbb{V})$, the Schouten bracket on $\Gamma(\wedge^{0,\bullet}\mathbb{V})$ coincides with the Schouten-Nijenhuis bracket of multivector fields restricted to $\Gamma(\wedge^\bullet V)$.

Finally, we remark that, for each $\eta \in \Gamma(\wedge^\bullet\mathbb{V})$, the adjoint operator $\mathrm{ad}_\eta : \Gamma(\wedge^\bullet\mathbb{V}) \to \Gamma(\wedge^\bullet\mathbb{V})$, given by $\mathrm{ad}_\eta(\eta') := [\eta,\eta']_{\mathbb{V}}$, is a graded derivation of the graded Poisson algebra $(\Gamma(\wedge^\bullet\mathbb{V}), \wedge, [\cdot,\cdot]_{\mathbb{V}})$ of bidegree $(p, q-1)$.

4.2.2 Derivations induced by bivectors and 3-forms.

Recall from Section 1.4 that the de Rham differential of the Lie algebroid of a twisted Poisson structure is the sum of two operators: the first of them is, as usual, the adjoint of the bivector with respect to the Schouten-Nijenhuis bracket of multivector fields. The other one is defined in terms of both the bivector and the background 3-form. This idea suggest that one can define bigraded derivations on the exterior algebra of the Dirac structure of a regular foliation, by means of a leaf-tangent bivector field and a 3-form.

In what follows, let $(M,\mathcal{V})$ be a regular foliation, and denote $V := T\mathcal{V}$, $V^\circ := \mathrm{Ann}(T\mathcal{V})$, $\mathbb{V} := V \oplus V^\circ$, $\wedge^{p,q}\mathbb{V} := \wedge^p V^\circ \otimes \wedge^q V$. Also, consider a leaf-tangent bivector field $P \in \Gamma(\wedge^2 V)$, and a 3-form $\psi \in \Gamma(\wedge^3 T^*M)$.

Derivations of bidegree (0,1). Consider the derivation $\mathrm{j}_{P,\psi} : \Gamma(\wedge^\bullet V) \to \Gamma(\wedge^\bullet V)$ of degree 1, defined as in (1.13). Let us extend it to a bigraded derivation $\mathrm{j}^{(0,1)}_{P,\psi} : \Gamma(\wedge^{\bullet,\bullet}\mathbb{V}) \to \Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ of bidegree $(0,1)$ by means of the formula

$$\mathrm{j}^{(0,1)}_{P,\psi}(\alpha \otimes A) := (-1)^p \alpha \otimes \mathrm{j}_{P,\psi} A, \qquad \forall \alpha \in \Gamma(\wedge^p V^\circ), A \in \Gamma(\wedge^q V). \tag{4.5}$$

The fact that $\mathrm{j}_{P,\psi}$ vanishes on $C^\infty(M)$ implies that $\mathrm{j}^{(0,1)}_{P,\psi}$ is well defined. On the other hand, the derivation property readily follows from (4.3). In fact, given $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$, and $\eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$ of the form $\eta = \alpha \otimes A$, and $\eta' = \beta \otimes B$, we have

$$\begin{aligned}
\mathrm{j}^{(0,1)}_{P,\psi}(\eta \wedge \eta') &= (-1)^{p'q}\,\mathrm{j}^{(0,1)}_{P,\psi}((\alpha \wedge \beta) \otimes (A \wedge B)) = (-1)^{p'q+p+p'}(\alpha \wedge \beta) \otimes \mathrm{j}_{P,\psi}(A \wedge B)\\
&= (-1)^{p'q+p+p'}(\alpha \wedge \beta) \otimes (\mathrm{j}_{P,\psi} A \wedge B + (-1)^q A \wedge \mathrm{j}_{P,\psi} B)\\
&= (-1)^p(\alpha \otimes \mathrm{j}_{P,\psi} A) \wedge (\beta \otimes B) + (-1)^{p+p'+q}(\alpha \otimes A) \wedge (\beta \otimes \mathrm{j}_{P,\psi} B)\\
&= \mathrm{j}^{(0,1)}_{P,\psi}(\alpha \otimes A) \wedge (\beta \otimes B) + (-1)^{p+q}(\alpha \otimes A) \wedge \mathrm{j}^{(0,1)}_{P,\psi}(\beta \otimes B)\\
&= \mathrm{j}^{(0,1)}_{P,\psi}\,\eta \wedge \eta' + (-1)^{|\eta|}\eta \wedge \mathrm{j}^{(0,1)}_{P,\psi}\,\eta'.
\end{aligned}$$

So, $\mathrm{j}^{(0,1)}_{P,\psi}$ is the graded derivation which coincides with $\mathrm{j}_{P,\psi}$ on $\Gamma(\wedge^\bullet V)$ and vanishing on $\Gamma(\wedge^\bullet V^\circ)$.

Derivations of degree (1,0). We now define a graded derivation $\mathrm{j}^{(1,0)}_{P,\psi} : \Gamma(\wedge^{\bullet,\bullet}\mathbb{V}) \to \Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ of bidegree $(1,0)$ as follows. For each $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$, $u_1, \dots, u_{p+1} \in \Gamma(V)$, and $\mu_1, \dots, \mu_q \in \Gamma(T^*M)$, set

$$\mathrm{j}^{(1,0)}_{P,\psi}\,\eta(u_1, \dots, u_{p+1}; \mu_1, \dots, \mu_q) := (-1)^p \sum_{\substack{\sigma \in S_{(p,1)}\\ \tau \in S_{(1,q)}}} (-1)^\sigma (-1)^\tau \eta(u_{\sigma_1}, \dots, u_{\sigma_p}; \mathrm{i}_{P^\sharp \mu_{\tau_1}}\, \mathrm{i}_{u_{\sigma_{p+1}}}\, \psi, \mu_{\tau_2}, \dots, \mu_{\tau_q}).$$

Since $P \in \Gamma(\wedge^2 V)$, it is clear that $\mathrm{j}^{(1,0)}_{P,\psi}\,\eta$ vanishes if any of μ_i lies in $\Gamma(V^\circ)$.

Lemma 4.6. *For each $P \in \Gamma(\wedge^2 V)$ and $\psi \in \Gamma(\wedge^3 T^*M)$, the operator $\mathrm{j}^{(1,0)}_{P,\psi} : \Gamma(\wedge^{\bullet,\bullet}\mathbb{V}) \to \Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ is a derivation.*

Proof. To save some space, denote $(-1)^{\sigma+\tau} := (-1)^{\sigma}(-1)^{\tau}$, and consider the tensor field $\mathbb{t} \equiv \mathbb{t}_{P,\psi}$ given by

$$\mathbb{t} : \Gamma(T^*M) \times \Gamma(TM) \to \Gamma(T^*M), \qquad \mathbb{t}_{\mu,u} := \mathrm{i}_{P^\sharp\mu}\, \mathrm{i}_u\, \psi.$$

Now, take $\eta \in \Gamma(\wedge^{p,q}\mathbb{V}), \eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$, $u_1, \ldots, u_{p+p'+1} \in \Gamma(TM)$, and $\mu_1, \ldots, \mu_{q+q'} \in \Gamma(T^*M)$. By definition, we have

$$\begin{aligned}
&\mathrm{j}^{(1,0)}_{P,\psi}(\eta \wedge \eta')(u_1, \ldots, u_{p+p'+1}; \mu_1, \ldots, \mu_{q+q'}) = \\
&(-1)^{p+p'} \sum_{\substack{\sigma\in S_{(p+p',1)}\\ \tau\in S_{(1,q+q'-1)}}} (-1)^{\sigma+\tau}(\eta \wedge \eta')(u_{\sigma_1}, \ldots, u_{\sigma_{p+p'}}; \mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+p'+1}}}, \mu_{\tau_2}, \ldots, \mu_{\tau_{q+q'}}) = \\
&(-1)^{p+p'+p'q} \sum_{\substack{\sigma\in S_{(p,p',1)}\\ \tau\in S_{(1,q+q'-1)}}} (-1)^{\sigma+\tau}[\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}) \wedge \eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p'}})](\mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+p'+1}}}, \mu_{\tau_2}, \ldots, \mu_{\tau_{q+q'}}).
\end{aligned}$$

Now, observe that $[\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}) \wedge \eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p'}})](\mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+p'+1}}}, \mu_{\tau_2}, \ldots, \mu_{\tau_{q+q'}})$ is the insertion of $\mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+p'+1}}}$ in the exterior product of the multivector fields $\eta(u_{\sigma_1}, \ldots, u_{\sigma_p})$, and $\eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p'}})$, all evaluated in $\mu_{\tau_2}, \ldots, \mu_{\tau_{q+q'}}$. Since the insertion is a graded derivation on multivector fields, the previous sum splits into two, namely,

$$\begin{aligned}
&\sum_{\substack{\sigma\in S_{(p,p',1)}\\ \tau\in S_{(1,q+q'-1)}}} (-1)^{\sigma+\tau}[\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}) \wedge \eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p'}})](\mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+p'+1}}}, \mu_{\tau_2}, \ldots, \mu_{\tau_{q+q'}}) = \\
&\sum_{\substack{\sigma\in S_{(p,p',1)}\\ \tau\in S_{(1,q-1,q')}}} (-1)^{\sigma+\tau}\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}; \mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+p'+1}}}, \mu_{\tau_2}, \ldots, \mu_{\tau_q})\eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p+p'}}; \mu_{\tau_{q+1}}, \ldots, \mu_{\tau_{q+q'}}) + \\
&(-1)^q \sum_{\substack{\sigma\in S_{(p,p',1)}\\ \tau\in S_{(1,q,q'-1)}}} (-1)^{\sigma+\tau}\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}; \mu_{\tau_2}, \ldots, \mu_{\tau_{q+1}})\eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p+p'}}; \mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+p'+1}}}, \mu_{\tau_{q+2}}, \ldots, \mu_{\tau_{q+q'}}) \\
&= (-1)^{p'} \sum_{\substack{\sigma\in S_{(p,1,p')}\\ \tau\in S_{(1,q-1,q')}}} (-1)^{\sigma+\tau}\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}; \mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+1}}}, \mu_{\tau_2}, \ldots, \mu_{\tau_q})\eta'(u_{\sigma_{p+2}}, \ldots, u_{\sigma_{p+p'+1}}; \mu_{\tau_{q+1}}, \ldots, \mu_{\tau_{q+q'}}) \\
&+ \sum_{\substack{\sigma\in S_{(p,p',1)}\\ \tau\in S_{(q,1,q'-1)}}} (-1)^{\sigma+\tau}\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}; \mu_{\tau_1}, \ldots, \mu_{\tau_q})\eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p+p'}}; \mathbb{t}_{\mu_{\tau_{q+1}}, u_{\sigma_{p+p'+1}}}, \mu_{\tau_{q+2}}, \ldots, \mu_{\tau_{q+q'}}).
\end{aligned}$$

Here, we have rearranged the indices of each of the two sums, while keeping track of the corresponding changes of sign. On the other hand, straightforward computations give

$$\begin{aligned}
&(\mathrm{j}^{(1,0)}_{P,\psi}\, \eta \wedge \eta')(u_{\sigma_1}, \ldots, u_{\sigma_{p+p'+1}}; \mu_{\tau_1}, \ldots, \mu_{\tau_{q+q'}}) = \\
&(-1)^{p+p'q} \sum_{\substack{\sigma\in S_{(p,1,p')}\\ \tau\in S_{(1,q-1,q')}}} (-1)^{\sigma+\tau}\eta(u_{\sigma_1}, \ldots, u_{\sigma_p}; \mathbb{t}_{\mu_{\tau_1}, u_{\sigma_{p+1}}}, \mu_{\tau_2}, \ldots, \mu_{\tau_q})\eta'(u_{\sigma_{p+2}}, \ldots, u_{\sigma_{p+p'+1}}; \mu_{\tau_{q+1}}, \ldots, \mu_{\tau_{q+q'}}),
\end{aligned}$$

and

$$(-1)^{p'+q+p'q}(\eta \wedge \mathrm{j}^{(1,0)}_{P,\psi}\, \eta')(u_{\sigma_1}, \ldots, u_{\sigma_{p+p'+1}}; \mu_{\tau_1}, \ldots, \mu_{\tau_{q+q'}}) =$$
$$\sum_{\substack{\sigma \in S_{(p,p',1)} \\ \tau \in S_{(q,1,q'-1)}}} (-1)^{\sigma+\tau} \eta(u_{\sigma_1}, \ldots, u_{\sigma_p}; \mu_{\tau_1}, \ldots, \mu_{\tau_q}) \eta'(u_{\sigma_{p+1}}, \ldots, u_{\sigma_{p+p'}}; \mathfrak{t}_{\mu_{\tau_{q+1}}, u_{\sigma_{p+p'+1}}}, \mu_{\tau_{q+2}}, \ldots, \mu_{\tau_{q+q'}}).$$

Taking into account the fact that the sum at the beginning of the proof has the factor $(-1)^{p+p'+p'q}$, the result follows. ■

Derivations of bidegree $(\mathbf{k}, -1)$. Now suppose we are given a $(k+1)$-form $\Theta \in \Gamma(\wedge^{k+1} T^*M)$ vanishing whenever two of its arguments are leaf-tangent vector fields, that is, $\Theta \in \Gamma(\wedge^k V^\circ \otimes T^*M)$. Let us define $\mathrm{j}^{(k,-1)}_\Theta : \Gamma(\wedge^{\bullet,\bullet}\mathbb{V}) \to \Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ on $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$ and $u_1, \ldots, u_{k+p} \in \Gamma(TM)$ by

$$\mathrm{j}^{(k,-1)}_\Theta\, \eta(u_1, \ldots, u_{k+p}) := (-1)^{p+k-1} \sum_{\sigma \in S_{(k,p)}} (-1)^\sigma\, \mathrm{i}_{\Theta(u_{\sigma_1}, \ldots, u_{\sigma_k})}\, \eta(u_{\sigma_{k+1}}, \ldots, u_{\sigma_{k+p}}). \tag{4.6}$$

Observe that each summand is the insertion of the 1-form $\Theta(u_{\sigma_1}, \ldots, u_{\sigma_k}) := \mathrm{i}_{u_{\sigma_k}} \ldots \mathrm{i}_{u_{\sigma_1}} \Theta$ in the q-vector field $\eta(u_{\sigma_{k+1}}, \ldots, u_{\sigma_{k+p}})$. Since Θ vanishes whenever two of its arguments lie in $\Gamma(V)$, we get that $\mathrm{j}^{(k,-1)}_\Theta\, \eta \in \Gamma(\wedge^{p+k,q-1}\mathbb{V})$. Moreover, we observe that $\mathrm{j}^{(k,-1)}_\Theta = 0$ if and only if $\Theta \in \Gamma(\wedge^3 V^\circ)$.

To show the graded derivation property, note that

$$\mathrm{j}^{(k,-1)}_{f\Theta} = f\, \mathrm{j}^{(k,-1)}_\Theta \qquad \forall f \in C^\infty(M).$$

On the other hand, it is clear that Θ is locally the sum of elements of the form $f\, \mathrm{d}\, \alpha$, where $f \in C^\infty(M)$, and $\alpha \in \Gamma(\wedge^k V^\circ)$. This can be observed, for instance, in a coordinate chart adapted to the foliation $\mathcal{V}$. So, to show that $\mathrm{j}^{(k,-1)}_\Theta$ is a graded derivation for every $\Theta \in \Gamma(\wedge^k V^\circ \otimes T^*M)$, it suffices to verify this property in the case $\Theta = \mathrm{d}\, \alpha$, where $\alpha \in \Gamma(\wedge^p V^\circ)$.

Lemma 4.7. *Let $\alpha \in \Gamma(\wedge^k V^\circ)$ be a k-form vanishing on leaf-tangent vector fields, and consider its adjoint operator $\mathrm{ad}_\alpha : \Gamma(\wedge^{\bullet,\bullet}\mathbb{V}) \to \Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ with respect to the Schouten bracket of $\mathbb{V}$. Then,*

$$\mathrm{j}^{(k,-1)}_{\mathrm{d}\,\alpha} = \mathrm{ad}_\alpha .$$

In particular, $\mathrm{j}^{(k,-1)}_{\mathrm{d}\,\alpha}$ is a derivation of bidegree $(k, -1)$.

Proof. Fix $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$, and $\mathcal{V}$-projectable vector fields $u_1, \ldots, u_{p+k} \in \mathrm{aut}(M, \mathcal{V})$. Then,

$$\begin{aligned} \mathrm{ad}_\alpha\, \eta(u_1, \ldots, u_{p+k}) &= (-1)^p \sum_{\sigma \in S_{(p,k)}} (-1)^\sigma [\alpha(u_{\sigma_1}, \ldots, u_{\sigma_k}), \eta(u_{\sigma_{k+1}}, \ldots, u_{\sigma_{k+p}})] \\ &= -(-1)^p \sum_{\sigma \in S_{(p,k)}} (-1)^\sigma\, \mathrm{i}_{\mathrm{d}(\alpha(u_{\sigma_1}, \ldots, u_{\sigma_k}))}\, \eta(u_{\sigma_{k+1}}, \ldots, u_{\sigma_{k+p}}). \end{aligned}$$

Since each u_{σ_i} is $\mathcal{V}$-projectable, $\alpha \in \Gamma(\wedge^k V^\circ)$, and η values on $\Gamma(\wedge^q V)$, the following equality holds for each $\sigma \in S_{(p,k)}$:

$$\mathrm{i}_{\mathrm{d}(\alpha(u_{\sigma_1}, \ldots, u_{\sigma_k}))}\, \eta(u_{\sigma_{k+1}}, \ldots, u_{\sigma_{k+p}}) = (-1)^k\, \mathrm{i}_{\mathrm{d}\,\alpha(u_{\sigma_1}, \ldots, u_{\sigma_k})}\, \eta(u_{\sigma_{k+1}}, \ldots, u_{\sigma_{k+p}}).$$

Hence,

$$\mathrm{ad}_\alpha \eta(u_1,\ldots,u_{p+k}) = (-1)^{p+k-1} \sum_{\sigma\in S_{(p,k)}} (-1)^\sigma \, \mathrm{i}_{\mathrm{d}\,\alpha(u_{\sigma_1},\ldots,u_{\sigma_k})} \eta(u_{\sigma_{k+1}},\ldots,u_{\sigma_{k+p}}) = \mathrm{j}^{(k,-1)}_{\mathrm{d}\,\alpha}\, \eta(u_1,\ldots,u_{p+k}).$$

■

4.2.3 Extension of the Schouten bracket

The present construction, given in the context of foliated manifolds, is based on the one in [50, Subsection 4.2.1] for the case of fiber bundles (see also [16, Proposition 3.1]). Since this work is devoted to the study of coupling structures on foliated manifolds, we find convenient to give this construction in this setting.

Let $(M,\mathcal{V})$ be a regular foliated manifold, and denote $V := T\mathcal{V}$, $V^\circ := \mathrm{Ann}(V)$. Recall that the algebra of $\mathcal{V}$-projectable vector fields $\mathrm{aut}(M,\mathcal{V})$ consists of the vector fields $u\in\Gamma(TM)$ such that $[u,Y]\in\Gamma(V)$ for all $Y\in\Gamma(V)$. As a consequence of this property, we get that $\mathrm{aut}(M,\mathcal{V})$ preserves $\Gamma(\wedge^\bullet\mathbb{V})$, where $\mathbb{V} := V\oplus V^\circ$. Therefore, the Lie bracket $[\cdot,\cdot]_{\mathbb{V}}$ on $\Gamma(\mathbb{V})$, given by the restriction of the Dorfman bracket, can be extended to $\mathrm{aut}(M,\mathcal{V})\oplus\Gamma(V^\circ)$ (just recall that $\Gamma(V)\subset\mathrm{aut}(M,\mathcal{V})$). Indeed, for $u,v\in\mathrm{aut}(M,\mathcal{V})$ and $\alpha,\beta\in\Gamma(V^\circ)$, one can set

$$[u\oplus\alpha, v\oplus\beta]_{\mathbb{V}} := [u,v]\oplus(\mathrm{L}_u\beta - \mathrm{L}_v\alpha),$$

which defines an $\mathbb{R}$-Lie bracket on $\mathrm{aut}(M,\mathcal{V})\oplus\Gamma(V^\circ)$ coinciding with the Dorfman bracket on $\Gamma(\mathbb{V})$.

In what follows, we generalize this extension to the Schouten bracket of $[\cdot,\cdot]_{\mathbb{V}}$ in $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$. To do so, we give an equivalent definition of it by using use some Vinogradov calculus. Our goal is to replace the description (4.2), which was given in terms of graded endomorphisms on $\Gamma(\wedge^\bullet\mathbb{V}^*)$, to a new one presented in terms of graded operators on $\Gamma(\wedge^\bullet T^*M)$.

Insertion operators and the Vinogradov bracket. Recall that, given two graded operators D and E on $\Gamma(\wedge^\bullet T^*M)$ with corresponding degrees d and e, their *graded commutator* is defined by $[D,E] := D\circ E - (-1)^{d\cdot e}E\circ D$. It is well-known that this bracket induces a graded Lie algebra structure of degree zero on the space of graded operators, which means that the graded versions of the skew-symmetry and the Jacobi identity hold. On the other hand, the *Vinogradov bracket* of D and E is given by

$$[\![D,E]\!] := [[D,\mathrm{d}],E],$$

where d denotes the exterior differential on $\Gamma(\wedge^\bullet T^*M)$. This is a bracket of degree 1 which satisfies the graded Jacobi identity (see, for instance, [80, Section 2.2]), but is not graded skew-symmetric. However, if the graded operators in question commute, then their Vinogradov bracket satisfies skew-symmetry.

Lemma 4.8. *If $[D,E]=0$, then $[\![D,E]\!] = -(-1)^{(d+1)(e+1)}[\![E,D]\!]$.*

Proof. By the graded Jacobi identity and skew-symmetry of the graded commutator, we get

$$\begin{aligned} 0 = [[D,E],\mathrm{d}] = [D,[E,\mathrm{d}]] + (-1)^e[[D,\mathrm{d}],E] &= -(-1)^{d(e+1)}[[E,\mathrm{d}],D] + (-1)^e[[D,\mathrm{d}],E] \\ &= -(-1)^{d(e+1)}[\![E,D]\!] + (-1)^e[\![D,E]\!], \end{aligned}$$

which proves the statement. ■

Given a multivector field $A \in \Gamma(\wedge^q TM)$, the operator i_A is the insertion of multivector fields, and is characterized by the relation $\mathrm{i}_A = \mathrm{i}_{a_1} \circ \cdots \circ \mathrm{i}_{a_q}$, if $A = a_1 \wedge \cdots \wedge a_q$. Furthermore, for each multivector-valued form $\eta \in \Gamma(\wedge^p T^*M \otimes \wedge^q TM)$, and each $\omega \in \Gamma(\wedge^k T^*M)$, define $\mathrm{i}_\eta \omega \in \Gamma(\wedge^{k+p-q} T^*M)$ by

$$\mathrm{i}_\eta \omega(X_1, \ldots, X_{k+p-q}) := \sum_{\sigma \in S_{(p,k-q)}} (-1)^\sigma \, \mathrm{i}_{\eta(X_{\sigma_1}, \ldots, X_{\sigma_p})} \, \omega(X_{\sigma_{p+1}}, \ldots, X_{\sigma_{k+p-q}}) \qquad \forall X_i \in \Gamma(TM).$$

This gives a graded operator i_η on $\Gamma(\wedge^\bullet T^*M)$ of degree $p - q$, called the *insertion* of η. The correspondence $\eta \mapsto \mathrm{i}_\eta$ is an injection of $\Gamma(\wedge^\bullet T^*M \otimes \wedge^\bullet TM)$ into the space of graded endomorphisms of $\Gamma(\wedge^\bullet T^*M)$. We also define the *Lie operator* $\mathrm{L}_\eta := [\mathrm{i}_\eta, \mathrm{d}]$, where in the right-hand side we have the bracket of graded endomorphisms on $\Gamma(\wedge^\bullet T^*M)$. In the case when $\eta = \alpha \otimes A$, with $\alpha \in \Gamma(\wedge^\bullet T^*M)$ and $A \in \Gamma(\wedge^\bullet TM)$, one has $\mathrm{i}_\eta = m_\alpha \circ \mathrm{i}_A$, where m_α is the left multiplication by α, and i_A is the insertion of A. Furthermore, since the exterior product of multivector-valued forms satisfies formula (4.3), we have the property $\mathrm{i}_{\eta \wedge \eta} = \mathrm{i}_\eta \circ \mathrm{i}_{\eta'}$.

It is well known that, in this generality, the space of insertion operators is not closed under the Vinogradov bracket. However, in the particular case when the multivector-valued differential forms are taken from $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$, viewed as a subset of $\Gamma(\wedge^\bullet T^*M \otimes \wedge^\bullet TM)$, we recover the bracket (4.2).

Lemma 4.9. *The bracket* $[\eta, \eta']_{\mathbb{V}}$ *of* $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$ *and* $\eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$*, given in* (4.2)*, is characterized in terms of the Vinogradov bracket of their insertions on* $\Gamma(\wedge^\bullet T^*M \otimes \wedge^\bullet TM)$*, namely*

$$\mathrm{i}_{[\eta,\eta']_{\mathbb{V}}} = [\mathrm{L}_\eta, \mathrm{i}_{\eta'}]. \tag{4.7}$$

Proof. By the injectivity of the correspondence $\eta \mapsto \mathrm{i}_\eta$, and the $\mathbb{R}$-bilinearity of both sides, it suffices to check the identity for $\eta = \alpha \otimes A$ and $\eta' = \beta \otimes B$, where $\alpha, \beta \in \Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ)$ are basic, and $A, B \in \Gamma(\wedge^\bullet V)$. By standard computations with the commutator of graded endomorphisms (see, for instance, [80, Lemma 2.2]),

$$\begin{aligned}
[\mathrm{L}_\eta, \mathrm{i}_{\eta'}] &= [[\mathrm{i}_\eta, \mathrm{d}], \mathrm{i}_{\eta'}] = [[m_\alpha \circ \mathrm{i}_A, \mathrm{d}], m_\beta \circ \mathrm{i}_B] = [m_\alpha \circ \mathrm{L}_A + (-1)^{p+q+1} m_{\mathrm{d}\alpha} \circ \mathrm{i}_A, m_\beta \circ \mathrm{i}_B] \\
&= [m_\alpha \circ \mathrm{L}_A, m_\beta \circ \mathrm{i}_B] + (-1)^{p+q+1} [m_{\mathrm{d}\alpha} \circ \mathrm{i}_A, m_\beta \circ \mathrm{i}_B].
\end{aligned}$$

Since $\beta, \mathrm{d}\alpha \in \Gamma(\wedge^\bullet V^\circ)$, and $A, B \in \Gamma(\wedge^\bullet V)$, it follows that $m_{\mathrm{d}\alpha}$, m_β, i_B and i_A commute with each other, so $[m_{\mathrm{d}\alpha} \circ \mathrm{i}_A, m_\beta \circ \mathrm{i}_B] = 0$. Similarly, $[m_\alpha \circ \mathrm{L}_A, m_\beta \circ \mathrm{i}_B] = (-1)^{p'(q-1)} m_\alpha \circ m_\beta \circ [\mathrm{L}_A, \mathrm{i}_B]$. Hence,

$$[\mathrm{L}_\eta, \mathrm{i}_{\eta'}] = (-1)^{p'(q-1)} m_\alpha \circ m_\beta \circ [\mathrm{L}_A, \mathrm{i}_B] = (-1)^{p'(q-1)} m_{\alpha \wedge \beta} \circ \mathrm{i}_{[A,B]} = \mathrm{i}_{[\eta,\eta']_{\mathbb{V}}},$$

where in the last step we have applied (4.4). ■

The bracket $[\cdot,\cdot]_{\mathscr{V}}$**.** Our goal is to extend the bracket $[\cdot,\cdot]_{\mathbb{V}}$ on $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ to a bracket $[\cdot,\cdot]_{\mathscr{V}}$ on a slightly larger bigraded space $\mathscr{V}^{\bullet,\bullet}$, via the characterization (4.7). This new bigraded space is defined as follows: For each $p \in \mathbb{Z}$, denote by $\mathscr{V}^{p,1}$ the space of p-forms vanishing along $\mathcal{V}$ and valued in $\mathcal{V}$-projectable vector fields,

$$\mathscr{V}^{p,1} := \{\eta \in \Gamma(\wedge^p V^\circ \otimes TM) \mid \eta(u_1, \ldots, u_p) \in \mathrm{aut}(M, \mathcal{V}), \ \forall u_1, \ldots, u_p \in \mathrm{aut}(M, \mathcal{V})\}.$$

The local generators of $\mathscr{V}^{p,1}$ are given as $\alpha \otimes u$, for $\alpha \in \Gamma_{\mathrm{b}}(\wedge^p V^\circ)$ and $u \in \mathrm{aut}(M, \mathcal{V})$. In particular, $\mathscr{V}^{0,1} = \mathrm{aut}(M, \mathcal{V})$. For $p, q \in \mathbb{Z}$ with $q \neq 1$, put $\mathscr{V}^{p,q} := \Gamma(\wedge^{p,q}\mathbb{V})$, and define $\mathscr{V}^{\bullet,\bullet} := \bigoplus_{p,q \in \mathbb{Z}} \mathscr{V}^{p,q}$. Since $\Gamma(V) \subseteq \mathrm{aut}(M, \mathcal{V})$, we have $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V}) \subseteq \mathscr{V}^{\bullet,\bullet}$.

To extend $[\cdot,\cdot]_{\mathbb{V}}$ to the hole bigraded space $\mathscr{V}$, we first show that (4.7) works for $\eta \in \mathscr{V}^{p,1}$ and $\eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$.

Lemma 4.10. *Fix $\eta \in \mathscr{V}^{p,1}$ and $\eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$. Then, there exists a unique $[\eta,\eta']_{\mathscr{V}} \in \Gamma(\wedge^{p+p',q'}\mathbb{V})$ such that $\mathrm{i}_{[\eta,\eta']_{\mathscr{V}}} = [\mathrm{L}_\eta, \mathrm{i}_{\eta'}]$.*

Proof. The uniqueness part, and the fact that it suffices to consider local generators, follow from the injectivity of the correspondence $\eta \mapsto \mathrm{i}_\eta$. So, without lost of generality suppose that $\eta = \alpha \otimes u$ and $\eta' = \beta \otimes B$, where $\alpha, \beta \in \Gamma_{\mathrm{b}}(\wedge^\bullet V^\circ)$, $u \in \mathrm{aut}(M, \mathcal{V})$, and $B \in \Gamma(\wedge^q V)$. By repeating the computations of the first part of the proof of Lemma 4.9, we get

$$\begin{aligned}
[\mathrm{L}_\eta, \mathrm{i}_{\eta'}] &= [m_\alpha \circ \mathrm{L}_u, m_\beta \circ \mathrm{i}_B] + (-1)^p [m_{\mathrm{d}\,\alpha} \circ \mathrm{i}_u, m_\beta \circ \mathrm{i}_B] \\
&= m_\alpha \circ m_\beta \circ [\mathrm{L}_u, \mathrm{i}_B] + m_\alpha \circ [\mathrm{L}_u, m_\beta] \circ \mathrm{i}_B + (-1)^p m_{\mathrm{d}\,\alpha} \circ [\mathrm{i}_u, m_\beta] \circ \mathrm{i}_B \\
&= m_{\alpha\wedge\beta} \circ \mathrm{i}_{[u,B]} + m_{\alpha \wedge \mathrm{L}_u \beta} \circ \mathrm{i}_B + (-1)^p m_{\mathrm{d}\,\alpha \wedge \mathrm{i}_u \beta} \circ \mathrm{i}_B\,.
\end{aligned}$$

Hence, $[\mathrm{L}_\eta, \mathrm{i}_{\eta'}]$ is the insertion of $(\alpha \wedge \mathrm{L}_u\,\beta) \otimes B + (\alpha \wedge \beta) \otimes [u, B] + (-1)^p (\mathrm{d}\,\alpha \wedge \mathrm{i}_u\,\beta) \otimes B$, which is an element of $\Gamma(\wedge^{p+p',q}\mathbb{V})$, due to the $\mathcal{V}$-projectability of u. This means that one can set

$$[\eta,\eta']_{\mathscr{V}} := (\alpha\wedge\beta)\otimes[u,B] + (\alpha \wedge \mathrm{L}_u\,\beta)\otimes B + (-1)^p(\mathrm{d}\,\alpha \wedge \mathrm{i}_u\,\beta)\otimes B,$$

and extend to any pair of elements by $\mathbb{R}$-bilinearity. ■

Remark 4.11. *If $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$ and $\eta' \in \mathscr{V}^{p',1}$, then Lemma 4.10 implies that there exists a unique $[\eta,\eta']_{\mathscr{V}} \in \Gamma(\wedge^{p+p',q}\mathbb{V})$ such that $\mathrm{i}_{[\eta,\eta']_{\mathscr{V}}} = [\mathrm{i}_\eta, \mathrm{L}_{\eta'}]$. By defining $[\eta,\eta']_{\mathscr{V}}$ in this manner, we obtain that the skew-symmetry of the graded commutator induces the graded skew-symmetry on $[\cdot,\cdot]_{\mathscr{V}}$.*

As a consequence of Lemmas 4.9 and 4.10, if $\eta \in \mathscr{V}^{p,q}$ and $\eta' \in \Gamma(\wedge^{p',q'}\mathbb{V})$, we can define $[\eta,\eta']_{\mathscr{V}} \in \Gamma(\wedge^{p+p',q+q'-1}\mathbb{V})$ by the relation

$$\mathrm{i}_{[\eta,\eta']_{\mathscr{V}}} := [\mathrm{L}_\eta, \mathrm{i}_{\eta'}] \tag{4.8}$$

Now, observe from the graded Jacobi identity that $\mathrm{L}_{[\eta,\eta']_{\mathscr{V}}} = [\mathrm{i}_{[\eta,\eta']_{\mathscr{V}}}, \mathrm{d}] = [[\mathrm{L}_\eta, \mathrm{i}_{\eta'}], \mathrm{d}] = [\mathrm{L}_\eta, \mathrm{L}_{\eta'}]$. Therefore, if we use

$$\mathrm{L}_{[\eta,\eta']_{\mathscr{V}}} := [\mathrm{L}_\eta, \mathrm{L}_{\eta'}] \tag{4.9}$$

as a definition of $[\eta,\eta']_{\mathscr{V}}$, provided the injectivity of the Lie operator correspondence $\eta \mapsto \mathrm{L}_\eta$, we obtain the same bracket as in Lemmas 4.9 and 4.10. In particular, we can use (4.9) to extend $[\eta,\eta']_{\mathscr{V}}$ to the case when $\eta, \eta' \in \mathscr{V}^{\bullet,1}$, even if non of them lies in $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$.

Lemma 4.12. *If $\eta \in \mathscr{V}^{p,1}$ and $\eta' \in \mathscr{V}^{p',1}$, then there exists a unique $[\eta,\eta']_{\mathscr{V}} \in \mathscr{V}^{p+p',1}$ such that* (4.9) *holds, and coincides with the Frölicher-Nijenhuis bracket: $[\eta,\eta']_{FN} = [\eta,\eta']_{\mathscr{V}}$.*

Proof. Since η and η' are vector-valued forms, it is well known that there exists a unique $[\eta,\eta']_{\mathscr{V}}$ such that (4.9) holds, namely, their Frölicher-Nijenhuis bracket [35, Section 2.8]: $[\eta,\eta']_{\mathscr{V}} = [\eta,\eta']_{FN}$. It is left to show that, in fact, $[\eta,\eta']_{FN} \in \mathscr{V}^{p+p',1}$. Pick $u_1, \ldots, u_{p+p'} \in \mathrm{aut}(M,\mathcal{V})$. Since $\eta \in \mathscr{V}^{p,1}$ and $\eta' \in \mathscr{V}^{p',1}$, it follows that each of

$$\begin{array}{ll}
[\eta(u_1,\ldots,u_p), \eta'(u_{p+1},\ldots,u_{p+p'})], & \eta'([\eta(u_1,\ldots,u_p),u_{p+1}],u_{p+2},\ldots,u_{p+p'}), \\
\eta([\eta'(u_1,\ldots,u_{p'}),u_{p'+1}],u_{p'+2},\ldots,u_{p+p'}), & \eta'(\eta([u_1,u_2],u_3,\ldots,u_p),u_{p+1},\ldots,u_{p+p'}), \\
\eta(\eta'([u_1,u_2],u_3,\ldots,u_{p'}),u_{p'+1},\ldots,u_{p+p'}) &
\end{array}$$

is a projectable vector field. By [35, Theorem 8.9], $[\eta,\eta']_{FN}(u_1,\dots,u_{p+p'})$ equals to a sum of some terms of the form given in above. Hence, $[\eta,\eta']_{FN}(u_1,\dots,u_{p+p'}) \in \mathrm{aut}(M,\mathcal{V})$. If, in addition, any of the arguments is leaf-tangent, say, $u_1 \in \Gamma(V)$, then each of the above terms vanish due to the projectability of each involved vector field and the fact that $\eta \in \Gamma(\wedge^p V^\circ \otimes TM)$ and $\eta' \in \Gamma(\wedge^{p'} V^\circ \otimes TM)$. Therefore, $[\eta,\eta']_{FN} \in \mathscr{V}$, as desired. ∎

The Lie algebra $(\mathscr{V},[\cdot,\cdot]_{\mathscr{V}})$. Up to this point, we have defined a bracket $[\cdot,\cdot]_{\mathscr{V}}$ on $\mathscr{V}$. In order to show that $(\mathscr{V},[\cdot,\cdot]_{\mathscr{V}})$ is indeed a graded Lie algebra, we require the following property:

$$[\mathrm{i}_\xi,\mathrm{i}_{\xi'}]=0, \qquad \forall \xi \in \Gamma(\wedge^{p,q}\mathbb{V}), \xi' \in \Gamma(\wedge^{p',q'}\mathbb{V}). \tag{4.10}$$

To see this, take $\xi=\alpha\otimes A$, and $\beta\otimes B$, with $\alpha,\beta\in\Gamma(\wedge^\bullet V^\circ)$, and $A,B\in\Gamma(\wedge^\bullet V)$. Then $\mathrm{i}_\xi = m_\alpha \circ \mathrm{i}_A$ and $\mathrm{i}_{\xi'} = m_\beta\circ \mathrm{i}_B$. Since $m_\alpha, \mathrm{i}_A, m_\beta, \mathrm{i}_B$ commute with each other, it follows that $[\mathrm{i}_\xi,\mathrm{i}_{\xi'}]=0$.

Theorem 4.13. *On the bigraded $\mathbb{R}$-space $\mathscr{V}^{\bullet,\bullet}$, the following relations give a well-defined Lie bracket $[\cdot,\cdot]_{\mathscr{V}}$ of bidegree $(0,-1)$ such that $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ is an ideal:*

$$\begin{aligned} \mathrm{i}_{[\eta,\eta']_{\mathscr{V}}} &:= [\mathrm{L}_\eta,\mathrm{i}_{\eta'}] && \text{if } \eta' \text{ lies in } \Gamma(\wedge^{\bullet,\bullet}\mathbb{V}),\\ \mathrm{L}_{[\eta,\eta']_{\mathscr{V}}} &:= [\mathrm{L}_\eta,\mathrm{L}_{\eta'}] && \text{if } \eta,\eta' \in \mathscr{V}^{\bullet,1}. \end{aligned}$$

Proof. Recall that (4.8) implies (4.9), so there is no ambiguity on the definition of $[\cdot,\cdot]_{\mathscr{V}}$. Moreover, the graded skew-symmetry holds due to Remark 4.11 and the skew-symmetry of the Frölicher-Nijenhuis bracket. We now verify the Jacobi identity on $\eta\in\mathscr{V}^{p,q}$, $\eta'\in\mathscr{V}^{p',q'}$, and $\eta''\in\mathscr{V}^{p'',q''}$. Since formula (4.8) applies whenever any of the arguments lies in $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$, the Jacobi identity follows from the one of the Vinogradov bracket in this case. This means that the only nontrivial case in which the graded Jacobi identity must be verified is when $\eta\in\mathscr{V}^{p,1}$, $\eta'\in\mathscr{V}^{p',1}$, and $\eta''\in\mathscr{V}^{p'',q''}$. Furthermore, if $q''\geq 1$, then the Lie operator correspondence $\eta\mapsto \mathrm{L}_\eta$ is injective, and we can apply the relation (4.9) and the Jacobi identity follows from the one of the graded commutator. So, it is left to consider the case $\eta''\in\mathscr{V}^{p'',0}$.

$$\begin{aligned} \mathrm{i}_{[[\eta,\eta']_{\mathscr{V}},\eta'']_{\mathscr{V}}} &= [\mathrm{L}_{[\eta,\eta']_{\mathscr{V}}},\mathrm{i}_{\eta''}] = [[\mathrm{L}_\eta,\mathrm{L}_{\eta'}],\mathrm{i}_{\eta''}] = [\mathrm{L}_\eta,[\mathrm{L}_{\eta'},\mathrm{i}_{\eta''}]] + (-1)^{p''p'}[[\mathrm{L}_\eta,\mathrm{i}_{\eta''}],\mathrm{L}_{\eta'}]\\ &= [\mathrm{L}_\eta,\mathrm{i}_{[\eta',\eta'']_{\mathscr{V}}}] + (-1)^{p''p'}[\mathrm{i}_{[\eta,\eta'']_{\mathscr{V}}},\mathrm{L}_{\eta'}] = \mathrm{i}_{[\eta,[\eta',\eta'']_{\mathscr{V}}]_{\mathscr{V}}} + (-1)^{(p''-1)p'}\,\mathrm{i}_{[[\eta,\eta'']_{\mathscr{V}},\eta']_{\mathscr{V}}}, \end{aligned}$$

where in the last step we have taken into account Remark 4.11. This proves that the graded Jacobi identity always holds, so $(\mathscr{V},[\cdot,\cdot]_{\mathscr{V}})$ is indeed a Lie algebra of degree -1. Furthermore, the fact that $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ is an ideal simply follows from Lemma 4.10. ∎

Since $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$ is an ideal of $(\mathscr{V},[\cdot,\cdot]_{\mathscr{V}})$, the adjoint operator of every $\eta\in\mathscr{V}$, $\mathrm{ad}_\eta := [\eta,\ \cdot\ ]_{\mathscr{V}}$, is a derivation of $(\Gamma(\wedge^{\bullet,\bullet}\mathbb{V}),\wedge,[\cdot,\cdot]_{\mathbb{V}})$. Indeed, the derivation property of the bracket simply follows from the graded Jacobi identity of $[\cdot,\cdot]_{\mathscr{V}}$. In the case of the exterior product, for $\eta\in\mathscr{V}^{p,q}$, $\eta'\in\Gamma(\wedge^{p',q'}\mathbb{V})$, and $\eta''\in\Gamma(\wedge^{p'',q''}\mathbb{V})$ we get

$$\begin{aligned} \mathrm{i}_{\mathrm{ad}_\eta(\eta'\wedge\eta'')} &= [\mathrm{L}_\eta,\mathrm{i}_{\eta'\wedge\eta''}] = [\mathrm{L}_\eta,\mathrm{i}_{\eta'}\circ\mathrm{i}_{\eta''}] = [\mathrm{L}_\eta,\mathrm{i}_{\eta'}]\circ\mathrm{i}_{\eta''} + (-1)^{(p-q+1)(p'-q')}\,\mathrm{i}_{\eta'}\circ[\mathrm{L}_\eta,\mathrm{i}_{\eta''}]\\ &= \mathrm{i}_{[\eta,\eta']_{\mathscr{V}}}\circ\mathrm{i}_{\eta''} + (-1)^{(p+q-1)(p'+q')}\,\mathrm{i}_{\eta'}\circ[\mathrm{L}_\eta,\mathrm{i}_{\eta''}]\\ &= \mathrm{i}_{\mathrm{ad}_\eta(\eta')\wedge\eta''} + (-1)^{|\mathrm{ad}_\eta||\eta'|}\,\mathrm{i}_{\eta'\wedge\mathrm{ad}_\eta(\eta'')}, \end{aligned}$$

which implies $\mathrm{ad}_\eta(\eta'\wedge\eta'') = \mathrm{ad}_\eta(\eta')\wedge\eta'' + (-1)^{|\mathrm{ad}_\eta||\eta'|}\eta'\wedge\mathrm{ad}_\eta(\eta'')$.

Proposition 4.14. *The center of* $(\mathscr{V}^{\bullet,\bullet},[\cdot,\cdot]_{\mathscr{V}})$ *lies in the space of closed forms in* $\Gamma(\wedge^\bullet V^\circ)$. *In the case when the leaves of* $\mathcal{V}$ *are the fibers of a submersion* $M\xrightarrow{\pi}B$, *the only central elements of* $(\mathscr{V}^{\bullet,\bullet},[\cdot,\cdot]_{\mathscr{V}})$ *are the locally constant functions.*

Proof. Let $\eta\in\mathscr{V}^{p,q}$ be a central element. Then, for $f\in C^\infty(M)=\mathscr{V}^{0,0}$, and $u_1,\dots,u_p\in\operatorname{aut}(M,\mathcal{V})$,

$$0=[\eta,f](u_1,\dots,u_p)=[\eta(u_1,\dots,u_p),f]=-(-1)^q\,\mathrm{i}_{\mathrm{d}f}\,\eta(u_1,\dots,u_p).$$

If $q\geq 1$, then the above relation implies that $\eta(u_1,\dots,u_p)=0$, so $\eta=0$. If $q=0$, then $\eta\in\Gamma(\wedge^pV^\circ)$ is a differential form. For each $Y\in\Gamma(V)\subset\mathscr{V}^{0,1}$, one has

$$0=[Y,\eta]=\mathrm{L}_Y\,\eta=\mathrm{i}_Y\,\mathrm{d}\,\eta,$$

which means that η must be a basic form, $\eta\in\Gamma_{\mathrm{b}}(\wedge^pV^\circ)$. This property implies that η commutes with every element in $\Gamma(\wedge^{\bullet,\bullet}\mathbb{V})$. On the other hand, if γ is a connection on $(M,\mathcal{V})$, then $\mathrm{Id}-\gamma\in\mathscr{V}^{1,1}$, and

$$0=[\mathrm{Id}-\gamma,\eta]=\partial^\gamma\eta=\mathrm{d}^\gamma_{1,0}\,\eta,$$

where we are using the decomposition of the de Rham differential $\mathrm{d}=\mathrm{d}^\gamma_{0,1}+\mathrm{d}^\gamma_{1,0}+\mathrm{d}^\gamma_{2,-1}$. Since $\eta\in\Gamma(\wedge^pV^\circ)$, one has $\mathrm{d}^\gamma_{2,-1}\,\eta=0$ and, since η is basic, $\mathrm{d}^\gamma_{0,1}\,\eta=0$. Therefore, $\mathrm{d}\,\eta=0$, proving that η is a closed form. Furthermore, in the case when the foliation $\mathcal{V}$ is given by the fibers of a submersion, then M is locally generated by global projectable vector fields. Thus, the fact that $\mathrm{L}_u\,\eta=[u,\eta]=0$ for all $u\in\operatorname{aut}(M,\mathcal{V})=\mathscr{V}^{0,1}$ implies that η must be a function $\eta\in C^\infty(M)$ which is locally constant by the closedness property. ∎

We now present an explicit formula for the computation of $[\eta,\eta']_{\mathscr{V}}$ in the particular case described in Lemma 4.10.

Lemma 4.15. *If* $\eta\in\mathscr{V}^{p,1}$ *and* $\eta'\in\Gamma(\wedge^{p',q'}\mathbb{V})$, *then for* $u_1,\dots,u_{p+p'}\in\operatorname{aut}(M,\mathcal{V})$, *we have*

$$\begin{aligned}[\eta,\eta']_{\mathscr{V}}(u_1,\dots,u_{p+p'}):=&\sum_{\sigma\in S_{(p,p')}}(-1)^\sigma[\eta(u_{\sigma_1},\dots,u_{\sigma_p}),\eta'(u_{\sigma_{p+1}},\dots,u_{\sigma_{p+p'}})]\\&-\sum_{\sigma\in S_{(p,1,p'-1)}}(-1)^\sigma\eta'([\eta(u_{\sigma_1},\dots,u_{\sigma_p}),u_{\sigma_{p+1}}],u_{\sigma_{p+2}},\dots,u_{\sigma_{p+p'}})\\&-(-1)^p\sum_{\sigma\in S_{(2,p-1,p'-1)}}(-1)^\sigma\eta'(\eta([u_{\sigma_1},u_{\sigma_2}],u_{\sigma_3},\dots,u_{\sigma_{p+1}}),u_{\sigma_{p+2}},\dots,u_{\sigma_{p+p'}}).\end{aligned}\tag{4.11}$$

Proof. Following the proof of Lemma 4.10, for $\eta=\alpha\otimes u$, and $\eta'=\beta\otimes B$, we have

$$[\eta,\eta']_{\mathscr{V}}=(\alpha\wedge\beta)\otimes[u,B]+(\alpha\wedge\mathrm{L}_u\,\beta)\otimes B+(-1)^p(\mathrm{d}\,\alpha\wedge\mathrm{i}_u\,\beta)\otimes B.$$

Let us show that this coincides with (4.11). By straightforward computations,

$$\begin{aligned}[\eta(u_{\sigma_1},\dots,u_{\sigma_p}),\eta'(u_{\sigma_{p+1}},\dots,u_{\sigma_{p+p'}})]=&\alpha(u_{\sigma_1},\dots,u_{\sigma_p})\beta(u_{\sigma_{p+1}},\dots,u_{\sigma_{p+p'}})[u,B]\\&+\alpha(u_{\sigma_1},\dots,u_{\sigma_p})\,\mathrm{L}_u(\beta(u_{\sigma_{p+1}},\dots,u_{\sigma_{p+p'}}))B\\&-\beta(u_{\sigma_{p+1}},\dots,u_{\sigma_{p+p'}})u\wedge\mathrm{i}_{\mathrm{d}(\alpha(u_{\sigma_1},\dots,u_{\sigma_p}))}\,B,\end{aligned}$$

where the last term vanishes due to the fact that $\alpha \in \Gamma_{\mathrm{b}}(\wedge^p V^\circ)$ and $B \in \Gamma(\wedge^q V)$. Also,

$$\begin{aligned}\eta'([\eta(u_{\sigma_1},\ldots,u_{\sigma_p}),u_{\sigma_{p+1}}],u_{\sigma_{p+2}},\ldots,u_{\sigma_{p+p'}}) = {}& \alpha(u_{\sigma_1},\ldots,u_{\sigma_p})\beta([u,u_{\sigma_{p+1}}],u_{\sigma_{p+2}},\ldots,u_{\sigma_{p+p'}})B \\ & - \mathrm{L}_{u_{\sigma_{p+1}}}(\alpha(u_{\sigma_1},\ldots,u_{\sigma_p}))\beta(u,u_{\sigma_{p+2}},\ldots,u_{\sigma_{p+p'}})B, \\ \eta'(\eta([u_{\sigma_1},u_{\sigma_2}],u_{\sigma_3},\ldots,u_{\sigma_{p+1}}),u_{\sigma_{p+2}},\ldots,u_{\sigma_{p+p'}}) = {}& \alpha([u_{\sigma_1},u_{\sigma_2}],u_{\sigma_3},\ldots,u_{\sigma_{p+1}})\beta(u,u_{\sigma_{p+2}},\ldots,u_{\sigma_{p+p'}})B.\end{aligned}$$

It is straightforward to check that the evaluation of $(\alpha\wedge\beta)\otimes[u,B]+(\alpha\wedge\mathrm{L}_u\,\beta)\otimes B+(-1)^p(\mathrm{d}\,\alpha\wedge\mathrm{i}_u\,\beta)\otimes B$ on $(u_1,\ldots,u_{p+p'})$ gives the corresponding signed sums of the above terms. ■

Covariant exterior derivatives. An special class of elements in $\mathscr{V}^{1,1}$ are given as follows. Consider a connection γ on $(M,\mathcal{V})$, that is, a vector valued 1-form $\gamma \in \Gamma(T^*M\otimes TM)$ which, viewed as a vector bundle map $\gamma : TM \to TM$, has the following properties: $\mathrm{im}(\gamma) = V$, and $\gamma^2 = \gamma$. We claim that $(\mathrm{Id}-\gamma) \in \mathscr{V}^{1,1}$. Indeed, since $\gamma(Y) = Y$ for all $Y \in \Gamma(V)$, it follows that $(\mathrm{Id}-\gamma)(Y) = 0$. Furthermore, since $\Gamma(V) \subset \mathrm{aut}(M,\mathcal{V})$, we have for all $u \in \mathrm{aut}(M,\mathcal{V})$ that $(\mathrm{Id}-\gamma)(u) = u-\gamma(u) \in \mathrm{aut}(M,\mathcal{V})$, proving the claim.

For a connection γ on $(M,\mathcal{V})$, and denote the adjoint operator of $(\mathrm{Id}-\gamma)$ by $\partial^\gamma := \mathrm{ad}_{\mathrm{Id}-\gamma}$. For $\eta \in \Gamma(\wedge^{p,q}\mathbb{V})$, and $u_1,\ldots,u_{p+1} \in \mathrm{aut}(M,\mathcal{V})$, we get from (4.11) that

$$\begin{aligned}\partial^\gamma\eta(u_1,\ldots,u_{p+1}) = {}& \sum_{\sigma\in S_{(1,p)}}(-1)^\sigma[(\mathrm{Id}-\gamma)(u_{\sigma_1}),\eta(u_{\sigma_2},\ldots,u_{\sigma_{p+1}})] \\ & - \sum_{\sigma\in S_{(1,1,p-1)}}(-1)^\sigma\eta([(\mathrm{Id}-\gamma)(u_{\sigma_1}),u_{\sigma_2}],u_{\sigma_3},\ldots,u_{\sigma_{p+1}}) \\ & + \sum_{\sigma\in S_{(2,p-1)}}(-1)^\sigma\eta((\mathrm{Id}-\gamma)([u_{\sigma_1},u_{\sigma_2}]),u_{\sigma_3},\ldots,u_{\sigma_{p+1}}) \\ = {}& \sum_{\sigma\in S_{(1,p)}}(-1)^\sigma[(\mathrm{Id}-\gamma)(u_{\sigma_1}),\eta(u_{\sigma_2},\ldots,u_{\sigma_{p+1}})] - \sum_{\sigma\in S_{(2,p-1)}}(-1)^\sigma\eta([u_{\sigma_1},u_{\sigma_2}],u_{\sigma_3},\ldots,u_{\sigma_{p+1}}).\end{aligned} \tag{4.12}$$

This formula is the so-called *covariant exterior derivative* of the connection γ [77, Subsection 2.2], [15, Section 4], [50, Subsection 4.2.2], [71, Section 2]. On the other hand, recall that the curvature of γ is the vector-valued 2-form $R^\gamma \in \Gamma(\wedge^{2,1}\mathbb{V})$, given on $X,Y \in \Gamma(TM)$ by $R^\gamma(X,Y) = \gamma[(\mathrm{Id}-\gamma)X,(\mathrm{Id}-\gamma)Y]$. Since $R^\gamma = \frac{1}{2}[\mathrm{Id}-\gamma,\mathrm{Id}-\gamma]_{FN}$, we get that the square of the covariant exterior derivative is just the adjoint operator of the curvature,

$$(\partial^\gamma)^2 = \tfrac{1}{2}[\partial^\gamma,\partial^\gamma] = \tfrac{1}{2}[\mathrm{ad}_{\mathrm{Id}-\gamma},\mathrm{ad}_{\mathrm{Id}-\gamma}] = \tfrac{1}{2}\,\mathrm{ad}_{[\mathrm{Id}-\gamma,\mathrm{Id}-\gamma]_{FN}} = \mathrm{ad}_{R^\gamma}.$$

Explicitly,

$$(\partial^\gamma)^2\eta(u_1,\ldots,u_{p+2}) = \sum_{\sigma\in S_{(2,p)}}(-1)^\sigma[R^\gamma(u_{\sigma_1},u_{\sigma_2}),\eta(u_{\sigma_3},\ldots,u_{\sigma_{p+2}})].$$

Eqiuvalence equations revisted. Let us consider a vertical bivector field $P \in \Gamma(\wedge^2 V)$. Associated with P, there exists an operation between horizontal forms $Q \in \Gamma(V^\circ)$, $\theta \in \Gamma(\wedge^p V^\circ)$, given by [78, Eq. (2.57)]

$$\{Q\wedge\theta\}_P(u_0,\ldots,u_p) := \sum_{i=0}^{p}(-1)^i\{Q(u_i),\theta(u_0,\ldots,\widehat{u}_i,\ldots,u_p)\}_P, \qquad u_0,\ldots,u_p \in \mathrm{aut}(M,\mathcal{V}).$$

where the bracket on the right-hand sums is $\{f,g\}_P := P(\mathrm{d}\, f, \mathrm{d}\, g)$. It is straightforward to verify that

$$\{Q \wedge \theta\}_P = [[P,Q],\theta].$$

This bracket appears in the equations arising in Theorem 2.30, which can be rewritten as

$$\gamma - g^*\widetilde{\gamma} = -[P,Q], \qquad \sigma - g^*\widetilde{\sigma} = [(\mathrm{Id} - \gamma) + \tfrac{1}{2}[P,Q], Q].$$

These equations arise in the context of equivalence of Poisson structures over symplectic leaves [78].

4.3 Coupling twisted Dirac structures and its cohomology

4.3.1 Geometric data and coupling structures with background

Following Section 2.3, we introduce, for the case of twisted structures, some notions associated with Poisson foliations with background. As we will see, these notions are closely related with the description of coupling Dirac structures with background via geometric data.

Definition 4.16. *Let $\psi \in \Gamma(\wedge^3 T^*M)$ be a closed 3-form. A ψ-Poisson foliation is a triple $(M,\mathcal{V},P)$ consisting of a regular foliation $\mathcal{V}$ and a leaf-tangent bivector field $P \in \Gamma(\wedge^2 V)$ which is ψ-Poisson,*

$$\tfrac{1}{2}[P,P] = -P^\sharp \psi.$$

Additionally, a connection $\gamma \in \Gamma(T^*M \otimes V)$ is said to be ψ-Poisson on $(M,\mathcal{V},P)$ if

$$[X,P] = P^\sharp\, \mathrm{i}_X\, \psi, \qquad \forall X \in \Gamma(H^\gamma) \cap \mathrm{aut}(M,\mathcal{V}).$$

We remark from Section 1.4 that the condition on a connection γ to be ψ-Poisson on $(M,\mathcal{V},P)$ is just that the γ-horizontal $\mathcal{V}$-projectable vector fields are cocycles of the complex associated with the ψ-Poisson structure P, $(\Gamma(\wedge^\bullet TM), \eth_{P,\psi})$,

$$\Gamma(H^\gamma) \cap \mathrm{aut}(M,\mathcal{V}) \subseteq Z^1_{P,\psi}(M).$$

Equivalently, the ψ-Poisson connection condition is equivalent

$$[X,P](\mu,\nu) = P^\sharp\, \mathrm{i}_X\, \psi(\mu,\nu), \qquad \forall X \in \Gamma(H^\gamma), \text{ and } \mu,\nu \in \Gamma(A^\gamma).$$

Finally, we recall that the curvature $R^\gamma \in \Gamma(\wedge^2 TM \otimes V)$ of γ is defined by

$$R^\gamma(X,Y) = \gamma[(\mathrm{Id} - \gamma)X, (\mathrm{Id} - \gamma)Y], \qquad \forall X,Y \in TM.$$

Now, we describe the geometric data associated with $\mathcal{V}$-coupling Dirac structures with background.

Definition 4.17. *Let $\psi \in \Gamma(\wedge^3 T^*M)$ be a closed 3-form. A triple of geometric data (P,γ,σ) on (M,V) are said to be ψ-Dirac elements on $(M,\mathcal{V})$ if the following relations hold for all $X,Y,Z \in \Gamma(H^\gamma)$ and $\mu,\nu \in \Gamma(A^\gamma)$:*

$$\tfrac{1}{2}[P,P] = -P^\sharp \psi, \tag{4.13}$$

$$[X,P](\mu,\nu) = P^\sharp\, \mathrm{i}_X\, \psi(\mu,\nu), \tag{4.14}$$

$$R^\gamma(X,Y) = -P^\sharp\, \mathrm{i}_Y\, \mathrm{i}_X(\mathrm{d}\,\sigma + \psi), \tag{4.15}$$

$$(\mathrm{d}\,\sigma + \psi)(X,Y,Z) = 0. \tag{4.16}$$

Condition (4.13) says that the triple $(M, \mathcal{V}, P)$ is a ψ-Poisson foliation, and the property (4.14) is equivalent to the fact that the connection γ is ψ-Poisson on $(M, \mathcal{V}, P)$. Also, condition (4.15) means that the curvature of γ is locally Hamiltonian, via the closed form $\mathrm{d}\,\sigma + \psi$.

The goal of this Section is to show that Dirac elements are precisely the geometric data associated with a Dirac structure with background. To do so, fix a maximally isotropic subbundle $D \subset \mathbb{T}M$. Let (P, γ, σ) be its associated geometric data, in the sense of Propositions 2.23 and 2.4. Then, $D = D^P_{A^\gamma} \oplus D^\sigma_{H^\gamma}$, where

$$\begin{aligned} D^P_{A^\gamma} &:= \mathrm{Graph}(P^\sharp|_{A^\gamma}) = \{P^\sharp\mu \oplus \mu \mid \mu \in A^\gamma\}, \\ D^\sigma_{H^\gamma} &:= \mathrm{Graph}(-\sigma^\flat|_{H^\gamma}) = \{X \oplus (-\,\mathrm{i}_X\,\sigma) \mid X \in H^\gamma\}. \end{aligned}$$

Now, fix a closed 3-form $\psi \in \Gamma(\wedge^3 T^*M)$. We now proceed to find necessary and sufficient conditions for the closedness of $\Gamma(D)$ under the ψ-Dorfman bracket (1.9) in terms of the splitting $D = D^P_{A^\gamma} \oplus D^\sigma_{H^\gamma}$.

Lemma 4.18. *Property* $[\![\Gamma(D^P_{A^\gamma}), \Gamma(D^P_{A^\gamma})]\!]_\psi \subseteq \Gamma(D)$ *is equivalent to* (4.13) *and* (4.14).

Proof. Given $\mu, \nu \in \Gamma(A^\gamma)$, we have

$$[\![P^\sharp\mu \oplus \mu, P^\sharp\nu \oplus \nu]\!]_\psi = [P^\sharp\mu, P^\sharp\nu] \oplus \{\mu, \nu\}_{P,\psi},$$

where $\{\cdot,\cdot\}_{P,\psi}$ is defined as in (1.11). Since, $P \in \Gamma(\wedge^2 V)$, and V is involutive by hypothesis, it follows that $[P^\sharp\mu, P^\sharp\nu] \in \Gamma(V)$. Therefore, $[\![P^\sharp\mu \oplus \mu, P^\sharp\nu \oplus \nu]\!]_\psi \in \Gamma(D)$ if and only if $[P^\sharp\mu, P^\sharp\nu] \oplus \{\mu, \nu\}_{P,\psi} \in \Gamma(D^P_{A^\gamma})$, which, by definition of $D^P_{A^\gamma}$, is equivalent to

$$[P^\sharp\mu, P^\sharp\nu] = P^\sharp\{\mu, \nu\}_{P,\psi} \qquad \text{and} \qquad \{\mu, \nu\}_{P,\psi} \in \Gamma(A^\gamma).$$

The first equality holds for all $\mu, \nu \in \Gamma(A^\gamma)$ if and only if P is a ψ-Poisson structure (in fact, since $P \in \Gamma(\wedge^2 V)$, both sides vanish whenever $\mu \in \Gamma(V^\circ)$ or $\nu \in \Gamma(V^\circ)$). The second identity is just $\{\mu, \nu\}_{P,\psi}(X) = 0$ for all $X \in H^\gamma$, which, by Lemma B.2, is equivalent to (4.14). ■

Lemma 4.19. *Property* $[\![\Gamma(D^\sigma_{H^\gamma}), \Gamma(D^\sigma_{H^\gamma})]\!]_\psi \subseteq \Gamma(D)$ *is equivalent to* (4.15) *and* (4.16).

Proof. Given $X, Y \in \Gamma(H^\gamma)$, we have $X \oplus (-\,\mathrm{i}_X\,\sigma), Y \oplus (-\,\mathrm{i}_Y\,\sigma) \in \Gamma(D^\sigma_{H^\gamma})$. Then,

$$\begin{aligned} [\![X \oplus (-\,\mathrm{i}_X\,\sigma), Y \oplus (-\,\mathrm{i}_Y\,\sigma)]\!]_\psi &= [X, Y] \oplus (\mathrm{L}_X(-\,\mathrm{i}_Y\,\sigma) - \mathrm{i}_Y\,\mathrm{d}(-\,\mathrm{i}_X\,\sigma) - \mathrm{i}_Y\,\mathrm{i}_X\,\psi) \\ &= [X, Y] \oplus (-\,\mathrm{i}_{[X,Y]}\,\sigma - \mathrm{i}_Y\,\mathrm{i}_X(\mathrm{d}\,\sigma + \psi)) \\ &= (\mathrm{Id}_{TM} - \gamma)[X, Y] \oplus (-\,\mathrm{i}_{(\mathrm{Id}_{TM} - \gamma)[X,Y]}\,\sigma) \\ &\qquad + R^\gamma(X, Y) \oplus (-\,\mathrm{i}_Y\,\mathrm{i}_X(\mathrm{d}\,\sigma + \psi)). \end{aligned}$$

Observe that $(\mathrm{Id}_{TM} - \gamma)[X, Y] \oplus (-\,\mathrm{i}_{(\mathrm{Id}_{TM} - \gamma)[X,Y]}\,\sigma) \in \Gamma(D^\sigma_{H^\gamma}) \subset \Gamma(D)$. Therefore,

$$[\![X \oplus (-\,\mathrm{i}_X\,\sigma), Y \oplus (-\,\mathrm{i}_Y\,\sigma)]\!]_\psi \in \Gamma(D)$$

if and only if

$$R^\gamma(X, Y) \oplus (-\,\mathrm{i}_Y\,\mathrm{i}_X(\mathrm{d}\,\sigma + \psi)) \in \Gamma(D^P_{A^\gamma}).$$

This is equivalent to

$$R^\gamma(X, Y) = -P^\sharp\,\mathrm{i}_Y\,\mathrm{i}_X(\mathrm{d}\,\sigma + \psi), \qquad \text{and} \qquad \mathrm{i}_Y\,\mathrm{i}_X(\mathrm{d}\,\sigma + \psi)(Z) = 0 \qquad \forall Z \in \Gamma(H^\gamma).$$

■

Lemma 4.20. *Property* $[\![\Gamma(D^{\sigma}_{H^\gamma}),\Gamma(D^{P}_{A^\gamma})]\!]_\psi \subseteq \Gamma(D)$ *is equivalent to* (4.14) *and* (4.15).

Proof. Take $X \in \Gamma(H^\gamma)$ and $\mu \in \Gamma(A^\gamma)$. By straightforward computations and Lemma B.4, we obtain

$$\begin{aligned}[\![X \oplus (-\mathrm{i}_X \sigma), P^\sharp\mu \oplus \mu]\!]_\psi =&(\mathrm{i}_\mu[X,P] - \mathrm{i}_\mu P^\sharp \mathrm{i}_X \psi) \oplus (\mathrm{i}_{P^\sharp\mu} \mathrm{i}_X \mathrm{d}\,\sigma + \mathrm{i}_{P^\sharp\mu} \mathrm{d}\,\mathrm{i}_X \sigma) + \\ &+ \mathrm{i}_\mu P^\sharp \mathrm{i}_X \psi \oplus (-\nu) + P^\sharp \mathrm{L}_X \mu \oplus (\mathrm{L}_X \mu + \mathrm{i}_{\mathrm{i}_X R^\gamma} \mu) + \\ &+ 0 \oplus (-\mathrm{i}_{P^\sharp\mu} \mathrm{i}_X \mathrm{d}\,\sigma - \alpha - \mathrm{i}_{\mathrm{i}_X R^\gamma} \mu).\end{aligned}$$

Here, $\alpha \in \Gamma(V^\circ)$ and $\nu \in \Gamma(A^\gamma)$ are such that $\alpha + \nu = \mathrm{i}_{P^\sharp\mu} \mathrm{i}_X \psi$. Observe that

$$\begin{aligned}\mathrm{i}_\mu P^\sharp \mathrm{i}_X \psi \oplus (-\nu) &\in \Gamma(D^P_{A^\gamma}),\\ P^\sharp \mathrm{L}_X \mu \oplus (\mathrm{L}_X \mu + \mathrm{i}_{\mathrm{i}_X R^\gamma} \mu) &\in \Gamma(D^P_{A^\gamma}),\\ \mathrm{i}_{P^\sharp\mu} \mathrm{i}_X \mathrm{d}\,\sigma + \mathrm{i}_{P^\sharp\mu} \mathrm{d}\,\mathrm{i}_X \sigma &= -\sigma^\flat(\mathrm{i}_\mu[X,P] - \mathrm{i}_\mu P^\sharp \mathrm{i}_X \psi).\end{aligned}$$

Indeed, just observe that $P^\sharp(-\nu) = -P^\sharp \mathrm{i}_{P^\sharp\mu} \mathrm{i}_X \psi = \mathrm{i}_\mu P^\sharp \mathrm{i}_X \psi$, due to the definition of the map $P^\sharp : \wedge^2 T^*M \to \wedge^2 TM$. The second claim follows from the properties of the curvature R^γ. The proof of the third claim is,

$$-\sigma^\flat(\mathrm{i}_\mu[X,P] - \mathrm{i}_\mu P^\sharp \mathrm{i}_X \psi) = -\sigma^\flat \mathrm{i}_\mu[X,P] = \sigma^\flat(P^\sharp \mathrm{L}_X \mu - [X, P^\sharp\mu]) = -\sigma^\flat[X, P^\sharp\mu] = \mathrm{i}_{P^\sharp\mu} \mathrm{L}_X \sigma,$$

where in the second equality we have applied Lemma B.4. Therefore, $[\![\Gamma(D^{\sigma}_{H^\gamma}),\Gamma(D^{P}_{A^\gamma})]\!]_\psi \subseteq \Gamma(D)$ if and only if, for all $X \in \Gamma(H^\gamma)$ and $\mu \in \Gamma(A^\gamma)$, we have

$$\begin{aligned}(\mathrm{i}_\mu[X,P] - \mathrm{i}_\mu P^\sharp \mathrm{i}_X \psi) \oplus (\mathrm{i}_{P^\sharp\mu} \mathrm{i}_X \mathrm{d}\,\sigma + \mathrm{i}_{P^\sharp\mu} \mathrm{d}\,\mathrm{i}_X \sigma) &\in \Gamma(D^\sigma_{H^\gamma}),\\ -\mathrm{i}_{P^\sharp\mu} \mathrm{i}_X \mathrm{d}\,\sigma - \alpha - \mathrm{i}_{\mathrm{i}_X R^\gamma} \mu &= 0.\end{aligned}$$

The first assertion is equivalent to (4.14). The second one is equivalent to (4.15). ■

These lemmas can be summarized in the following result:

Theorem 4.21. *Let* $\psi \in \Gamma(\wedge^3 T^*M)$ *be a closed 3-form, and let* $D \subset \mathbb{T}M$ *be a* V*-coupling Lagrangian subbundle with associated geometric data* (P,γ,σ)*. Then,* D *is a* ψ*-Dirac structure on* M *if and only if* (P,γ,σ) *are* ψ*-Dirac elements.*

We remark that the correspondence between V-coupling ψ-Dirac structures and ψ-Dirac elements is one to one. In the case $\psi = 0$, we recover Theorem 2.24.

4.3.2 The bigraded cochain complex

Let $(M,\mathcal{V})$ be a foliated manifold, and fix a closed 3-form $\psi \in \Gamma(\wedge^3 T^*M)$. Let us denote by $V := T\mathcal{V}$ the tangent bundle of $\mathcal{V}$ and by $V^\circ := \mathrm{Ann}(V)$ its annihilator.

Let (P,γ,σ) be ψ-Dirac elements, and let D be its correspondent ψ-Dirac structure,

$$D = \mathrm{Graph}(-\sigma^\flat|_{H^\gamma}) \oplus \mathrm{Graph}(P^\sharp|_{A^\gamma})$$

Let us denote $\mathbb{H} := H^\gamma \oplus A^\gamma$. Observe that the map $\phi : \mathbb{H} \to D$, given by

$$\phi(X \oplus \mu) := (X + P^\sharp\mu) \oplus (-\mathrm{i}_X \sigma + \mu)$$

is an isomorphism. On the other hand, recall that D has a Lie algebroid structure, where the anchor $p_D : D \to TM$ is the restriction of the canonical projection $\mathbb{T}M \to TM$, and the bracket $[\cdot,\cdot]_D$ is the restriction of the ψ-Dorfman bracket

$$[\![X \oplus \alpha, Y \oplus \beta]\!]_\psi := [X,Y] \oplus (\mathrm{L}_X \beta - \mathrm{i}_Y \mathrm{d}\,\alpha - \mathrm{i}_Y \mathrm{i}_X \psi), \qquad \forall\, X \oplus \alpha, Y \oplus \beta \in \Gamma(\mathbb{T}M).$$

Since $\phi : \mathbb{H} \to D$ is a vector bundle isomorphism, we can pull back the Lie algebroid structure of D to $\mathbb{H}$.

Lemma 4.22. *Consider the vector bundle map $q : \mathbb{H} \to TM$ given by $q(X \oplus \mu) := X + P^\sharp \mu$, and the bracket $[\cdot,\cdot]_{\mathbb{H}} : \Gamma(\mathbb{H}) \times \Gamma(\mathbb{H}) \to \Gamma(\mathbb{H})$ given on $X, X' \in \Gamma(H^\gamma)$ and $\mu,\nu \in \Gamma(A^\gamma)$ by the relations*

$$\begin{aligned}
[X,X']_{\mathbb{H}} &:= (\mathrm{Id}_{TM} - \gamma)[X,X'] \oplus (-\,\mathrm{i}_{X'} \mathrm{i}_X(\mathrm{d}\,\sigma + \psi)),\\
[X,\nu]_{\mathbb{H}} &:= (\mathrm{Id}_{TM} - \gamma)[X,P^\sharp \nu] \oplus (\mathrm{L}_X \nu - \mathrm{i}_{P^\sharp \nu} \mathrm{i}_X(\mathrm{d}\,\sigma + \psi)),\\
[\mu,\nu]_{\mathbb{H}} &:= 0 \oplus \{\mu,\nu\}_{P,\psi}.
\end{aligned}$$

The isomorphism $\phi : \mathbb{H} \to D$ satisfies $q = \phi \circ p_D$, and $\phi[\xi,\eta]_{\mathbb{H}} = [\phi(\xi),\phi(\eta)]_D$ for all $\xi,\eta \in \mathbb{H}$. In particular, $(\mathbb{H}, q, [\cdot,\cdot]_{\mathbb{H}})$ is a Lie algebroid isomorphic to D.

Proof. The fact that the bracket $[\cdot,\cdot]_{\mathbb{H}}$ is well defined follows from Lemmas 4.18-4.20. Now, note that

$$(p_D \circ \phi)(X \oplus \mu) = p_D(X + P^\sharp \mu) \oplus (-\,\mathrm{i}_X \sigma + \mu) = X + P^\sharp \mu = q(X \oplus \mu).$$

On the other hand, by applying (4.15) and the computations of Lemma 4.19, we get

$$\begin{aligned}
\phi[X,X']_{\mathbb{H}} &= \phi((\mathrm{Id}_{TM} - \gamma)[X,X'] \oplus (-\,\mathrm{i}_{X'} \mathrm{i}_X(\mathrm{d}\,\sigma + \psi)))\\
&= ((\mathrm{Id}_{TM} - \gamma)[X,X'] + P^\sharp(-\,\mathrm{i}_{X'} \mathrm{i}_X(\mathrm{d}\,\sigma + \psi))) \oplus (-\,\mathrm{i}_{(\mathrm{Id}_{TM}-\gamma)[X,X']}\,\sigma - \mathrm{i}_{X'} \mathrm{i}_X(\mathrm{d}\,\sigma + \psi))\\
&= ((\mathrm{Id}_{TM} - \gamma)[X,X'] + R^\gamma(X,X')) \oplus (-\,\mathrm{i}_{(\mathrm{Id}_{TM}-\gamma)[X,X']}\,\sigma - \mathrm{i}_{X'} \mathrm{i}_X(\mathrm{d}\,\sigma + \psi))\\
&= [\![X \oplus (-\,\mathrm{i}_X \sigma), X' \oplus (-\,\mathrm{i}_{X'} \sigma)]\!]_\psi\\
&= [\phi(X),\phi(X')]_D.
\end{aligned}$$

Similarly, since P is ψ-Poisson,

$$\begin{aligned}
\phi[\mu,\nu]_{\mathbb{H}} &= \phi(0 \oplus \{\mu,\nu\}_{P,\psi}) = P^\sharp\{\mu,\nu\}_{P,\psi} \oplus \{\mu,\nu\}_{P,\psi}\\
&= [P^\sharp \mu, P^\sharp \nu] \oplus \{\mu,\nu\}_{P,\psi} = [\![P^\sharp \mu \oplus \mu, P^\sharp \nu \oplus \nu]\!]_\psi\\
&= [\phi(\mu),\phi(\nu)]_D.
\end{aligned}$$

Finally, the fact that $\phi[X,\nu]_{\mathbb{H}} = [\phi(X),\phi(\nu)]_D$ follows from (4.15), (4.16) and arguments similar to the proof of Lemma 4.20. ∎

Due to the previous lemma, the cochain complex $(\Gamma(\wedge^\bullet D^*), \partial_D)$ of the Lie algebroid $(D, p_D, [\cdot,\cdot]_D)$ of the Dirac structure D is isomorphic to the one of $(\mathbb{H}, q, [\cdot,\cdot]_{\mathbb{H}})$. On the other hand, observe that the dual space $\mathbb{H}^*$ is isomorphic to $\mathbb{V} = V^\circ \oplus V$ via the canonical pairing. In fact, one can think of $\alpha \oplus v \in \mathbb{V}$ as an element $\Phi_{\alpha \oplus v} \in \mathbb{H}^*$ by setting

$$\Phi_{\alpha \oplus v}(X \oplus \mu) = \alpha(X) + \mu(v).$$

This map can be extended to an isomorphism $\Phi : \Gamma(\wedge^\bullet \mathbb{V}) \to \Gamma(\wedge^\bullet \mathbb{H}^*)$, which induce a cochain complex structure on $\Gamma(\wedge^\bullet \mathbb{V})$.

Let us describe the coboundary operator on $\Gamma(\wedge^\bullet \mathbb{V})$, in terms of the geometric data (P, γ, σ) and the background 3-form ψ. To do so, we need to consider the derivations of $\Gamma(\wedge^{\bullet,\bullet} \mathbb{V})$ described in Subsections 4.2.2, and 4.2.3, namely, the derivations induced by 3-forms, and the adjoint operators with respect to the bracket of $(\mathscr{V}, [\cdot,\cdot])$.

For the rest of this section, let us fix $f \in C^\infty(M)$, $\alpha \in \Gamma(V^\circ)$, $Y \in \Gamma(V)$, $\mu, \nu \in \Gamma(T^*M)$, and $u, v \in \mathrm{aut}(M, \mathcal{V})$. Associated with P and ψ, consider the operator $\partial_{0,1} := \partial_P + \mathrm{j}^{(0,1)}_{P,\psi}$, where $\partial_P := \mathrm{ad}_P$ is the adjoint of $P \in \Gamma(\wedge^{2,0}\mathbb{V})$, and $\mathrm{j}^{(0,1)}_{P,\psi}$ is defined in terms of (1.13) by (4.5).

$$
\begin{aligned}
\partial_{0,1} f &= -P^\sharp \,\mathrm{d}\, f, \\
\partial_{0,1}\alpha(u) &= P^\sharp \,\mathrm{d}(\alpha(u)), && (4.17)\\
\partial_{0,1} Y(\mu,\nu) &= \mathrm{L}_{P^\sharp\mu}(\nu(Y)) - \mathrm{L}_{P^\sharp\nu}(\mu(Y)) - \{\mu,\nu\}_{P,\psi}(Y). && (4.18)
\end{aligned}
$$

The operator $\partial_{1,0} := \partial_\gamma + \mathrm{j}^{(1,0)}_{P,\psi}$ is the sum of the covariant derivative of γ, $\partial_\gamma := \mathrm{ad}_{(\mathrm{Id}-\gamma)}$ defined by (4.12), and the operator $\mathrm{j}^{(0,1)}_{P,\psi}$ is defined as in Section 4.2.2. Then,

$$
\begin{aligned}
\partial_{1,0} f(u) &= \mathrm{L}_{(\mathrm{Id}-\gamma)(u)}\, f, \\
\partial_{1,0}\alpha(u,v) &= \mathrm{L}_{(\mathrm{Id}-\gamma)(u)}(\alpha(v)) - \mathrm{L}_{(\mathrm{Id}-\gamma)(v)}(\alpha(u)) - \alpha[u,v], && (4.19)\\
\partial_{1,0} Y(u,\mu) &= [(\mathrm{Id}-\gamma)(u), Y] + \mathrm{i}_Y\, \mathrm{i}_{P^\sharp\mu}\, \mathrm{i}_u\, \psi. && (4.20)
\end{aligned}
$$

Now, let $\psi = \psi^\gamma_{3,0} + \psi^\gamma_{2,1} + \psi^\gamma_{1,2} + \psi^\gamma_{0,3}$ be the bigraded decomposition of ψ with respect to γ. Then, the 3-form $\psi^\gamma_{2,1}$ is a section of $\wedge^2 V^\circ \otimes T^*M$. As described in Section 4.2.2, there is an algebraic derivation induced by $\psi^\gamma_{2,1}$, namely, $\mathrm{j}^{(2,-1)}_{\psi^\gamma_{2,1}}$ defined by (4.6). Since this derivation depends on the choice of ψ and γ, let us denote it by $\mathrm{j}^{(2,-1)}_{\psi,\gamma}$.

Finally, the operator of bidegree $(2,-1)$ is $\partial_{2,-1} := \partial_\sigma + \mathrm{j}^{(2,-1)}_{\gamma,\psi}$, which is the sum of the negative adjoint operator $\partial_\sigma := -\,\mathrm{ad}_\sigma$ with the algebraic derivation $\mathrm{j}^{(2,-1)}_{\gamma,\psi}$. Then,

$$
\partial_{2,-1} f = 0, \qquad \partial_{2,-1}\alpha = 0, \qquad \partial_{2,-1} Y = \mathrm{L}_Y\, \sigma - \mathrm{i}_Y\, \psi^\gamma_{2,1} \tag{4.21}
$$

Proposition 4.23. *The de Rham complex of the Lie algebroid $\mathbb{H}$ isomorphic to* $(\Gamma(\wedge^\bullet\mathbb{V}), \partial)$*, where* $\partial = \partial_{0,1} + \partial_{1,0} + \partial_{2,-1}$ *are the bigraded operators defined above.*

Proof. We need to show that $\Phi \circ \partial = \partial_{\mathbb{H}} \circ \Phi$, where $\partial_{\mathbb{H}}$ is the de Rham differential of $\mathbb{H}$. Recall that the adjoint operators are derivations of the exterior algebra $\Gamma(\wedge^\bullet\mathbb{V})$. Therefore, it suffices to show that $\Phi \circ \partial = \partial_{\mathbb{H}} \circ \Phi$ holds on $\Gamma(\mathbb{V})$. Pick $\alpha \in \Gamma(V^\circ)$, and $v \in \Gamma(V)$. Then, for $u, v \in \Gamma(H^\gamma) \cap \mathrm{aut}(M, \mathcal{V})$ and $\mu, \nu \in \Gamma(A^\gamma)$, we get

$$
\begin{aligned}
(\Phi \circ \partial)\alpha(u,v) &= \Phi(\partial\alpha)(u,v) = \Phi(\partial_{1,0}\alpha)(u,v) = \mathrm{L}_u(\alpha(v)) - \mathrm{L}_v(\alpha(u)) - \alpha[u,v] \\
&= \mathrm{L}_{q(u)}(\Phi_\alpha(v)) - \mathrm{L}_{q(v)}(\Phi_\alpha(u)) - \Phi_\alpha[u,v]_{\mathbb{H}} = \partial_{\mathbb{H}}\Phi_\alpha(u,v) \\
&= (\partial_{\mathbb{H}} \circ \Phi)\alpha(u,v).
\end{aligned}
$$

Here, we have applied (4.19). Now, by (4.17), we get

$$\begin{aligned}(\Phi\circ\partial)\alpha(u,\nu)&=\Phi(\partial\alpha)(u,\nu)=\Phi(\partial_{0,1}\alpha)(u,\nu)=\nu(P^\sharp\,\mathrm{d}(\alpha(u)))=-\,\mathrm{d}(\alpha(u))(P^\sharp\nu)\\&=-\,\mathrm{L}_{P^\sharp\nu}(\alpha(u))=0-\mathrm{L}_{P^\sharp\nu}(\alpha(u))-0\\&=\mathrm{L}_{q(u)}(\Phi_\alpha(\nu))-\mathrm{L}_{q(\nu)}(\Phi_\alpha(u))-\Phi_\alpha[u,\nu]_{\mathbb{H}}=\partial_{\mathbb{H}}\Phi_\alpha(u,\nu)\\&=(\partial_{\mathbb{H}}\circ\Phi)\alpha(u,\nu).\end{aligned}$$

Moreover,

$$\begin{aligned}(\Phi\circ\partial)\alpha(\mu,\nu)&=\Phi(\partial\alpha)(\mu,\nu)=\Phi(\partial_{-1,2}\alpha)(\mu,\nu)=0=\mathrm{L}_{P^\sharp\mu}(0)-\mathrm{L}_{P^\sharp\nu}(0)-0\\&=\mathrm{L}_{q(\mu)}(\Phi_\alpha(\nu))-\mathrm{L}_{q(\nu)}(\Phi_\alpha(\mu))-\Phi_\alpha[\mu,\nu]_{\mathbb{H}}=\partial_{\mathbb{H}}\Phi_\alpha(\mu,\nu)\\&=(\partial_{\mathbb{H}}\circ\Phi)\alpha(\mu,\nu).\end{aligned}$$

On the other hand, by (4.21),

$$\begin{aligned}(\Phi\circ\partial)Y(u,v)&=\Phi(\partial Y)(u,v)=\Phi(\partial_{2,-1}Y)(u,v)=\mathrm{L}_Y\,\sigma(u,v)-\mathrm{i}_Y\,\psi^{\gamma}_{2,1}(u,v)\\&=\mathrm{L}_u(0)-\mathrm{L}_v(0)-Y(-\,\mathrm{i}_v\,\mathrm{i}_u(\mathrm{d}\,\sigma+\psi))\\&=\mathrm{L}_{q(u)}(\Phi_Y(v))-\mathrm{L}_{q(v)}(\Phi_Y(u))-\Phi_Y[u,v]_{\mathbb{H}}=\partial_{\mathbb{H}}\Phi_Y(u,v)\\&=(\partial_{\mathbb{H}}\circ\Phi)Y(u,v).\end{aligned}$$

By applying (4.20)

$$\begin{aligned}(\Phi\circ\partial)Y(u,\nu)&=\Phi(\partial Y)(u,\nu)=\Phi(\partial_{1,0}Y)(u,\nu)=[(\mathrm{Id}-\gamma)(u),Y]+\mathrm{i}_Y\,\mathrm{i}_{P^\sharp\nu}\,\mathrm{i}_u\,\psi\\&=\mathrm{L}_u(\nu(Y))-\mathrm{L}_u\,\nu(Y)+\mathrm{i}_{P^\sharp\nu}\,\mathrm{i}_u\,\psi(Y)\\&=\mathrm{L}_u(\nu(Y))-\mathrm{L}_{P^\sharp\nu}(0)-(\mathrm{L}_u\,\nu-\mathrm{i}_{P^\sharp\nu}\,\mathrm{i}_u(\mathrm{d}\,\sigma+\psi))(Y)\\&=\mathrm{L}_{q(u)}(\Phi_Y(\nu))-\mathrm{L}_{q(\nu)}(\Phi_Y(u))-\Phi_Y[u,\nu]_{\mathbb{H}}=\partial_{\mathbb{H}}\Phi_Y(u,\nu)\\&=(\partial_{\mathbb{H}}\circ\Phi)Y(u,\nu).\end{aligned}$$

Finally, by (4.18),

$$\begin{aligned}(\Phi\circ\partial)Y(\mu,\nu)&=\Phi(\partial Y)(\mu,\nu)=\partial_{1,0}Y(\mu,\nu)=\mathrm{L}_{P^\sharp\mu}(\nu(Y))-\mathrm{L}_{P^\sharp\nu}(\mu(Y))-\{\mu,\nu\}_{P,\psi}(Y)\\&=\mathrm{L}_{q(\mu)}(\Phi_Y(\nu))-\mathrm{L}_{q(\nu)}(\Phi_Y(\mu))-\Phi_Y[\mu,\nu]_{\mathbb{H}}=\partial_{\mathbb{H}}\Phi_Y(\mu,\nu)\\&=(\partial_{\mathbb{H}}\circ\Phi)Y(\mu,\nu),\end{aligned}$$

■

Summarizing the above results, we arrive at the main result.

Theorem 4.24. *Let $\psi\in\Gamma(\wedge^3T^*M)$ be a closed 3-form, and $\mathcal{V}$ be a regular foliation on M. Denote $\mathbb{V}:=V\oplus V^\circ$, with $V:=T\mathcal{V}$ and $V^\circ:=\mathrm{Ann}(V)$. Let also $D\subset\mathbb{T}M$ be a $\mathcal{V}$-coupling Dirac ψ-structure with associated ψ-Dirac elements (P,γ,σ). On $\Gamma(\wedge^\bullet\mathbb{V})$, consider the coboundary operator $\partial\equiv\partial^{\psi}_{P,\gamma,\sigma}$ with bigraded components $\partial_{0,1}$, $\partial_{1,0}$, and $\partial_{2,-1}$ given by*

$$\partial_{2,-1}:=\partial_P+\mathrm{j}^{(0,1)}_{P,\psi},\qquad\partial_{2,-1}:=\partial_\gamma+\mathrm{j}^{(1,0)}_{P,\psi},\qquad\partial_{2,-1}:=\partial_\sigma+\mathrm{j}^{(2,-1)}_{\gamma,\psi}\,.$$

Then, the de Rham complex $(\Gamma(\wedge^\bullet D^*),\partial_D)$ of the Lie algebroid of D is isomorphic to the bigraded cochain complex $(\Gamma(\wedge^\bullet\mathbb{V}),\partial)$.

Proof. Because of Proposition 4.23, the pair $(\Gamma(\wedge^\bullet \mathbb{V}), \partial)$ is a cochain complex isomorphic to the de Rham complex of the Lie algebroid $\mathbb{H}$. On the other hand, $\phi : \mathbb{H} \to D$ is an isomorphism of Lie algebroids, which implies that their corresponding cochain complexes are isomorphic. ■

As a consequence of Theorem 4.24, we have the following results [15, Prop. 5.3], [50, Prop. 4.2.11].

Corollary 4.25. *Let S be an embedded symplectic leaf of a Poisson manifold (M, Π), and let N be a coupling neighborhood of S. Let (P, γ, σ) be the associated geometric data of $\Pi|_N$. Then, the Lichnerowicz-Poisson complex $(\Gamma(\wedge^\bullet TN, \partial_\Pi)$ of Π in N is isomorphic to the bigraded cochain complex $(\Gamma(\wedge^\bullet \mathbb{V}), \partial_{P,\gamma,\sigma} = \partial_P + \partial_\gamma + \partial_\sigma)$, where $\partial_P := \mathrm{ad}_P$, $\partial_\gamma := \mathrm{ad}_{\mathrm{Id}-\gamma}$, $\partial_\sigma := \mathrm{ad}_{-\sigma}$ are the adjoint operators with respect to the bracket in Theorem 4.13.*

Part III

Applications

Introduction to Part III

This part is devoted to the study of infinitesimal automorphisms of Poisson manifolds carrying a singular symplectic foliation, with applications to the description of the first Poisson cohomology and the modular class in the semilocal context, around possibly singular symplectic leaves.

Given a Poisson manifold (M, Ψ), the first Poisson cohomology $H^1_\Psi(M)$ is the quotient of the Lie algebra $\mathrm{Poiss}(M, \Psi)$ of Poisson vector fields by its ideal $\mathrm{Ham}(M, \Psi)$ of Hamiltonian vector fields (see Section 1.4). The Hamiltonian vector fields generate the distribution $\Psi^\sharp(T^*M) \subseteq TM$, which integrates to a possibly singular symplectic foliation $(\mathcal{S}, \omega)$.

In the regular case, when $\operatorname{rank}\Psi$ is locally constant on M, the choice of a subbundle $\nu(\mathcal{S}) \subset TM$ normal to $\mathcal{S}$, $TM = T\mathcal{S} \oplus \nu(\mathcal{S})$, induces the following short exact sequence which describes the first Poisson cohomology [62, Section 2], [74]:

$$0 \to H^1_{\mathrm{dR}}(\mathcal{S}) \overset{\Psi^\sharp}{\hookrightarrow} H^1_\Psi(M) \xrightarrow{\mathrm{pr}_{\nu(\mathcal{S})}} \ker(\rho) \to 0.$$

Here, the first term is the tangential component, which is independent of ω, and just coincides with the first *foliated de Rham cohomology* of $\mathcal{S}$. The second term is the transversal component, consisting of the $\nu(\mathcal{S})$-valued infinitesimal automorphisms of the symplectic foliation $Y \in \Gamma_{\mathcal{S}\text{-pr}}(\nu(\mathcal{S}))$ which correspond to the cohomologically trivial in $H^2_{\mathrm{dR}}(\mathcal{S})$ transversal variations $\mathrm{L}_Y\,\omega$ of ω. In other words, $\rho : \Gamma_{\mathcal{S}\text{-pr}}(\nu(\mathcal{S})) \to H^2_{\mathrm{dR}}(\mathcal{S})$ is given by $\rho(Y) := [\mathrm{L}_Y\,\omega]$. The above short exact sequence allows to compute the first Poisson cohomology in some particular cases [25, 34, 64, 62, 74, 84].

One of our purposes is to generalize this result in the singular case. To this end, recall from Section 2.5 that each embedded symplectic leaf S admits a tubular neighborhood $N \subset M$ in which the Poisson structure Ψ is coupling. Then, the Poisson structure splits into "regular" and "singular" parts, $\Psi = \Psi_H + \Psi_V$, where Ψ_V is the *transverse Poisson structure* of S. Furthermore, as we show in Theorem 5.12, the following holds:

$$0 \to H^1(\mathcal{N}_0, \overline{\partial}) \overset{\Psi^\sharp}{\hookrightarrow} H^1_\Psi(N) \xrightarrow{\mathrm{pr}_V} \frac{\ker(\rho : \mathcal{A} \longrightarrow H^2(\mathcal{N}_0, \overline{\partial}))}{\mathrm{Ham}(N, \Psi_V)} \to 0.$$

The first term, which is the tangential component, is the first cohomology of the *de Rham - Casimir complex* $(\mathcal{N}_0^\bullet, \overline{\partial})$, consisting of the differential forms on S with values in Casimir functions of Ψ_V. The coboundary operator $\overline{\partial}$ is defined by the covariant exterior derivative associated with $\Psi_H^\sharp(T^*N)$. The second "transversal" term involves the kernel of an intrinsic morphism ρ from a Lie subalgebra $\mathcal{A}$ of $\mathrm{Poiss}(N, \Psi_V)$ to the second cohomology group $H^2(\mathcal{N}_0, \overline{\partial})$.

We present a geometric derivation of this result in the context of the description of infinitesimal automorphisms of coupling Poisson structures [65, 77, 78]. Algebraically, this is Theorem 3.11 applied to the bigraded cochain complex in Corollary 4.25. In the case when the symplectic leaf S is regular, this result coincides with the short exact sequence of the regular case.

We also apply our result to two particular cases related with singular symplectic foliations. Firstly, in Theorems 5.15 and 5.19, we formulate some sufficient criteria for the triviality of the first Poisson cohomology of coupling Poisson structures. The key point is to regard $(\mathcal{N}^\bullet, \overline{\partial})$ as a subcomplex of some foliated de Rham complex. Under some geometric conditions, the natural morphism between its cohomologies in degree 1 is injective, so the triviality of the first foliated de Rham cohomology implies the triviality of the tangential term. These criterion realize under some “flatness condition” for the symplectic leaf. Combining this with the results of Conn [12], we derive Theorem 5.22.

Secondly, the above result is also applied to the case when the Casimir functions of the transversal Poisson structure Ψ_V are projectable (foliated) with respect to the vertical foliation of the tubular neighborhood. Using the fact that the de Rham - Casimir complex is isomorphic to the de Rham complex of the symplectic leaf S, we illustrate the computation of $H^1_\Psi(N)$ by some examples.

We remark that the results presented in this context can be generalized to the Dirac case in order to describe the first Lie algebroid cohomology of Dirac structures around presymplectic leaves [50, 67].

We also apply our results to the description of the semilocal properties of the modular class.

The modular class $\mathrm{Mod}(M,\Pi)$ of an orientable Poisson manifold (M,Π) is a distinguished element of the first Poisson cohomology group $H^1_\Pi(M)$ and gives an obstruction to the existence of a volume form which is invariant under the flow of every Hamiltonian vector field [41, 83]. If the modular class is trivial, then such an invariant volume form exists and the Poisson manifold is said to be unimodular.

In the regular case, when the rank of the Poisson tensor Π is locally constant, we have the following fact [1]: the modular class $\mathrm{Mod}(M,\Pi)$ is equivalent to the Reeb class $\mathrm{Mod}(\mathcal{S})$ of the regular symplectic foliation $\mathcal{S}$ of Π (for the case $\operatorname{codim}\mathcal{S}=1$, see [30]). For a transversally orientable regular foliation, the Reeb class is the obstruction to the existence of a closed transversal volume element [31]. This relationship leads to a geometric criterion: the triviality of the Reeb class of $\mathcal{S}$ is equivalent to the unimodularity of the regular Poisson manifold (M,Π). As a consequence, the unimodularity of a regular Poisson manifold only depends on its characteristic (symplectic) foliation rather than the leaf-wise symplectic form (see also [14, Corollary 9]). Along with the standard approaches [31, 1], one can characterize the Reeb class in different ways, for example, by using the Bott connection [83, Section 5] or, as the modular class of the associated Lie algebroid [22, 36].

We are interested in a generalization of these results to the case of Poisson manifolds with singular symplectic foliations, for which it does not exist a direct analog of the Reeb class. Our goal is to study the behavior of the modular class of an orientable Poisson manifold (M,Π) and formulate some unimodularity criteria in the semilocal context, around a possibly singular symplectic leaf S.

Due to local Weinstein's splitting theorem [82], the unimodularity of Π in a neighborhood of a singular point is provided by the unimodularity of the transverse Poisson structure of the point. In the nonzero dimensional case, we describe some obstructions to the semilocal unimodularity of the leaf which are related to some “tangential” and “transversal” characteristics of S. In particular, in Proposition 6.18 we show that the unimodularity of a transverse Poisson structure P of S is a necessary condition for $\mathrm{Mod}(M,\Pi)=0$. Moreover, it is proved in Theorem 6.19 that, under the vanishing of the modular class of P, some cohomological obstructions possibly appear in the first cohomology of the associated cochain complex [71, 78]. It is shown in Theorem 6.16 that, in the case when the neighborhood of the leaf is “flat”, these obstructions are directly related to the Reeb class of a foliation. In particular, this occurs in the regular case.

Our main results are based on the formula of Proposition 6.5 for a bigraded decomposition of the modular vector fields of coupling Poisson structures on a foliated manifolds, which involves the modular vector field of the associated Poisson foliation. We also show in Proposition 6.3 that the modular vector field of the Poisson foliation is related to the Reeb class. Also, in Proposition 6.9, we study the behavior of the unimodularity property under gauge equivalence [79, 9]. A similar problem for the Morita equivalence of Poisson structures was studied in [28, 14].

Chapter 5

Semilocal Splitting of the First Cohomology

A geometric description of the first Poisson cohomology group is given in the semilocal context, around possibly singular symplectic leaves. This result is based on the splitting theorems for infinitesimal automorphisms of coupling Poisson structures which describe the interaction between the tangential and transversal data of the characteristic distributions. As a consequence, we derive some criteria of vanishing of the first Poisson cohomology group and apply the general splitting formulas to some particular classes of Poisson structures associated with singular symplectic foliations.

This chapter is organized as follows. In Section 5.1, we briefly recall some notions and facts about Ehresmann connections. In Section 5.2, we review some properties of coupling Poisson structures and formulate the result on the Lichnerowicz-Poisson complex in coupling neighborhoods of a symplectic leaf. In Section 5.3, we present our main results on the infinitesimal automorphisms of coupling Poisson structures and give a proof of Theorem 5.12. In the last three sections, 5.4-5.6, the general results are applied to some particular cases.

The contents of this chapter, with exception of Sections 5.7 and 5.8, were published in [71].

5.1 Covariant exterior derivatives

In this section, we recall some notions and facts in the theory of Ehresmann connections on fiber bundles which will be used throughout the text (for more details, see [35, 65, 78]).

Let $E \xrightarrow{\pi} B$ be a fiber bundle (a surjective submersion) over a manifold B. Denote by $V := \ker d\pi \subset TE$ the vertical subbundle and by $V^\circ \subset T^*E$ its annihilator. The sections of the vector bundles $\wedge^q V$ and $\wedge^p V^\circ$ are called the vertical q-vector fields and horizontal p-forms on with respect to $E \xrightarrow{\pi} B$, respectively. In particular, $\Gamma(\wedge^0 V^\circ) = C^\infty(E)$.

Recall that a vector field $X \in \Gamma(TE)$ is said to be π-related to $u \in \Gamma(TB)$, if $d\pi \circ X = u \circ \pi$. In this case, we say that X is *π-projectable.* It is clear that π-projectable vector fields form an $\mathbb{R}$-Lie subalgebra of $\Gamma(TE)$, which will be denoted by $\Gamma_{\pi\text{-pr}}(TE)$. Given any vector subbundle $F \subset TE$, we will also use the notation $\Gamma_{\pi\text{-pr}}(F) := \Gamma_{\pi\text{-pr}}(TE) \cap \Gamma(F)$ throughout this text. Note that every π-projectable vector field $X \in \Gamma_{\pi\text{-pr}}(TE)$ has the property $[X, \Gamma(V)] \subseteq \Gamma(V)$. Conversely, in the case when the fibers of $E \xrightarrow{\pi} B$ are connected, this property characterizes the π-projectability of vector fields.

On the other hand, recall that an *Ehresmann connection* on E is a vector bundle morphism $\gamma : TE \to TE$ such that $\gamma^2 = \gamma$ and is the identity on the vertical subbundle, $\gamma(Y) = Y$ for all $Y \in \Gamma(V)$. Therefore, γ induces a splitting

$$TE = H \oplus V, \tag{5.1}$$

where $H = H^\gamma := \ker\gamma$ is the horizontal subbundle associated with γ. Conversely, given a subbundle H complementary to V, one can recover the Ehresmann connection from H by setting $\gamma = \mathrm{pr}_V : TE \to V$ (the projection along H).

Suppose we are given an Ehresmann connection γ. The smooth sections of the vector bundles $\wedge^p H$ and $\wedge^q H^0$ are said to be horizontal p-vector fields and the vertical q-forms on E, respectively. The *horizontal lift* of $u \in \Gamma(TB)$ with respect to γ is the unique horizontal vector field $\mathrm{hor}^\gamma(u) \in \Gamma(H)$ which is π-related with u. Therefore, the horizontal lifts are π-projectable and hence, satisfy the following condition:

$$[\mathrm{hor}^\gamma(u), \Gamma(V)] \subset \Gamma(V). \tag{5.2}$$

Consider the $C^\infty(B)$-module $\Omega_B^{p,q}(E) := \Gamma(\wedge^p T^*B) \otimes_{C^\infty(B)} \Gamma(\wedge^q V)$ of all p-forms on B with values in vertical q-vector fields on E. In particular, by property (5.2), the Lie derivative $L_{\mathrm{hor}^\gamma(u)}$ leaves invariant the subspaces $\Gamma(\wedge^q V)$ of vertical tensor fields. Hence, the γ-*covariant exterior derivative* $\partial_{1,0}^\gamma : \Omega_B^{p,q}(E) \to \Omega_B^{p+1,q}(E)$ is defined by the standard formula

$$(\partial_{1,0}^\gamma \eta)(u_0, u_1, \dots, u_p) := \sum_{i=0}^{p} (-1)^i L_{\mathrm{hor}^\gamma(u_i)} \eta(u_0, u_1, \dots, \hat{u}_i, \dots, u_p) \tag{5.3}$$
$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \eta([u_i, u_j], u_0, \dots, \hat{u}_i, \dots, \hat{u}_j, \dots, u_p).$$

The curvature form $\mathrm{Curv}^\gamma \in \Omega_B^{2,1}(E)$ of the connection γ is given on $u_1, u_2 \in \Gamma(TB)$ by

$$\mathrm{Curv}^\gamma(u_1, u_2) := [\mathrm{hor}^\gamma(u_1), \mathrm{hor}^\gamma(u_2)] - \mathrm{hor}^\gamma([u_1, u_2]).$$

The Bianchi identity reads $\partial_{1,0}^\gamma \mathrm{Curv}^\gamma = 0$. Moreover, we have the identity

$$((\partial_{1,0}^\gamma)^2 \eta)(u_0, \dots, u_{p+1}) = - \sum_{0 \le i < j \le p+1} (-1)^{i+j} L_{\mathrm{Curv}^\gamma(u_i, u_j)} \eta(u_0, u_1, \dots, \hat{u}_i, \dots, \hat{u}_j, \dots, u_{p+1}) \tag{5.4}$$

which says that $\partial_{1,0}^\gamma$ is a coboundary operator if and only if the connection γ is *flat*, i.e., $\mathrm{Curv}^\gamma = 0$. Geometrically, the zero curvature condition is equivalent to the integrability of the horizontal subbundle H.

The splitting (5.1) induces the following H-dependent bigrading of multivector fields on E:

$$\Gamma(\wedge^k TE) = \bigoplus_{p+q=k} \Gamma(\wedge^{p,q} TE), \tag{5.5}$$

where $\wedge^{p,q} TE := \wedge^p H \otimes \wedge^q V$. For any k-vector field A on E, the term of bidegree (p,q) in decomposition (5.5) is denoted by $A_{p,q}$. Moreover, the dual splitting $T^*E = V^\circ \oplus H^0$ induces a bigrading of differential forms on E, as follows:

$$\Gamma(\wedge^k T^*E) = \bigoplus_{p+q=k} \Gamma(\wedge^p V^\circ \otimes \wedge^q H^0).$$

We observe that there exists a natural identification

$$\Omega_B^{p,0}(E) = \Gamma(\wedge^p T^*B) \otimes_{C^\infty(B)} C^\infty(E) \cong \Gamma(\wedge^p V^\circ).$$

Indeed, one can associate to every $\eta \in \Omega_B^{p,0}(E)$ a horizontal p-form $\pi^*\eta \in \Gamma(\wedge^p V^\circ)$, given for $X_1,\ldots,X_p \in \Gamma(TE)$ and $e \in E$ by

$$(\pi^*\eta)(X_1,\ldots,X_p)(e) := \eta(\mathrm{d}_e\,\pi(X_1),\ldots,\mathrm{d}_e\,\pi(X_p)).$$

Since $V = \ker d\pi$, it is clear that $\pi^*\eta \in \Gamma(\wedge^p V^\circ)$. Therefore, if we fix an Ehresmann connection γ, then $\pi^*\eta$ is uniquely determined by its values on horizontal lifts, namely,

$$(\pi^*\eta)(\mathrm{hor}^\gamma(u_1),\ldots,\mathrm{hor}^\gamma(u_p)) = \eta(u_1,\ldots,u_p).$$

Moreover, from (5.3), we get the relation

$$\pi^*(\partial^\gamma_{1,0}\eta) = (d(\pi^*\eta))_{p+1,0}, \tag{5.6}$$

where d is the exterior differential for forms on E.

5.2 Coupling neighborhoods

In this section, we recall some properties of coupling Poisson structures on fiber bundles and their applications to describe the geometry of Poisson manifolds around its symplectic leaves. For more details, see [50, 65, 77, 78].

Coupling Poisson structures. Let $E \xrightarrow{\pi} B$ be a fiber bundle and $V^\circ \subset T^*E$ the annihilator of the vertical subbundle V.

Definition 5.1. *The Poisson structure defined by a bivector field* $\Pi \in \Gamma(\wedge^2 TE)$ *is said to be a* coupling Poisson structure *on the fiber bundle if*

$$TE = H \oplus V, \ \ \textit{where } H := \Pi^\sharp(V^\circ). \tag{5.7}$$

Note that every coupling Poisson structure Π has the bigraded decomposition of the form $\Pi = \Pi_{2,0} + \Pi_{0,2}$, where $\Pi_{2,0} \in \Gamma(\wedge^2 H)$ is a horizontal bivector field of constant rank, $\operatorname{rank}\Pi_{2,0} = \operatorname{rank} H$, and $\Pi_{0,2} \in \Gamma(\wedge^2 V)$ is a *vertical Poisson tensor*. The characteristic distribution of Π is the direct sum of the horizontal bundle H and the characteristic distribution of $\Pi_{0,2}$,

$$\Pi^\sharp(T^*E) = H \oplus \Pi^\sharp_{0,2}(H^0).$$

It follows that the fibers of the projection π intersect the symplectic leaves of Π transversally and symplectically. Moreover, the restriction of $\Pi^\sharp_{2,0} : T^*E \to TE$ to V° is a vector bundle isomorphism onto H.

One can associate to a given coupling Poisson tensor Π the geometric data (P,γ,σ) consisting of the Ehresmann connection $\gamma \in \Gamma(T^*E \otimes B)$ associated with the horizontal subbundle $H = \Pi^\sharp(V^\circ)$, a nondegenerated 2-form $\sigma \in \Gamma(\wedge^2 T^*B) \otimes_{C^\infty(B)} C^\infty(E)$, called the *coupling form*, and the vertical Poisson bivector field $P := \Pi_{0,2} \in \Omega_B^{0,2}(E)$. The nondegeneracy of the 2-form σ means that the vector bundle morphism $(\pi^*\sigma)^\flat : H \to V^\circ$ is an isomorphism. In terms of the horizontal part of Π, the coupling form is given by $(\pi^*\sigma)^\flat = -\left(\Pi^\sharp_{2,0}\mid_{V^\circ}\right)^{-1}$. One can show that the geometric data satisfy the *structure equations*

$$[P,P] = 0, \tag{5.8}$$

$$L_{\mathrm{hor}^\gamma(u)} P = 0, \tag{5.9}$$

$$\mathrm{Curv}^\gamma(u,v) = -P^\sharp d\sigma(u,v), \tag{5.10}$$

$$\partial^\gamma_{1,0}\sigma = 0, \tag{5.11}$$

for any $u, v \in \Gamma(TB)$, which give a factorization of the Jacobi identity for Π. Condition (5.9) means that the connection γ on the Poisson fiber bundle $(E \xrightarrow{\pi} B, P)$ is Poisson. In general, the curvature $\mathrm{Curv}^\gamma \in \Omega^{2,1}_B(E)$ of a Poisson connection takes values in the space of vertical Poisson vector fields of P. The curvature identity (5.10) says that $\mathrm{Curv}^\gamma(u,v)$ is a Hamiltonian vector field for any $u, v \in \Gamma(TB)$. Moreover, the coupling 2-form σ must be γ-covariantly constant (condition (5.11)). We say that some geometric data are *integrable* if they satisfy (5.8)-(5.11).

Conversely, every integrable geometric data (P, γ, σ) defines a coupling Poisson tensor Π on E under the nondegeneracy condition for σ.

Bigrading of the Lichnerowicz-Poisson complex. Following [15], let us associate to the geometric data (P, γ, σ) of a coupling Poison tensor $\Pi \in \Gamma(\wedge^2 TE)$ the following cochain complex. Consider the Schouten-Nijenhuis bracket $[\cdot,\cdot] : \Gamma(\wedge^{k_1}TE) \times \Gamma(\wedge^{k_2}TE) \to \Gamma(\wedge^{k_1+k_2-1}TE)$ for multivector fields on the total space E defined in such a way that the triple $(\Gamma(\wedge TE), \wedge, [\cdot,\cdot])$ is a graded Poisson algebra of degree -1 (see [21]). It is clear that the Schouten-Nijenhuis bracket of two vertical multivector fields on E is again vertical. As a consequence, we can endow the bigraded $C^\infty(B)$-module

$$\mathfrak{M}^{\bullet\bullet} = \bigoplus_{k=0}^{\infty} \mathfrak{M}^k, \qquad \mathfrak{M}^k := \bigoplus_{p+q=k} \Omega^{p,q}_B(E).$$

with a structure of graded Poisson algebra of degree -1, $(\mathfrak{M}^{\bullet\bullet}, \wedge, [\cdot,\cdot])$. Explicitly, for $\eta \in \Omega^{p,q}_B(E)$ and $\theta \in \Omega^{p',q'}_B(E)$, we have [15, 50]

$$(\eta\wedge\theta)(u_1,\ldots,u_{p+p'}) := (-1)^{p'q}\sum_\tau \mathrm{sgn}(\tau)\eta(u_{\tau(1)},\ldots,u_{\tau(p)})\wedge\theta(u_{\tau(p+1)},\ldots,u_{\tau(p+p')}),$$

$$[\eta,\theta](u_1,\ldots,u_{p+p'}) := (-1)^{p'(q-1)}\sum_\tau \mathrm{sgn}(\tau)[\eta(u_{\tau(1)},\ldots,u_{\tau(p)}),\theta(u_{\tau(p+1)},\ldots,u_{\tau(p+p')})],$$

where $u_i \in \Gamma(TB)$. Here, in the right-hand sides of these equalities, the symbols $\wedge$ and $[\cdot,\cdot]$ denote the exterior product and the Schouten-Nijenhuis bracket on $\Gamma(\wedge^* V)$, respectively. Thus, every element $\theta \in \Omega^{p,q}_B(E)$ induces a graded derivation ad_θ of bidegree $(p, q-1)$, defined by the adjoint operator $\mathrm{ad}_\theta(\cdot) = [\theta, \cdot\,]$. In particular, the vertical Poisson bivector field $P \in \Gamma(\wedge^2 V)$ induces the derivation $\delta_P := \mathrm{ad}_P : \Omega^{p,q}_B(E) \to \Omega^{p,q+1}_B(E)$ of bidegree $(0,1)$ given by

$$(\mathrm{ad}_P\,\eta)(u_1,\ldots,u_p) := (-1)^p[P, \eta(u_1,\ldots,u_p)].$$

This is a coboundary operator which gives rise to the vertical Poisson complex $(\oplus_{q=0}^\infty \Omega^{0,q}_B(E), \delta_P)$.

Now, using the geometric data (P,γ,σ), we can define an operator $\eth : \mathfrak{M}^{\bullet\bullet} \to \mathfrak{M}^{\bullet\bullet}$ as the sum of bigraded operators

$$\eth := \eth^{\sigma}_{2,-1} + \eth^{\gamma}_{1,0} + \eth^{P}_{0,1}, \tag{5.12}$$

where $\eth^{\sigma}_{2,-1} := -\operatorname{ad}_{\sigma}$, $\eth^{\gamma}_{1,0}$ is the covariant exterior derivative (see Section 5.1), and $\eth^{P}_{0,1} := \delta_P$. Observe that the integrability conditions for the geometric data (P,γ,σ) mean that $\eth$ is a coboundary operator, $\eth^2 = 0$. Indeed, computing the bigraded components of $\eth^2$, we get that equations (5.8)-(5.11) are equivalent to the following relations:

$$(\eth^{P}_{0,1})^2 = 0, \tag{5.13}$$

$$\eth^{\gamma}_{1,0}\eth^{P}_{0,1} + \eth^{P}_{0,1}\eth^{\gamma}_{1,0} = 0, \tag{5.14}$$

$$\eth^{\sigma}_{2,-1}\eth^{P}_{0,1} + \eth^{P}_{0,1}\eth^{\sigma}_{2,-1} + (\eth^{\gamma}_{1,0})^2 = 0, \tag{5.15}$$

$$\eth^{\sigma}_{2,-1}\eth^{\gamma}_{1,0} + \eth^{\gamma}_{1,0}\eth^{\sigma}_{2,-1} = 0. \tag{5.16}$$

Moreover, by the Jacobi identity for the bracket on $\mathfrak{M}$, one can show that $(\eth^{\sigma}_{2,-1})^2 = 0$.

Various versions of the following fact can be found in [15, 50].

Proposition 5.2. *Let $\Pi \in \Gamma(\wedge^2 TE)$ be a coupling Poisson tensor on $E \xrightarrow{\pi} B$ and let (P,γ,σ) be the geometric data associated with Π. Then the Lichnerowicz-Poisson complex $(\Gamma(\wedge^{\bullet}TE),\delta_{\Pi})$ is isomorphic to the cochain complex $(\mathfrak{M}^{\bullet\bullet},\eth)$.*

Proof. Consider the decomposition of multivector fields (5.5). Note that each $A \in \Gamma(\wedge^{p,q}TE)$ can be viewed as a $C^{\infty}(B)$ p-linear skew-symmetric map $A : \Gamma(V^{\circ}) \times \cdots \times \Gamma(V^{\circ}) \to \Gamma(\wedge^{q}V)$. Define $\flat_{\sigma}A \in \mathfrak{M}^{p,q}$ by

$$(\flat_{\sigma}A)(u_1,\ldots,u_p) := (-1)^p A((\pi^*\sigma)^{\flat}\operatorname{hor}^{\gamma}u_1,\ldots,(\pi^*\sigma)^{\flat}\operatorname{hor}^{\gamma}u_p). \tag{5.17}$$

for any $u_i \in \Gamma(TB)$. We claim that the map $\flat_{\sigma} : \Gamma(\wedge TE) \to \mathfrak{M}$ is a cochain complex isomorphism. Since $(\pi^*\sigma)^{\flat}|_H = -\left(\Pi_{2,0}|_{V^{\circ}}\right)^{-1}$ is a vector bundle isomorphism, it follows that $\flat_{\sigma}$ is an exterior algebra isomorphism. By the property that every graded derivation of $\Gamma(\wedge TE)$ is determined by its action on $C^{\infty}(E)$ and $\Gamma(TE)$, it suffices to show that $\flat_{\sigma} \circ \delta_{\Pi} = \eth \circ \flat_{\sigma}$ holds on $C^{\infty}(E)$, $\Gamma(V)$ and $\Gamma(H)$.

For every $f \in C^{\infty}(E)$, we have $(\flat_{\sigma} \circ \delta_{\Pi})(f) = \flat_{\sigma}[\Pi_{2,0}, f] + [P,f]$ and $(\eth \circ \flat_{\sigma})(f) = \eth^{\gamma}_{1,0}f + [P,f]$. Moreover, $\flat_{\sigma}[\Pi_{2,0}, f](u) = df(\operatorname{hor}^{\gamma}u) = \eth^{\gamma}_{1,0}f(u)$.

Next, let $X \in \Gamma(V)$. By bigrading arguments, the equality $(\flat_{\sigma} \circ \delta_{\Pi})(X) = (\eth \circ \flat_{\sigma})(X)$ splits into three equations: $\flat_{\sigma}[\Pi, X]_{1,1} = \eth^{\gamma}_{1,0}X$, $\flat_{\sigma}[\Pi, X]_{0,2} = \delta_P X$, and $\flat_{\sigma}[\Pi, X]_{2,0} = -\operatorname{ad}_{\sigma}X$. For the first equation, by definition, we have $\flat_{\sigma}[\Pi, X]_{1,1}(u) = [\operatorname{hor}^{\gamma}u, X] = \eth^{\gamma}_{1,0}X(u)$. The second one holds because of $[\Pi_{2,0}, X]_{0,2} = 0$. The last equation follows from $\flat_{\sigma}[\Pi, X]_{2,0}(u,v) = [X, \sigma(u,v)] = -\operatorname{ad}_{\sigma}X(u,v)$.

Finally, for $X = \operatorname{hor}^{\gamma}u$, $u \in \Gamma(TB)$, the equality $(\flat_{\sigma} \circ \delta_{\Pi})(X) = (\eth \circ \flat_{\sigma})(X)$ splits into the following relations: $\eth^{\gamma}_{1,0}\flat_{\sigma}(X) = \flat_{\sigma}[\Pi, X]_{2,0}$, $\delta_P\flat_{\sigma}(X) = \flat_{\sigma}[\Pi, X]_{1,1}$, and $0 = \flat_{\sigma}[\Pi, X]_{0,2}$. The verification of these equalities is straightforward by using the structure equations (5.8)-(5.11). ■

As a consequence of Proposition 5.2, we conclude that the infinitesimal automorphisms of the coupling Poisson structure Π are determined by the 1-cocycles $\eta = \eta_{1,0} + \eta_{0,1} \in Z^1_{\eth}$ of $\eth$ which are the solutions to the equations:

$$\partial_{0,1}^{P}(\eta_{0,1}) = 0, \tag{5.18}$$

$$\partial_{1,0}^{\gamma}(\eta_{0,1}) + \partial_{0,1}^{P}(\eta_{1,0}) = 0, \tag{5.19}$$

$$\partial_{1,0}^{\gamma}(\eta_{1,0}) + \partial_{2,-1}^{\sigma}(\eta_{0,1}) = 0. \tag{5.20}$$

In the next section, we describe the infinitesimal automorphisms of coupling Poisson structures in terms of the solutions of these equations.

Coupling neighborhood of a symplectic leaf. Let (M,Ψ) be a Poisson manifold and $B \subset M$ an embedded symplectic leaf. Let $\pi : E \to B$, $E = T_B M \diagup TB$ be the normal bundle of the leaf. By a tubular neighborhood of a symplectic leaf B, we mean an open neighborhood N of B in M together with an exponential map $\mathbf{f} : U \to N$, that is, a diffeomorphism from an open neighborhood U of the zero section $B \hookrightarrow E$ onto N satisfying the conditions: $\mathbf{f}|_B = \mathrm{id}_B$ and $\nu \circ d_B\mathbf{f} = \tau$. Here, $\tau : T_B E \to E$ is the projection along TB according to the decomposition $T_B E = TB \oplus E$ and $\nu : T_B M \to E$ is the natural projection. These properties imply that the differential $d_B\mathbf{f} : T_B E \to T_B M$ sends the fibers of the normal bundle to transverse subspaces to the leaf $B \subset M$, $T_B M = TB \oplus (d_B\mathbf{f})(E)$.

Definition 5.3. *A tubular neighborhood $(N,\mathbf{f})$ of the symplectic leaf B of (M,Ψ) is said to be a* coupling neighborhood *if the pull-back $\Pi := \mathbf{f}^*(\Psi|_N)$ is a coupling Poisson structure on the fiber bundle $\pi_U : U \to B$.*

Given a coupling neighborhood $(N,\mathbf{f})$ of B, we have the bigraded decomposition $\Pi = \Pi_{2,0} + \Pi_{0,2}$. Hence, $\Psi|_N = \Psi_H + \Psi_V$, where $\Psi_H = \mathbf{f}_*\Pi_{2,0}$ is a bivector field on N of constant rank, $\operatorname{rank}\Psi_H = \dim B$, and $\Psi_V = \mathbf{f}_*\Pi_{0,2}$ is a Poisson tensor on N vanishing at B and tangent to the vertical subbundle $d\mathbf{f}(\ker d\pi_U) \subset T_N M$ over the tubular neighborhood. The bivector field Ψ_V is said to be a *transverse Poisson structure* around the leaf B and can be viewed as the result of gluing the local transverse Poisson structures on the vertical fibers due to the local splitting Weinstein theorem [82]. Furthermore, one can show that the different choices of exponential maps lead to isomorphic transverse Poisson structures. Notice that, in the case when the symplectic leaf B is regular, the coupling neighborhood N may be chosen in such a way that the transverse Poisson structure is identically zero, $\Psi_V \equiv 0$. This follows from the property: $\operatorname{rank}_m \Psi = \dim B + \operatorname{rank}_m \Psi_V$ for every $m \in N$. Observe also that the linearization of $\Pi_{0,2}$ at B gives a vertical fiberwise linear Poisson structure $\Pi_{0,2}^{(1)}$. This Lie algebra is called the *linearized transverse Poisson structure* of the leaf B [82], which is well defined on the whole total space E. As a consequence, we get an intrinsic locally trivial Lie-Poisson bundle $(E, \Pi_{0,2}^{(1)})$ over B whose typical fiber is the co-algebra $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ called the *isotropy algebra* of the symplectic leaf.

As is known [77], each embedded symplectic leaf B admits a coupling neighborhood and hence, by Proposition 5.2, the computation of the Poisson cohomology around B is reduced to the study of the bigraded cochain complex $(\mathfrak{M}^{\bullet\bullet}, \partial)$ attributed to a coupling Poisson structure Π.

5.3 Infinitesimal automorphisms of coupling Poisson structures

Suppose we are given a coupling Poisson tensor Π on a fiber bundle $\pi : E \to B$ associated with an integrable geometric data (P, γ, σ). As we saw in the previous section, the infinitesimal Poisson automorphisms of Π are related to the solutions of equations (5.18)-(5.20). Our goal is to describe these solutions in terms of the geometric data (P, γ, σ). To formulate the main results, let us introduce the following objects.

The coboundary operator $\overline{\partial}^{\gamma}$**.** Consider the space $\mathrm{Casim}(E, P)$ of all Casimir functions of the vertical Poisson tensor P on E. It is clear that $\pi^* C^\infty(B) \subseteq \mathrm{Casim}(E, P)$. For each $p \in \mathbb{Z}$, define the $C^\infty(B)$-submodule $\mathcal{N}^p \subseteq \Omega_B^{p,0}(E)$ by

$$\mathcal{N}^p := \Omega^p(B) \otimes_{C^\infty(B)} \mathrm{Casim}(E, P).$$

In particular, $\mathcal{N}^0 = \mathrm{Casim}(E, P)$. Since the Poisson vector fields of P preserve the space of Casimir functions, by (5.9) and definition (5.3), we have $\partial_{1,0}^{\gamma}(\mathcal{N}^p) \subset \mathcal{N}^{p+1}$. Hence one can define the operator

$$\overline{\partial}^{\gamma} := \partial_{1,0}^{\gamma}|_{\mathcal{N}^p}. \tag{5.21}$$

Then, by (5.4) and the curvature identity (5.10), we conclude that $\overline{\partial}^{\gamma}$ is a coboundary operator, $\overline{\partial}^{\gamma} \circ \overline{\partial}^{\gamma} = 0$. The pair $(\mathcal{N}^\bullet, \overline{\partial}^{\gamma})$ is call the *de Rham-Casimir complex*. The p-cohomology space of $\overline{\partial}^{\gamma}$ is $H^p_{\overline{\partial}^{\gamma}} := \frac{Z^p_{\overline{\partial}^{\gamma}}}{B^p_{\overline{\partial}^{\gamma}}}$, where $Z^p_{\overline{\partial}^{\gamma}}$ and $B^p_{\overline{\partial}^{\gamma}}$ are the spaces of $\overline{\partial}^{\gamma}$-closed and $\overline{\partial}^{\gamma}$-exact p-forms, respectively.

Consider the Lie algebra $\mathrm{Poiss}(E, \Pi)$ of Poisson vector fields of the coupling Poisson structure Π. Let $\sharp_H : \Omega_B^{1,0}(E) \to \Gamma(H)$ be a linear mapping given by

$$\sharp_H(\alpha) := \Pi_{2,0}^{\sharp}(\pi^*\alpha).$$

By the horizontal nondegeneracy of $\Pi_{2,0}$, it follows that $\sharp_H : \Omega_B^{1,0}(E) \to \Gamma(H)$ is an isomorphism.

Lemma 5.4. *The image of the space of 1-cocycles $Z^1_{\overline{\partial}^{\gamma}}$ under the isomorphism $\sharp_H$ coincides with the space of Poisson vector fields of Π tangent to the horizontal distribution,*

$$\sharp_H(Z^1_{\overline{\partial}^{\gamma}}) = \Gamma(H) \cap \mathrm{Poiss}(E, \Pi). \tag{5.22}$$

Proof. Consider the vector bundle morphism $\Pi^\sharp : T^*M \to TM$ associated with the bivector field Π. For every 2-form $\mu \in \Omega^2(E)$, one can associate a bivector field $\Pi^\sharp\mu \in \Gamma(\wedge^2 TE)$ defined by $(\Pi^\sharp\mu)(\eta_1, \eta_2) := \mu(\Pi^\sharp\eta_1, \Pi^\sharp\eta_2)$. Then, one has $[\Pi, \Pi^\sharp(\pi^*\alpha)] = -\Pi^\sharp(d\pi^*\alpha)$. Moreover, we observe that, for any $\eta \in \Omega^1(E)$ and $\mu \in \Omega^2(E)$ such that $\eta_{0,1} = 0$ and $\mu_{0,2} = 0$, the following identities hold: $\Pi^\sharp\eta = \Pi_{2,0}^\sharp\eta$ and $\Pi_{2,0}^\sharp\mu = \Pi_{2,0}^\sharp\mu_{2,0}$. Setting $\eta = \pi^*\alpha$, and $\mu = d\pi^*\alpha$ and combining these properties with (5.6), we get

$$[\Pi, \sharp_H(\alpha)] = [\Pi, \Pi_{2,0}^\sharp(\pi^*\alpha)] = [\Pi, \Pi^\sharp(\pi^*\alpha)] = -\Pi^\sharp(d\pi^*\alpha) \tag{5.23}$$

and

$$\Pi_{2,0}^\sharp(\pi^*(\overline{\partial}^{\gamma}\alpha)) = \Pi_{2,0}^\sharp(d\pi^*\alpha)_{2,0} = \Pi_{2,0}^\sharp(d\pi^*\alpha). \tag{5.24}$$

Finally, observe that $\alpha \in \mathcal{N}^1$ if and only if $\mathrm{i}_{P^\sharp\eta} d\pi^*\alpha = 0 \; \forall \eta \in \Omega^1(E)$, which is equivalent to $\Pi^\sharp(d\pi^*\alpha) = \Pi_{2,0}^\sharp(d\pi^*\alpha)$. Therefore, from (5.23) and (5.24), it follows that $[\Pi, \sharp_H(\alpha)] = 0$ if and only if $\overline{\partial}^{\gamma}\alpha = 0$. ■

Remark 5.5. *Notice that Lemma 5.4 can be deduced from Proposition 5.2. Indeed, the cochain complex isomorphism $\flat_\sigma : \Gamma(\wedge TE) \to \mathfrak{M}$ satisfies $\sharp_H = -(\flat_\sigma|_{\Gamma(H)})^{-1}$. Thus, $\sharp_H(\Omega_B^{1,0}(E) \cap Z^1_{\partial}) = \Gamma(H) \cap \mathrm{Poiss}(E, \Pi)$. Since $Z^1_{\overline{\partial}^{\gamma}} = \Omega_B^{1,0}(E) \cap Z^1_{\partial}$, the result follows.*

The Lie algebra $\mathcal{A}^\gamma$. Let $\mathrm{Ham}(E,P)\subset\Gamma(V)$ be the Lie algebra of Hamiltonian vector fields of the vertical Poisson tensor P on E. Consider the Poisson connection γ on (E,P). The set of all vertical Poisson vector fields is a Lie algebra

$$\mathrm{Poiss}_V(E,P):=\{Y\in\Gamma(V)\mid L_YP=0\}$$

for which $\mathrm{Ham}(E,P)$ is an ideal. Furthermore, by (5.2) and (5.9), we have

$$[\mathrm{hor}^\gamma(u),\mathrm{Poiss}_V(E,P)]\subseteq\mathrm{Poiss}_V(E,P)\qquad\forall u\in\Gamma(TB).$$

One can associate to the triple (E,P,γ) the subspace $\mathcal{A}^\gamma\subset\mathrm{Poiss}_V(E,P)$ of vertical Poisson vector fields determined by the condition

$$[\mathrm{hor}^\gamma(u),\mathcal{A}^\gamma]\subseteq\mathrm{Ham}(E,P)\qquad\forall u\in\Gamma(TB)$$

or, more precisely,

$$\mathcal{A}^\gamma:=\{Y\in\mathrm{Poiss}_V(E,P)\mid[\mathrm{hor}^\gamma(u),Y]\in\mathrm{Ham}(E,P)\quad\forall u\in\Gamma(TB)\}.\tag{5.25}$$

Observe that $\mathcal{A}^\gamma$ is a Lie algebra and $\mathrm{Ham}(E,P)\subseteq\mathcal{A}^\gamma$ is an ideal. These properties follow from the identity

$$[X,P^\sharp dF]=P^\sharp dL_XF$$

for any Poisson vector field X of P. Moreover, for every $Y\in\mathcal{A}^\gamma$ there exists a 1-form $\beta_Y\in\Omega_B^{1,0}(E)=\Omega^1(B)\otimes_{C^\infty(B)}C^\infty(E)$ such that

$$[\mathrm{hor}^\gamma(u),Y]=-P^\sharp d\beta_Y(u)\qquad\forall u\in\Gamma(TB).\tag{5.26}$$

This follows from a partition of unity argument applied to an open coordinate covering of the base B, and the fact that P is vertical.

The homomorphism $\rho^\gamma:\mathcal{A}^\gamma\to H^2_{\partial^\gamma}$. Given an arbitrary vector field $Y\in\mathcal{A}^\gamma$ and fixing a 1-form $\beta_Y\in\Omega^1(B)\otimes_{C^\infty(B)}C^\infty(E)$ in (5.26), we associate to Y an element $\tau_Y\in\Omega^2(B)\otimes_{C^\infty(B)}C^\infty(E)$ given by

$$\tau_Y:=\partial^\gamma_{1,0}\beta_Y+L_Y\sigma.\tag{5.27}$$

Here, the Lie derivative $L_Y:\Omega_B^{p,q}(E)\to\Omega_B^{p,q}(E)$ along an arbitrary vertical vector field Y is given on $u_1,\dots,u_k\in\Gamma(TB)$ by the formula $(L_Y\eta)(u_1,\dots,u_k):=L_Y(\eta(u_1,\dots,u_k))$. Note that we also have the equality $L_Y\sigma=\partial^\sigma_{2,-1}Y$.

By using the structure equations (5.8)-(5.11), one can show that the 2-form τ_Y takes values in Casimir functions, $\tau_Y\in\mathcal{N}^2$. Indeed, from (5.27), we have

$$\tau_Y(u_1,u_2)=L_{\mathrm{hor}^\gamma(u_1)}\beta_Y(u_2)-L_{\mathrm{hor}^\gamma(u_2)}\beta_Y(u_1)-\beta_Y([u_1,u_2])+L_Y\sigma(u_1,u_2).\tag{5.28}$$

Next, for every Poisson vector field Z of P, we have $L_Z\circ P^\sharp=P^\sharp\circ L_Z$. Using this property, equality (5.28), the curvature identity (5.10) and (5.26), by direct computation we verify that $P^\sharp d\tau_Y(u_1,u_2)=0$.

Now, we observe that the 2-form τ_Y is $\overline{\partial}^\gamma$-closed, $\overline{\partial}^\gamma \tau_Y = 0$. Indeed, this can be verified by straightforward computations and by applying again (5.27), (5.4), (5.10), (5.26), and (5.11). Moreover, the cohomology class $[\tau_Y] \in H^2_{\overline{\partial}^\gamma}$ is independent of the choice of β_Y in (5.26). To see this, observe that any other element $\beta'_Y \in \Omega_B^{1,0}(E)$ satisfying (5.26) is of the form $\beta'_Y = \beta_Y + c_Y$ for some $c_Y \in \mathcal{N}^1$. Then, the corresponding τ_Y and τ'_Y are related by $\tau'_Y = \tau_Y + \overline{\partial}^\gamma c_Y$ and hence, $[\tau'_Y] = [\tau_Y]$. So, we have proved the following fact.

Lemma 5.6. *There exists an intrinsic homomorphism*

$$\rho^\gamma : \mathcal{A}^\gamma \to H^2_{\overline{\partial}^\gamma} \tag{5.29}$$

which assigns to every vertical vector field $Y \in \mathcal{A}^\gamma$ the $\overline{\partial}^\gamma$-cohomology class of the 2-form τ_Y,

$$\rho^\gamma(Y) := [\tau_Y].$$

It is easy to see that every Hamiltonian vector field of P belongs to the kernel of ρ^γ and hence we have the inclusions:

$$\mathrm{Ham}(E,P) \subseteq \ker \rho^\gamma \subseteq \mathcal{A}^\gamma \subseteq \mathrm{Poiss}_V(E,P). \tag{5.30}$$

Consider the projection $\mathrm{pr}_V : \Gamma(TE) \to \Gamma(V)$ associated with the splitting $TE = H \oplus V$, $\mathrm{pr}_V(X) = X_{0,1}$. It is clear that $\ker \mathrm{pr}_V = \Gamma(H)$.

Lemma 5.7. *The image of* $\mathrm{Poiss}(E,\Pi)$ *under the projection* pr_V *coincides with the kernel of ρ^γ,*

$$\mathrm{pr}_V(\mathrm{Poiss}(E,\Pi)) = \ker \rho^\gamma.$$

Proof. Let $Z \in \mathrm{Poiss}(E,\Pi)$ be an infinitesimal automorphism of Π. Since the map $\sharp_H : \Omega_B^{1,0}(E) \to \Gamma(H)$ is an isomorphism, there exist unique $Y \in \Gamma(V)$ and $\beta \in \Omega_B^{1,0}(E)$ such that $Z = -\sharp_H \beta + Y$. Let us show that $Y \in \ker \rho^\gamma$. If $\flat_\sigma : (\Gamma(\wedge^\bullet TE), \delta_\Pi) \to (\mathfrak{M}^{\bullet\bullet}, \partial)$ is the cochain complex isomorphism (5.17), then $\flat_\sigma Z = \beta + Y$. Since $Z \in Z^1_\Pi(E)$, we have $\beta + Y \in Z^1_\partial$. Explicitly, this means that $\eta := \beta + Y$ must satisfy equations (5.18)-(5.20). Note that (5.18) means that $Y \in \mathrm{Poiss}_V(E,P)$. Moreover, by evaluating the left-hand side of (5.19) on $u \in \Gamma(TB)$, we get that β and Y satisfy (5.26), so $Y \in \mathcal{A}^\gamma$. Finally, (5.20) implies that $\tau_Y = 0$ and hence, $\rho^\gamma(Y) = 0$, as desired. Conversely, pick an arbitrary $Y \in \ker \rho^\gamma$. Since $\ker \rho^\gamma \subseteq \mathcal{A}^\gamma$, there exists a 1-form $\beta_Y \in \Omega_B^{1,0}(E)$ satisfying (5.26). Next, by the definition of ρ^γ, there exists a primitive $c_Y \in \mathcal{N}^1$ of the 2-cocycle $\tau_Y \in Z^2_{\overline{\partial}^\gamma}$ in (5.27) so that $\overline{\partial}^\gamma c_Y = \tau_Y$. Then, one can easily verify that

$$X_Y := -\sharp_H(\beta_Y - c_Y) + Y \in \mathrm{Poiss}(E,\Pi). \tag{5.31}$$

This means that every element $Y \in \ker \rho^\gamma$ can be extended to a Poisson vector field X of Π in the sense that $X_{0,1} = Y$. ■

Corollary 5.8. *The Poisson vector fields of the coupling Poisson structure* Π *are of the form*

$$\sharp_H(\alpha) + X_Y, \tag{5.32}$$

where $\alpha \in Z^1_{\overline{\partial}^\gamma}$ and $Y \in \ker \rho^\gamma \subset \mathcal{A}^\gamma$ are arbitrary elements.

In particular, it follows that a Poisson vector field $X \in \mathrm{Poiss}(E,\Pi)$ is tangent to the symplectic foliation of Π if and only if $\mathrm{pr}_V(X)$ is tangent to the symplectic foliation of the vertical Poisson structure P.

By Lemma 5.6 and Lemma 5.7, we conclude that $\mathrm{Poiss}(E,\Pi)$ fits into the short exact sequence of vector spaces

$$0 \to Z^1_{\overline{\partial}^\gamma} \xrightarrow{\sharp_H} \mathrm{Poiss}(E,\Pi) \xrightarrow{\mathrm{pr}_V} \ker \rho^\gamma \to 0. \tag{5.33}$$

Summarizing the above considerations, we get the following splitting theorem for infinitesimal automorphisms of a coupling Poisson structure.

Theorem 5.9. *Let $\Pi = \Pi_{2,0} + P$ be the coupling Poisson tensor on E and (P,γ,σ) its integrable geometric data. Let $(\mathcal{A}^\gamma, \overline{\partial}^\gamma, \rho^\gamma)$ be the associated set up. Then, there is a vector space isomorphism*

$$\mathrm{Poiss}(E,\Pi) \cong Z^1_{\overline{\partial}^\gamma} \oplus \ker \rho^\gamma. \tag{5.34}$$

Now, let us consider the spaces of Hamiltonian vector fields $\mathrm{Ham}(E,\Pi)$ and $\mathrm{Ham}(E,P)$ of the Poisson structures Π and P, respectively. Recall that the space $B^1_{\overline{\partial}^\gamma}$ consists of $\overline{\partial}^\gamma$-exact 1-forms $\bar{\partial}^\gamma k$, with $k \in \mathrm{Casim}(E,P)$. Then, by using (5.6) and the fact that $P^\sharp(dk) = 0$, we get

$$\sharp_H(\overline{\partial}^\gamma k) = \Pi^\sharp_{2,0}(\pi^*(\overline{\partial}^\gamma k)) = \Pi^\sharp_{2,0}(dk)_{1,0} = \Pi^\sharp_{2,0} dk = \Pi^\sharp dk.$$

This shows that the image of $B^1_{\overline{\partial}^\gamma}$ under the mapping $\sharp_H$ belongs to $\mathrm{Ham}(E,\Pi)$ and is of the form

$$\sharp_H(B^1_{\overline{\partial}^\gamma}) = \{\Pi^\sharp_{2,0} dk \mid k \in \mathrm{Casim}(E,P)\}.$$

Furthermore, we have the following result.

Proposition 5.10. *There is a short exact sequence:*

$$0 \to B^1_{\overline{\partial}^\gamma} \xrightarrow{\sharp_H} \mathrm{Ham}(E,\Pi) \xrightarrow{\mathrm{pr}_V} \mathrm{Ham}(E,P) \to 0. \tag{5.35}$$

Proof. By the nondegeneracy property of $\Pi^\sharp_{2,0}$, the mapping $\sharp_H$ is injective and hence, $\ker \sharp_H = \{0\}$. On the other hand, by the bigraded decomposition $\Pi = \Pi_{2,0} + P$, we conclude that $\Pi^\sharp df = \Pi^\sharp_{2,0} df + P^\sharp df$. This implies the equality $\mathrm{pr}_V(\mathrm{Ham}(E,\Pi)) = \mathrm{Ham}(E,P)$. It follows also that $\mathrm{pr}_V(\Pi^\sharp df) = P^\sharp df = 0$ if and only if $f \in \mathrm{Casim}(E,P)$ and hence, $\ker \mathrm{pr}_V = \mathrm{Im}\, \sharp_H$. ■

We observe that a necessary condition for a vector field X being Hamiltonian relative to Π and a function f is that, the vertical part $X_{0,1}$ of X is Hamiltonian relative to P and the same function f. Notice also that the Poisson vector field $\sharp_H(\alpha)$ is Hamiltonian relative to Π if and only if α is $\overline{\partial}^\gamma$-exact.

By (5.33), (5.35), we have the following short exact sequence of the cohomology spaces

$$0 \to \frac{Z^1_{\overline{\partial}^\gamma}}{B^1_{\overline{\partial}^\gamma}} \xrightarrow{\sharp_H} \frac{\mathrm{Poiss}(E,\Pi)}{\mathrm{Ham}(E,\Pi)} \xrightarrow{\mathrm{pr}_V} \frac{\ker \rho^\gamma}{\mathrm{Ham}(E,P)} \to 0.$$

So, we arrive at the main result.

Theorem 5.11. *Let Π be a coupling Poisson tensor on a fiber bundle $\pi : E \to B$, with associated geometric data (P, γ, σ). Let also $(\mathcal{N}_0^\bullet, \overline{\partial})$ be the de Rham - Casimir complex of P and γ, and consider the map $\rho^\gamma : \mathcal{A}^\gamma \to H^2_{\overline{\partial}^\gamma}$ in (5.29). The first Poisson cohomology of (E, Π) is*

$$H^1_\Pi(E) \cong H^1_{\overline{\partial}^\gamma} \oplus \frac{\ker \rho^\gamma}{\mathrm{Ham}(E, P)}. \tag{5.36}$$

By taking into account the facts in Section 5.2, we derive the statement of Theorem 5.12 as a consequence of this result.

Theorem 5.12. *Let $S \subset M$ be an embedded symplectic leaf of the Poisson manifold (M, Ψ). Then, there exists a tubular neighborhood N of S in M such that*

$$\Psi = \Psi_H + \Psi_V \ \textit{on}\ N,$$

where Ψ_H is a bivector field on N of constant rank, $\operatorname{rank} \Psi_H = \dim S$ and Ψ_V is a Poisson bivector field on N vanishing at S. Moreover, the first cohomology group of the Poisson structure Ψ fits in the following short exact sequence:

$$0 \to H^1(\mathcal{N}_0, \overline{\partial}) \overset{\Psi^\sharp}{\hookrightarrow} H^1_\Psi(N) \xrightarrow{\mathrm{pr}_V} \frac{\ker\{\rho : \mathcal{A} \longrightarrow H^2(\mathcal{N}_0, \overline{\partial})\}}{\mathrm{Ham}(N, \Psi_V)}.$$

Regular symplectic leaves. As mentioned in the introduction, formula (5.36) coincides with some well-known results [62, 74] in the regular case. Recall that the semilocal model for a Poisson structure Ψ on M around an embedded regular symplectic leaf (B, ω_B) is represented by a coupling Poisson structure Π on the normal vector bundle $\pi : E \to B$ with associated geometric data of the form $(P = 0, \gamma^0, \sigma = \omega_B \otimes 1 + C)$ and having the zero section $B \hookrightarrow E$ as a symplectic leaf. Here, γ^0 is a flat Ehresmann connection on E whose horizontal distribution is just the tangent bundle $T\mathcal{S}$ of the symplectic foliation $(\mathcal{S}, \omega)$ of Π. The coupling form σ is determined by the symplectic form ω_B of the leaf and a $\partial^{\gamma^0}_{1,0}$-closed 2-form $C \in \Omega^{2,0}_B(E)$ vanishing at B.

Proposition 5.13. *We have the following relations:*

$$H^1_{\overline{\partial}^{\gamma^0}} \cong H^1_{\mathrm{dR}}(\mathcal{S}) \qquad \textit{and} \qquad \ker \rho^{\gamma^0} = \{Y \in \Gamma_{\mathcal{S}\text{-pr}}(V) \mid L_Y\omega \ \textit{is}\ d_{\mathcal{S}}\textit{-exact}\}\,.$$

Proof. Because of the triviality of the transverse Poisson structure of B, we have $\mathrm{Casim}(E, P) = C^\infty(E)$. Then, taking into account that $(\mathcal{N}^\bullet, \overline{\partial}^{\gamma^0}) = (\Omega^{*,0}_B(E), \partial^{\gamma^0}_{1,0})$, we conclude that there exists a natural identification of the cochain complexes $(\Gamma(\wedge T^*\mathcal{S}), \mathrm{d}_{\mathcal{S}})$ and $(\mathcal{N}^\bullet, \overline{\partial}^{\gamma^0})$. In particular, the leafwise symplectic form ω of Π coincides with the coupling form σ. Moreover, by definition (5.25) and the relations $\mathrm{Poiss}_V(E, P) = \Gamma(V)$ and $\mathrm{Ham}(E, P) = \{0\}$, we get that

$$\mathcal{A}^{\gamma^0} = \{Y \in \Gamma(V) \mid [\mathrm{hor}^{\gamma_0}(u), Y] = 0 \quad \forall u \in \Gamma(TB)\}$$

coincides with the space of vertical vector fields preserving the symplectic foliation, $\mathcal{A}^{\gamma^0} = \Gamma_{\mathcal{S}\text{-pr}}(V)$. So, we can think of the homomorphism $\rho^{\gamma^0} : \mathcal{A}^{\gamma^0} \to H^2_{\overline{\partial}^{\gamma^0}}$ as a mapping $\Gamma_{\mathcal{S}\text{-pr}}(V) \to H^2_{\mathrm{dR}}(\mathcal{S})$ which sends an element Y to the $\mathrm{d}_{\mathcal{S}}$-cohomology class $[L_Y\omega]$. ■

In the next three sections, we discuss some other particular cases to which formula (5.36) can be effectively applied.

Spectral sequences. Here we briefly discuss an alternative algebraic approach to the computation of the first cohomology of the bigraded complex introduced in Section 5.2 (see also [62, 64], for the use of spectral sequences in the computation of Poisson cohomology).

Consider the nonnegative cochain complex $(\mathfrak{M}^\bullet = \bigoplus_{n\in\mathbb{Z}} \mathfrak{M}^n, \eth)$, where

$$\mathfrak{M}^n = \bigoplus_{p+q=n} \Omega_B^{p,q}(E)$$

and the differential operator $\eth$ is given by (5.12). Consider a filtration F of $\mathfrak{M}$ defined as

$$F^p\mathfrak{M}^n := \bigoplus_{p\leq k\leq n} \Omega_B^{k,n-k}(E).$$

Then,

$$\mathfrak{M}^n = F^0\mathfrak{M}^n \supset F^1\mathfrak{M}^n \supset \cdots \supset F^n\mathfrak{M}^n = \Omega_B^{n,0}(E) \supset F^{n+1}\mathfrak{M}^n = \{0\}$$

and hence the filtration F is *bounded* [47]. Moreover, the bigraded decomposition (5.12) provides the inclusions $\eth(F^p\mathfrak{M}^n) \subseteq F^p\mathfrak{M}^{n+1}$ for all p and n. Therefore, $(\mathfrak{M}, \eth, F)$ is a graded filtered complex.

Now, consider the spectral sequence $(E_r, \mathrm{d}_r)_{r\geq 0}$ associated with $(\mathfrak{M}, \eth, F)$. Observe that $(E_r, \mathrm{d}_r)_{r\geq 0}$ is a first quadrant spectral sequence.

Lemma 5.14. *The spectral sequence $(E_r, \mathrm{d}_r)_{r\geq 0}$ converges to the cohomology of the cochain complex $(\mathfrak{M}, \eth)$. Moreover, we have the following isomorphism of vector spaces*

$$H^1(\mathfrak{M}, \eth) \cong E_2^{1,0} \oplus E_3^{0,1}. \tag{5.37}$$

Proof. Since $(E_r, \mathrm{d}_r)_{r\geq 0}$ is a first quadrant spectral sequence, for every $p, q \in \mathbb{Z}$ we have $E_\infty^{p,q} = E_N^{p,q}$, where $N = \max\{p+1, q+2\}$. Then, the convergence and relation (5.37) follow from the fact that F is bounded (see [47]). ∎

To compute $E_2^{1,0}$ and $E_3^{0,1}$ in the notations introduced in Section 5.2, we use the explicit general formulas for the E-terms of a spectral sequence, given as in [21, Subsection 2.4.1]:

$$E_r^{p,q} = \frac{\ker(\eth|_{\mathfrak{M}^{p+q}}) \cap F^p\mathfrak{M}^{p+q} + F^{p+1}\mathfrak{M}^{p+q}}{\mathrm{Im}(\eth|_{\mathfrak{M}^{p+q-1}}) \cap F^p\mathfrak{M}^{p+q} + F^{p+1}\mathfrak{M}^{p+q}}, \qquad r \geq \max\{p+1, q+2\}.$$

The direct computations give

$$E_2^{1,0} = \frac{\ker(\eth|_{\mathfrak{M}^{1,0}})}{\mathrm{Im}(\eth^\gamma_{1,0}|_{\ker \eth^P_{0,1}\cap\mathfrak{M}^0})} = \frac{Z^1_{\eth^\gamma}}{B^1_{\eth^\gamma}} = H^1_{\eth^\gamma},$$

$$E_3^{0,1} = \frac{\ker(\eth|_{\mathfrak{M}^1}) + \mathfrak{M}^{1,0}}{\mathrm{Im}(\eth|_{\mathfrak{M}^0}) + \mathfrak{M}^{1,0}} = \frac{\mathrm{pr}_V(\ker(\eth|_{\mathfrak{M}^1})) \oplus \mathfrak{M}^{1,0}}{\mathrm{pr}_V(\mathrm{Im}(\eth|_{\mathfrak{M}^0})) \oplus \mathfrak{M}^{1,0}} \cong \frac{\ker\rho^\gamma}{\mathrm{Ham}(E,P)},$$

where we use the relations: $\mathrm{pr}_V(\ker(\eth|_{\mathfrak{M}^1})) = \ker\rho^\gamma$ and $\mathrm{pr}_V(\mathrm{Im}(\eth|_{\mathfrak{M}^0})) = \mathrm{Ham}(E,P)$, where the last one follows from Lemma 5.7. This shows that formula (5.37) coincides with (5.36) under the cochain complex isomorphism $\flat_\sigma$ (see the proof of Proposition 5.2).

5.4 Vanishing of the first Poisson cohomology

Here, by using the results of the previous section, we present some sufficient conditions for the vanishing of the first Poisson cohomology.

Let $\pi : E \to B$ be a fiber bundle. Suppose that we start again with a coupling Poisson tensor Π on E with associated geometric data (P, γ, σ). We make the following assumptions. Assume that the *first vertical cohomology group* of P is trivial, that is,

$$\mathrm{Poiss}_V(E, P) = \mathrm{Ham}(E, P). \tag{5.38}$$

It follows from (5.9) and (5.38) that the horizontal lifts of every $u \in \Gamma(TB)$ with respect to two Poisson connections on (E, P) differ by a Hamiltonian vector field. Then, by (5.3) and definition (5.21), we conclude that the coboundary operator $\overline{\partial}^{\gamma}$ is independent of the choice of the Poisson connection γ on $(E \xrightarrow{\pi} B, P)$ and will be simply denoted by $\overline{\partial}$. Therefore, under condition (5.38), one can associate to the Poisson bundle the intrinsic cochain complex $(\mathcal{N}^* = \oplus_p \mathcal{N}^p, \overline{\partial})$. Taking into account (5.38) and (5.30), we conclude that $\ker \rho^{\gamma} = \mathrm{Ham}(E, P)$ and hence,

$$\mathrm{Poiss}(E, \Pi) \cong Z^1_{\overline{\partial}} \oplus \mathrm{Ham}(E, P).$$

So, in this case, formula (5.36) for the first Poisson cohomology of Π reads

$$H^1_{\Pi}(E) \cong H^1_{\overline{\partial}}. \tag{5.39}$$

Next, let us assume that (E, P) is a *flat Poisson bundle*, that is,

$$\text{there exists a } \textit{flat Poisson connection } \gamma^0 \text{ on } (E, P). \tag{5.40}$$

Equivalently, condition (5.40) can be reformulated as follows: there exists a regular foliation $\mathcal{F}$ on E such that

$$TE = T\mathcal{F} \oplus V \quad (V := \ker d\pi) \tag{5.41}$$

and every π-projectable section of $T\mathcal{F}$ is a Poisson vector field on (E, P),

$$L_Z P = 0 \qquad \forall Z \in \Gamma_{\pi\text{-pr}}(T\mathcal{F}). \tag{5.42}$$

Then, the horizontal subbundle H^{γ^0} of the flat Poisson connection γ^0 is just the tangent bundle $T\mathcal{F}$ of the foliation. Recall that $\Gamma_{\pi\text{-pr}}(T\mathcal{F}) = \{\mathrm{hor}^{\gamma^0}(u) \mid u \in \Gamma(TB)\}$ denotes the space of all π-projectable, γ^0-horizontal vector fields on E.

Let $\Omega^p(\mathcal{F}) := \Gamma(\wedge^p T^*\mathcal{F})$ be the space of foliated p-forms on E. In particular, $\Omega^0(\mathcal{F}) = C^\infty(E)$. Consider the *foliated de Rham complex* $(\Omega^\bullet(\mathcal{F}) := \bigoplus_{p\in\mathbb{Z}} \Omega^p(\mathcal{F}), \mathrm{d}_{\mathcal{F}})$, where $d_{\mathcal{F}} : \Omega^p(\mathcal{F}) \to \Omega^{p+1}(\mathcal{F})$ is the *foliated exterior differential* given by the standard formula

$$\begin{aligned}(d_{\mathcal{F}}\mu)(X_0, \dots, X_p) := &\sum_{i=0}^{p} (-1)^i L_{X_i}(\mu(X_0, \dots, \hat{X}_i, \dots, X_p)) \\ &+ \sum_{i<j} (-1)^{i+j} \mu([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).\end{aligned}$$

The cohomology of $(\Omega^*(\mathcal{F}), \mathrm{d}_{\mathcal{F}})$ is called the *foliated de Rham cohomology* and denoted by $H^*_{\mathrm{dR}}(\mathcal{F})$. Observe that $(\Omega^*(\mathcal{F}), \mathrm{d}_{\mathcal{F}})$ is isomorphic to the cochain complex $(\Gamma(\wedge^* V^\circ), d^{\gamma^0}_{1,0})$, where $d^{\gamma^0}_{1,0}$ is the component of bidegree $(1,0)$ of the exterior differential relative to the flat connection γ^0 [65]. More precisely,

$$d^{\gamma^0}_{1,0}\beta(Y_0, \dots, Y_p) := d\beta(p_{T\mathcal{F}}Y_0, \dots, p_{T\mathcal{F}}Y_p)$$

for any $\beta \in \Gamma(\wedge^p V^\circ)$ and vector fields $Y_0, \dots, Y_p$ on E. Here, $p_{T\mathcal{F}} : TE \to T\mathcal{F}$ is the projection along V. It follows from (5.4) and (5.6) that $(\Omega^{*,0}_B(E), \partial^{\gamma^0}_{1,0})$ is a cochain complex is isomorphic to $(\Gamma(\wedge^* V^\circ), d^{\gamma^0}_{1,0})$.

Now let us associate to the flat Poisson bundle $(E \xrightarrow{\pi} B, P, \mathcal{F})$ the following subalgebra of Hamiltonian vector fields:

$$\mathrm{Ham}_{\mathcal{F}}(E, P) := \{Y \in \mathrm{Ham}(E, P) \mid [Y, \Gamma_{\pi\text{-pr}}(T\mathcal{F})] = 0\}.$$

Observe that a Hamiltonian vector field Y on (E, P) belongs to $\mathrm{Ham}_{\mathcal{F}}(E, P)$ if and only if the flow of Y preserves the foliation $\mathcal{F}$. Let $C^\infty_{\mathcal{F}}(E) := \{H \in C^\infty(E) \mid d_{\mathcal{F}}H = 0\}$ be the space of smooth functions on E which are constant along the leaves of $\mathcal{F}$.

Theorem 5.15. *Suppose that in addition to hypotheses* (5.38), (5.40)*, the following conditions hold:*

(i) the first foliated de Rham cohomology group of $(E, \mathcal{F})$ *is trivial,*

$$H^1_{\mathrm{dR}}(\mathcal{F}) = \{0\}; \tag{5.43}$$

(ii) the subalgebra of Hamiltonian vector fields on (E, P) *preserving the foliation* $\mathcal{F}$ *is generated by the subspace* $C^\infty_{\mathcal{F}}(E)$*,*

$$\mathrm{Ham}_{\mathcal{F}}(E, P) = \{P^\sharp dH \mid H \in C^\infty_{\mathcal{F}}(E)\}. \tag{5.44}$$

Then, the first Poisson cohomology of the coupling Poisson tensor Π *vanishes,*

$$H^1_\Pi(E) = \{0\}. \tag{5.45}$$

Proof. First, we observe that condition (5.44) can be reformulated as follows: For every function $f \in C^\infty_E$ with property:

$$L_Z f \in \mathrm{Casim}(E, P) \quad \forall Z \in \Gamma_{\pi\text{-pr}}(T\mathcal{F}), \tag{5.46}$$

there exists $g \in \mathrm{Casim}(E, P)$ such that

$$d_{\mathcal{F}} f = d_{\mathcal{F}} g. \tag{5.47}$$

Indeed, it follows from (5.46), (5.47) that $[P^\sharp df, Z] = -P^\sharp dL_Z f = 0$ and $P^\sharp df = P^\sharp dH$, where $H = f - g \in C^\infty_{\mathcal{F}}(E)$. Now, let us define

$$\bar{\Omega}^q(\mathcal{F}) := \left\{\beta \in \Omega^q(\mathcal{F}) \mid \mathbf{i}_{X_1} \cdots \mathbf{i}_{X_q}\beta \in \mathrm{Casim}(E, P) \quad \forall X_i \in \Gamma_{\pi\text{-pr}}(T\mathcal{F})\right\}.$$

In particular, $\bar{\Omega}^0(\mathcal{F}) = \mathrm{Casim}(E, P)$. Using the property that $\Gamma_{\pi\text{-pr}}(T\mathcal{F}) \subset \mathrm{Poiss}(E, P)$, it easy to see that the foliated differential $d_{\mathcal{F}}$ leaves invariant the subspaces $\bar{\Omega}^q(\mathcal{F})$ of $\Omega^q(\mathcal{F})$ and hence, the coboundary operator $\bar{d}_{\mathcal{F}} := d_{\mathcal{F}}|_{\bar{\Omega}^q(\mathcal{F})}$ is well defined. Then, $(\bar{\Omega}^\bullet(\mathcal{F}) := \oplus_q \bar{\Omega}^q(\mathcal{F}), \bar{d}_{\mathcal{F}})$ is a subcomplex of the cochain complex $(\Omega^*(\mathcal{F}), \mathrm{d}_{\mathcal{F}})$ and there is a natural homomorphism $H^q_{\bar{d}_{\mathcal{F}}} \to H^q_{\mathrm{dR}}(\mathcal{F})$ between the corresponding cohomology groups. One can show that conditions (5.46), (5.47) are equivalent to the following:

$$d_{\mathcal{F}}(\bar{\Omega}^0(\mathcal{F})) = d_{\mathcal{F}}(\Omega^0(\mathcal{F})) \cap \bar{\Omega}^1(\mathcal{F}).$$

This condition means that the natural homomorphism $H^1_{\bar{d}_{\mathcal{F}}} \to H^1_{\mathrm{dR}}(\mathcal{F})$ is injective. Therefore, the hypotheses (i), (ii) of the theorem imply that $H^1_{\bar{d}_{\mathcal{F}}} = \{0\}$. Finally, we observe that the cochain complexes associated with $\bar{d}_{\mathcal{F}}$ and $\bar{\partial} := \partial^{\gamma^0}_{1,0}|_{\mathcal{N}^\bullet}$ are isomorphic and hence, $H^1_{\bar{\partial}}$ is trivial. This, together with (5.39), proves (5.45). ∎

To get more insight for the criterion in Theorem 5.15, let us consider the situation when conditions (5.38), (5.40) are fulfilled and the foliation $\mathcal{F}$ is a fibration. In other words, we assume that the leaf space $K := E/\mathcal{F}$ of the foliation is a smooth manifold and the natural projection $\nu : E \to K$ is a surjective submersion. So, we have $T\mathcal{F} = \ker d\nu$.

Lemma 5.16. *There exists a Poisson structure Υ on K such that the projection $\nu : E \to K$ is a Poisson map. Moreover, condition* (5.44) *is equivalent to the following property: the Hamiltonian vector field $P^\sharp df$ of every function $f \in C^\infty(E)$ such that*

$$[Z, P^\sharp df] = 0 \quad \forall Z \in \Gamma_{\pi\text{-pr}}(T\mathcal{F}), \tag{5.48}$$

is ν-related with a Hamiltonian vector field on (K, Υ),

$$d\nu \circ P^\sharp df = \Upsilon^\sharp dh \circ \nu, \tag{5.49}$$

for a certain $h \in C^\infty(K)$.

Proof. Notice that $C^\infty_{\mathcal{F}}(E) = \nu^* C^\infty(K)$. This and condition (5.42) imply that

$$L_Z(P(d\nu^*\kappa_1, d\nu^*\kappa_2)) = P(dL_Z(\nu^*\kappa_1), d\nu^*\kappa_2) + P(d\nu^*\kappa_1, dL_Z(\nu^*\kappa_2)) = 0$$

for any $Z \in \Gamma_{\pi\text{-pr}}(T\mathcal{F})$ and $\kappa_1, \kappa_2 \in C^\infty(K)$. It follows that there exists a bivector field $\Upsilon \in \Gamma(\wedge^2 TK)$ which is uniquely determined by

$$\nu^*\Upsilon(d\kappa_1, d\kappa_2) = P(d\nu^*\kappa_1, d\nu^*\kappa_2)$$

and satisfies the Jacobi identity. Now, condition (5.48) for $f \in C^\infty(E)$ says that $P^\sharp df \in \mathrm{Ham}_{\mathcal{F}}(E, P)$ and hence (5.46) holds. It remains to show the equivalence of conditions (5.47) and (5.49). Indeed, by (5.48) the Hamiltonian vector field $P^\sharp df$ is ν-related with a vector field $w \in \Gamma(TK)$ which is an infinitesimal automorphism of Υ. Then, it is easy to see that w is Hamiltonian, $w = \Upsilon^\sharp dh$, $h \in C^\infty(K)$ if and only if f satisfies (5.47), where $g = f - \nu^*h \in \mathrm{Casim}(E, P)$. ∎

Observe that the hypotheses (i) and (ii) of Theorem 5.15 are independent in general. Here is a particular case in which condition (5.49) is satisfied but (5.38) or (5.43) do not necessarily hold.

Example 5.17. Let B be a manifold and consider a Poisson manifold K equipped with a Poisson tensor Υ. Let $(E = B \times K, P)$ be the product of Poisson manifolds, where B has the trivial Poisson structure. Let us think of (E, P) as the total space of a trivial Poisson bundle over B with projection $\pi = \mathrm{pr}_1$ and the vertical Poisson structure P. It is clear that P and Υ are pr_2-related and $\ker(\mathrm{pr}_2) \subset TE$ induces a flat Poisson connection for P. Fixing $x_0 \in B$, consider the section $s : K \to E$ of ν given by $s(y) = (x_0, y)$. Pick a function $f \in C^\infty(E)$ satisfying (5.48) and put $h = s^* f$. Then, one can easily verify that (5.49) holds. ▼

By the same arguments as in the proof of Lemma 5.16, we derive the following cohomological criterion.

Lemma 5.18. *Under hypotheses* (5.38), (5.40), *in the case when the foliation $\mathcal{F}$ is a fibration, condition* (5.44) *is equivalent to the triviality of the first Poisson cohomology group of* Υ.

Proof. First assume that $H^1_\Upsilon(K) = \{0\}$ and let $f \in C^\infty_E$ be such that $P^\sharp df$ preserves the foliation $\mathcal{F}$. As in the proof of Lemma 5.16, $P^\sharp df$ is ν-related to some infinitesimal automorphism $w \in \Gamma(TK)$ of Υ. By hypothesis, $w = \Upsilon^\sharp dh$, so $P^\sharp d(\nu^* h) = P^\sharp df$, by the uniqueness of the horizontal lift of $w \in \Gamma(TK)$ in the fibration $\nu : E \to K$ with horizontal distribution $\ker d\pi$. Therefore, condition (5.44) holds. Conversely, let $w \in \Gamma(TK)$ be an infinitesimal automorphism of Υ. If $W \in \Gamma(TE)$ is the horizontal lift of w as described in above, then $[W, P] \in \Gamma(V) \cap \Gamma(\ker d\nu) = \{0\}$, so $W \in \mathrm{Poiss}_V(E, P)$. By (5.38), W is Hamiltonian. Furthermore, it follows from (5.44) that, $W = P^\sharp d(\nu^* h)$ for some $h \in C^\infty(K)$. Hence, $w = \Upsilon^\sharp dh$. ■

Summarizing the above considerations, we arrive at the following result.

Theorem 5.19. *Let $K \xleftarrow{\nu} E \xrightarrow{\pi} B$ be a transversal bi-fibration, that is, ν and π are surjective submersions and*

$$TE = \ker d\nu \oplus \ker d\pi. \tag{5.50}$$

Suppose that ν has connected fibers and satisfies the following compatibility condition with a Poisson tensor $P \in \Gamma(\wedge^2 \ker d\pi)$:

$$\Gamma_{\pi\text{-pr}}(\ker d\nu) \subset \mathrm{Poiss}(E, P). \tag{5.51}$$

Let $\mathcal{F}$ be the regular foliation on E with $T\mathcal{F} = \ker d\nu$ and Υ be a unique Poisson structure on K for which the natural projection $\nu : E \to K$ is a Poisson map. Suppose we are given a coupling Poisson structure $\Pi = \Pi_{2,0} + \Pi_{0,2}$ on the fiber bundle $\pi : E \to B$ with vertical part $\Pi_{0,2} = P$. Then, the first Poisson cohomology group of Π on E is trivial if

$$\frac{\mathrm{Poiss}_V(E, P)}{\mathrm{Ham}(E, P)} = \{0\}, \tag{5.52}$$

$$H^1_{\mathrm{dR}}(\mathcal{F}) = \{0\}, \tag{5.53}$$

$$H^1_\Upsilon(K) = \{0\}. \tag{5.54}$$

We end this section with the following remarks about conditions (5.51)-(5.54). First we notice that if the ν-fibers are simply connected, then condition (5.53) holds (see [17, Subsection 7.4]).

Moreover, by (5.50), the restriction of the surjective submersion ν to each π-fiber $\nu|_{\pi^{-1}(x)}$: $(\pi^{-1}(x), P_x) \to (K, \Upsilon)$ is a local Poisson diffeomorphism. We claim that conditions (5.51) and (5.52) imply (5.54) if the restriction $\nu|_{E_0} : E_0 \to K$ is bijective for a single fiber $E_0 := \pi^{-1}(x_0)$. This fact is based on the following observation: the lifting $W \in \Gamma_{\nu\text{-pr}}(V)$ of every element $w \in \mathrm{Poiss}(K, \Upsilon)$, given as in Lemma 5.18, is a Poisson vector field on (E, P).

5.5 Isotropy algebras of compact semisimple type

The triviality condition (5.38) is realized in the following case. Let $(E \xrightarrow{\pi} B, P)$ be a locally trivial Lie-Poisson bundle whose typical fiber is the co-algebra $(\mathfrak{g}^*, \Lambda)$ of a semisimple Lie algebra $\mathfrak{g}$ of compact type. Recall that this condition means that the Killing form is negative definite or, equivalently, that the connected and simply connected Lie group integrating $\mathfrak{g}$ is compact. Due to [12, Theorem 4.1] (see, also [50, Theorem 1.3.1]), we have $H^1_\Lambda(\mathfrak{g}^*) = 0$. Moreover, there exist the linear homotopy operators for the Poisson complex of $(\mathfrak{g}^*, \delta_\Lambda)$ of degree 1:

$$C^\infty(\mathfrak{g}^*) \xleftarrow{h_0} \Gamma(T\mathfrak{g}^*) \xleftarrow{h_1} \Gamma(\wedge^2 T\mathfrak{g}^*),$$

$$\delta_\Lambda \circ h_0 + h_1 \circ \delta_\Lambda = \mathrm{Id}_{\Gamma(T\mathfrak{g}^*)}\,.$$

Observe that this fact remains true if instead of $\mathfrak{g}^*$ we take an open ball (with respect to the invariant inner product in $\mathfrak{g}^*$) centered at the origin. The existence of the homotopy operators imply the triviality of the parametrized first Poisson cohomology groups of the Lie-Poisson structure Λ. Combining this fact with the partition unity argument, we conclude that the first vertical cohomology group of P is also trivial.

Remark 5.20. *The triviality property of the parametrized first cohomology groups appears also in the context of the* tame Poisson structures*, introduced in [48].*

Now, as an illustration of Theorem 5.19, let us consider the following situation. Let $M = B \times \mathbb{R}^k$ be the product of a compact connected symplectic manifold B and the k-dimensional Euclidean space $\mathbb{R}^k = \{x = (x^1, \dots, x^k)\}$. Let us view M as the total space of the trivial vector bundle over B. Suppose we are given a Poisson tensor Π on M such that the zero section $B \times \{0\}$ is a symplectic leaf of Π. Assume that

$$c^{\alpha\beta}_\sigma := \frac{\partial}{\partial x^\sigma}\Pi(dx^\alpha, dx^\beta)\bigg|_{x=0} = \text{const on } B. \tag{5.55}$$

Then, $c^{\alpha\beta}_\sigma$ are the structure constants of a Lie algebra $\mathfrak{g}$ and condition (5.55) means that the isotropy bundle of the leaf $B \times \{0\}$ is just the trivial Lie bundle $B \times \mathfrak{g}$. Observe that after a change of coordinates on the fiber, condition (5.55) still holds. Combining Theorem 5.19 with Conn's results [12], we establish the following criterion which implies Theorem 5.22.

Proposition 5.21. *If B is simply connected and compact and the isotropy algebra $\mathfrak{g}$ is semisimple of compact type, then there exists an open neighborhood E of $B \times \{0\}$ in $M = B \times \mathbb{R}^k$ such that $H^1_\Pi(E) = \{0\}$.*

Proof. We have to verify that the hypotheses of the proposition imply conditions (5.52)-(5.54). First, we observe that $\Pi = \Pi_{2,0} + \Pi_{0,2}$ is a coupling Poisson structure in a neighborhood E of $B \times \{0\}$ in $M = B \times \mathbb{R}^k$ which is viewed as the total space of the fiber bundle $\pi := \mathrm{pr}_1|_E$ over B. By (5.55), the linearization of the vertical Poisson structure $P = \Pi_{0,2}$ at $B \times \{0\}$ gives

$$P^{(1)} = \frac{1}{2} c^{\alpha\beta}_\sigma x^\sigma \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}. \tag{5.56}$$

By the linearization Conn theorem, for every $b \in B$, the Poisson structure P_b on the fiber E_b around 0 is isomorphic to the Lie-Poisson structure Λ on $\mathfrak{g}^*$. Then, one can show [78] that the neighborhood E

can be chosen in such a way that there exists a fiber preserving diffeomorphism $g : E \to g(E)$ identical on B and $g_*P = P^{(1)}$. So, we obtain the coupling Poisson tensor $g_*\Pi = g_*\Pi_{2,0} + P^{(1)}$ defined on the neighborhood $g(E)$ of B. Then, as we mentioned above, condition (5.52) holds for $P^{(1)}$. Moreover, by the compactness of B, one can arrange the neighborhood E to have $g(E) = B \times K$, where K is an open ball centered at 0. Then, condition (5.54) holds for $\Upsilon = \Lambda$, $H^1_\Lambda(K) = \{0\}$. Finally, by condition (5.55), there exists a flat Poisson connection γ^0 on $(B\times K, P^{(1)})$ associated with the horizontal foliation $\mathcal{F}$ with leaves $B \times \{x\}$, $x \in K$. Then, the foliated de Rham cohomology of $d_{\mathcal{F}}$ is the same thing as the de Rham cohomology of the forms in B depending smoothly on $x \in K$ as a parameter. Since B is simply connected, according to results in [17, 29], we conclude that $H^1_{\mathrm{dR}}(\mathcal{F}) = \{0\}$. ■

Theorem 5.22. *Let $S \subset M$ be an embedded symplectic leaf of the Poisson manifold (M, Ψ) such that the normal bundle of S (viewing as a Lie-Poisson bundle) is trivial. Assume that the isotropy algebra of the symplectic leaf S is a semisimple Lie algebra of compact type. If S is compact and simply connected, then there exists a tubular neighborhood N of S in M such that every Poisson vector field of Ψ is Hamiltonian on N.*

Example 5.23. Consider the case when $E := \mathbb{S}^2 \times \mathbb{R}^3$, where $\mathbb{R}^3 = \{x = (x^1, x^2, x^3)\}$, and the base $B := \mathbb{S}^2 \subset \mathbb{R}^3$ is the unit 2-sphere equipped with the area form $\omega = dp \wedge dq$. Here, the Darboux coordinates p, q can be defined as the azimuthal angle $p = \varphi$ and the height function $q = h$ on the sphere. Then, given a vector valued 1-form ϱ on B,

$$\varrho = \varrho^{(1)}(p, q, x)dp + \varrho^{(2)}(p, q, x)dq,$$

with $\varrho^{(1)}, \varrho^{(2)} \in \mathbb{R}^3$, and a constant $c \in \mathbb{R}$, one can define the following Poisson tensor on E [78]:

$$\Pi_{\varrho,c} = \tfrac{1}{2(1-x\cdot\Delta_\varrho+c\|x\|^2)}\left(\frac{\partial}{\partial p} + (x\times\varrho^{(1)})\cdot\frac{\partial}{\partial x}\right)\wedge\left(\frac{\partial}{\partial q} + (x\times\varrho^{(2)})\cdot\frac{\partial}{\partial x}\right) + \frac{1}{2}\epsilon_{\alpha\beta\gamma}x^\gamma\frac{\partial}{\partial x^\alpha}\wedge\frac{\partial}{\partial x^\beta},$$

where $\Delta_\varrho := \frac{\partial\varrho^{(2)}}{\partial p} - \frac{\partial\varrho^{(1)}}{\partial q} + \varrho^{(1)}\times\varrho^{(1)}$. In this case, $\mathbb{S}^2 \times \{0\}$ is a simply connected, compact symplectic leaf of $\Pi_{\varrho,c}$ whose isotropy Lie algebra is $\mathfrak{g} = \mathfrak{so}(3)$. Therefore, by Proposition 5.21 we conclude that the first cohomology of $\Pi_{\varrho,c}$ vanishes for arbitrary data (ϱ, c). In particular, this is true for the product Poisson structure $\Pi_{0,0}$ on $\mathbb{S}^2 \times \mathfrak{so}^*(3)$. ▼

5.6 Projectability of Casimir functions

Let $(E \xrightarrow{\pi} B, P)$ be again a Poisson bundle and Π a coupling Poisson structure on E with associated geometric data (P, γ, σ). Let us consider another extreme situation, assuming that every Casimir function of the vertical Poisson structure P is *projectable* in the sense that

$$\mathrm{Casim}(E, P) = \pi^*C^\infty(B). \tag{5.57}$$

So, this means that $P^\sharp dF = 0$ if and only if $F = \pi^*f$ for a certain $f \in C^\infty(B)$.

Example 5.24. Let

$$\Lambda = \frac{\partial}{\partial x_1}\wedge\left(x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right) \tag{5.58}$$

be the Lie-Poisson structure on the co-algebra $\mathfrak{g}^* = \mathbb{R}^3$ of the 3-dimensional Lie algebra

$$[e_1, e_2] = e_2, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -e_3. \tag{5.59}$$

In this case, the foliation of $\mathbb{R}^3$ by the symplectic leaves (the co-adjoint orbits) is an open book type foliation. As a consequence, the corresponding Lie-Poisson structure Λ on $\mathfrak{g}^*$ does not admit any global nontrivial Casimir function on $\mathbb{R}^3$, that is, $\mathrm{Casim}(\Lambda, \mathbb{R}^3) = \mathbb{R}$. One can show that the first cohomology group of the Lie-Poisson structure Λ in (5.58) is generated by the Poisson vector fields

$$Z_1 = \frac{\partial}{\partial x_1}, Z_2 = x_2\frac{\partial}{\partial x_2} - x_3\frac{\partial}{\partial x_3}, Z_3 = x_3\frac{\partial}{\partial x_2}, Z_4 = x_2\frac{\partial}{\partial x_3} \tag{5.60}$$

and, hence, isomorphic to $\mathbb{R} \times \mathfrak{sl}(2, \mathbb{R})$ as Lie algebras. ▼

It follows that condition (5.57) holds for any locally trivial Lie-Poisson bundle $(\pi : E \to B, P)$ over B whose typical fiber is just $\mathbb{R}^3$ equipped with linear Poisson bracket (5.58).

Remark 5.25. *The fact that $H^1_\Lambda(\mathbb{R}^3)$ is generated by the basis* (5.60) *can be stated by direct computations. It is of interest to note that in the regular domain $\mathbb{R}^3\backslash\{x_1\text{-axis}\} \cong \mathbb{R}^2 \times \mathbb{S}^1$, the Poisson structure* (5.58) *has nontrivial Casimir functions and, as a consequence, the first Poisson cohomology group is infinite dimensional and isomorphic to $C^\infty(\mathbb{S}^1)$ [25]. More examples of explicit computations of the first cohomology of low-dimensional Poisson manifolds with singularities can be also found in [49, 54].*

Now, we observe that the condition (5.57) implies that the cochain complex $(\mathcal{N}^\bullet, \overline{\partial}^\gamma)$ is isomorphic to the de Rham complex $(\Omega^*(B), \mathrm{d}_B)$ on the base B. Therefore, in this case we have $H^k_{\overline{\partial}^\gamma} \cong H^k_{\mathrm{dR}}(B)$ $\forall k \geq 0$.

Proposition 5.26. *Let Π be a coupling Poisson structure on the fiber bundle $E \xrightarrow{\pi} B$ with associated geometric data (P, γ, σ). If, in addition to* (5.57)*, the second de Rham cohomology of the base B is trivial,*

$$H^2_{\mathrm{dR}}(B) = \{0\}, \tag{5.61}$$

then

$$\mathrm{Poiss}(E, \Pi) \cong \Omega^1_{\mathrm{cl}}(B) \oplus \mathcal{A}^\gamma, \tag{5.62}$$

and the first Poisson cohomology of Π is of the form

$$H^1_\Pi(E) \cong H^1_{\mathrm{dR}}(B) \oplus \frac{\mathcal{A}^\gamma}{\mathrm{Ham}(E, P)}. \tag{5.63}$$

Here, $\mathcal{A}^\gamma$ is a Lie subalgebra of vertical Poisson vector fields of P defined in (5.25).

Proof. If (5.61) holds, then $H^2_{\overline{\partial}_{1,0}} = \{0\}$. Therefore, by definition (5.29), we get $\ker \rho^\gamma = \mathcal{A}^\gamma$. Hence, the decompositions (5.34) and (5.36) coincide with (5.62) and (5.63), respectively. ■

As a consequence of Proposition 5.26, hypotheses (5.57) and (5.61) imply that the properties of the first Poisson cohomology of Π are controlled by the γ-dependent Lie algebra $\mathcal{A}^\gamma$. In fact, the algebra $\mathcal{A}^\gamma$ depends on an equivalence class of the Poisson connection γ on $(E \xrightarrow{\pi} B, P)$. Indeed, suppose we have another Poisson connection $\widetilde{\gamma}$ on $(E \xrightarrow{\pi} B, P)$ which is equivalent to γ in following sense: there exists a 1-form $\varrho \in \Omega^{1,0}_B(E)$ such that $\mathrm{hor}^{\widetilde{\gamma}}(u) = \mathrm{hor}^{\gamma}(u) + P^{\sharp} d\varrho(u)$, for every $u \in \Gamma(TB)$. Then, $\widetilde{\gamma} \sim \gamma$ implies $\mathcal{A}^{\widetilde{\gamma}} = \mathcal{A}^{\gamma}$.

It is useful also to single out the Lie subalgebra of $\mathcal{A}^{\gamma}$ consisting of all vertical Poisson vector fields on (E, P) that preserve the horizontal subbundle of γ,

$$\mathcal{A}^{\gamma}_0 := \{Y \in \mathrm{Poiss}_V(E, P) \mid [\mathrm{hor}^{\gamma}(u), Y] = 0 \ \forall u \in \Gamma(TB)\}.$$

Then, taking into account (5.57), we get

$$\mathrm{Ham}_0(E, P) := \mathcal{A}^{\gamma}_0 \cap \mathrm{Ham}(E, P) = \{P^{\sharp} dF \mid L_{\mathrm{hor}^{\gamma}(u)} F \in \pi^* C^{\infty}(B) \ \forall u \in \Gamma(TB)\}.$$

An interesting situation occurs when

$$\mathcal{A}^{\gamma} \cong \frac{\mathcal{A}^{\gamma}_0}{\mathrm{Ham}_0(E, P)} \oplus \mathrm{Ham}(E, P). \tag{5.64}$$

In this case,

$$H^1_{\Pi}(E) \cong H^1_{\mathrm{dR}}(B) \oplus \frac{\mathcal{A}^{\gamma}_0}{\mathrm{Ham}_0(E, P)}. \tag{5.65}$$

Lemma 5.27. *Condition* (5.64) *is equivalent to the following: every $Y \in \mathcal{A}^{\gamma}$ admits the decomposition*

$$Y = P^{\sharp} dG + Y_0, \tag{5.66}$$

where $Y_0 \in \mathcal{A}^{\gamma}_0$ and $G \in C^{\infty}(E)$. Furthermore, given $\beta \in \Omega^1(B) \otimes_{C^{\infty}(B)} C^{\infty}(E)$ in (5.26), *there exists $c \in \Omega^1(B)$ such that*

$$\partial^{\gamma}_{1,0} G = \beta - c \otimes 1. \tag{5.67}$$

Next, let us consider the following particular case. Let $E = B \times K$ be the product of a manifold B equipped with zero Poisson structure and a Poisson manifold (K, Υ). Let P be the product Poisson structure on E. Then, we have the trivial Poisson bundle $(E = B \times K, P)$ over B with projection $\pi_B = \mathrm{pr}_1$ and the typical fiber (K, Υ). Consider the trivial Poisson connection γ^0 on E associated with the canonical horizontal distribution $\ker(d\pi_K)$, where $\pi_K = \mathrm{pr}_2$.

Proposition 5.28. *Let Π be a compatible coupling Poisson tensor on the trivial Poisson bundle $(E = B \times K, P)$ in the sense that $\Pi_{0,2} = P$ and the associated Poisson connection γ is equivalent to the trivial one, $\gamma \sim \gamma^0$. Assume that B is connected,*

$$\mathrm{Casim}(K, \Upsilon) = \mathbb{R}, \tag{5.68}$$

condition (5.61) *holds, and $\mathcal{A}^{\gamma^0}$ admits splitting* (5.64)*. Then,*

$$H^1_{\Pi}(E) \cong H^1_{\mathrm{dR}}(B) \oplus H^1_{\Upsilon}(K). \tag{5.69}$$

Proof. By the connectedness of B, it is easy to see that $\mathcal{A}^{\gamma^0}_0 \cong \mathrm{Poiss}(K, \Upsilon)$, where the isomorphism is given by the horizontal lift in $E \xrightarrow{\pi_K} K$ with horizontal distribution $\ker d\pi_B$. Moreover, we claim that $\mathrm{Ham}_0(E, P) \cong \mathrm{Ham}(K, \Upsilon)$. Indeed, pick a $Y = P^{\sharp} dF \in \mathrm{Ham}_0(E, P)$. Since hypothesis (5.68) implies (5.57), we conclude that $L_{\mathrm{hor}^{\gamma^0}(u)} F \in \pi_B^* C^{\infty}(B)$ for any $u \in \Gamma(TB)$. Then, fixing $y^0 \in K$, we see that the function $\tilde{F} \in C^{\infty}(E)$ given by $\tilde{F}(x, y) = F(x, y) - F(x, y^0)$ for $x \in B, y \in K$ is of the form $\tilde{F} = \pi_K^* f$ for a certain $f \in C^{\infty}(K)$. Consequently, Y is π_K-related with the Hamiltonian vector field $\Upsilon^{\sharp} df$. ∎

Example 5.29. Consider the trivial Poisson bundle $(\pi = \mathrm{pr}_1 : E = B \times \mathbb{R}^3, P)$, where the base $B = \mathbb{R}^1 \times \mathbb{S}^1 = \{(t, \varphi \bmod 2\pi)\}$ is the 2-cylinder and the typical fiber $(\mathbb{R}^3, \Lambda)$ is given by the Lie-Poisson structure (5.58). We already know that in this case, the projectability condition (5.57) holds. It is clear that (5.61) is also satisfied. By analyzing equations (5.26), (5.66), (5.67), one can show that decomposition (5.64) is true for $\mathcal{A}^{\gamma^0}$. In this case, an arbitrary compatible coupling Poisson structure on the trivial Poisson bundle E such that $\gamma \sim \gamma^0$ has the form

$$\Pi_\varrho = \tfrac{1}{2(1-\psi(x)\cdot\Delta_\varrho)} \left(\frac{\partial}{\partial t} + (\psi \times \varrho^{(1)}) \cdot \frac{\partial}{\partial x}\right) \wedge \left(\frac{\partial}{\partial \varphi} + (\psi \times \varrho^{(2)}) \cdot \frac{\partial}{\partial x}\right) + \frac{\partial}{\partial x_1} \wedge \left(x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right),$$

where $\psi = (0, -x_3, x_2)$, $\varrho = \varrho^{(1)}(t, \varphi, x)dt + \varrho^{(2)}(t, \varphi, x)d\varphi$ is an arbitrary horizontal 1-form on E and

$$\Delta_\varrho := \frac{\partial \varrho^{(2)}}{\partial t} - \frac{\partial \varrho^{(1)}}{\partial \varphi} + \varrho^{(1)} \times \varrho^{(2)}$$

is the Hamiltonian of the curvature of γ [78]. Applying Proposition 5.28 to Π_ϱ and taking into account Example 5.24, we get

$$H^1_{\Pi_\varrho}(E) \cong \mathbb{R} \oplus \mathbb{R}^4.$$

▼

5.7 First cohomology of coupling Dirac structures

An example. Let (M, ω) be a presymplectic manifold and (N, Ψ) the Poisson manifold given by $N = \mathbb{R}^2_y$, $\Psi = \|y\|^2 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}$. Consider the product Dirac structure D on $M \times N$. Then, $M \times \{0\}$ is a presymplectic leaf of D and we can think of $M \times N \overset{\mathrm{pr}_M}{\to} M$ as a coupling neighborhood. More precisely, the vertical distribution is $V := \ker(\mathrm{pr}_M)_*$, and the associated geometric data (P, γ, σ) consists of the flat connection $\gamma := (\mathrm{pr}_N)_*$, given by the differential of the projection $\mathrm{pr}_N : M \times N \to N$; the pullback $\sigma := \mathrm{pr}_N^* \,\omega$ of the presymplectic structure on M; and the unique vertical Poisson bivector field P on $M \times N$ which is pr_N-related to Ψ, $P = \|y\|^2 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}$ on $M \times N$.

First we note that (N, Ψ) has two symplectic leaves: the origin $(0, 0)$, which is zero-dimensional, and the complement $N_{\mathrm{reg}} := \mathbb{R}^2 - \{(0, 0)\}$. Then, each Casimir funcion of (N, Ψ) is constant, $\mathrm{Casim}(N, \Psi) \cong \mathbb{R}$. This implies that $\mathrm{Casim}(M \times N, P) = \mathrm{pr}_M^* \, C^\infty(M)$ and

$$H^0(M \times N, D) \cong H^0_{\mathrm{dR}}(M).$$

Observe that in this case, the de Rham - Casimir complex $(\mathcal{N}^\bullet, \overline{\partial})$ is isomorphic to the de Rham complex $(\Gamma(\wedge^\bullet T^*M), \mathrm{d})$ of M. In particular, $H^1(\mathcal{N}^\bullet, \overline{\partial}) \cong H^1_{\mathrm{dR}}(M)$. We now proceed to describe $\ker(\rho_1 : \mathcal{A}^1 \to H^2(\mathcal{N}^\bullet, \overline{\partial}))$. Let $Y = Y_1 \frac{\partial}{\partial y_1} + Y_2 \frac{\partial}{\partial y_2} \in \Gamma(V)$ be a vertical vector field. Then, $\mathrm{L}_Y P = 0$ if and only if

$$y_1 Y_1 + y_2 Y_2 = \tfrac{1}{2} \|y\|^2 \operatorname{div}^y(Y). \tag{5.70}$$

Here, $\operatorname{div}^y(Y) := \frac{\partial Y_1}{\partial y_1} + \frac{\partial Y_2}{\partial y_2}$ denotes the divergence of Y with respect to the fiber-wise volume form $\mathrm{d}\, y_1 \wedge \mathrm{d}\, y_2$. In particular, the fiber-wise Euler and modular vector fields

$$Z_1 := y_1 \tfrac{\partial}{\partial y_1} + y_2 \tfrac{\partial}{\partial y_2} \quad \text{and} \quad Z_2 := -y_2 \tfrac{\partial}{\partial y_1} + y_1 \tfrac{\partial}{\partial y_2} \tag{5.71}$$

are Poisson vector fields of P, respectively.

Proposition 5.30. *Consider the Poisson manifold $(M \times N, P)$ given as in above. Let Z_1, Z_2 be defined as in (5.71), and $Y \in \Gamma(V)$. Then, $Y \in \mathrm{Poiss}(M \times N, P)$ if and only if $Y = a_1 Z_1 + a_2 Z_2$ for unique $a_1, a_2 \in C^\infty(M)$ satisfying $L_{Z_1} a_1 + L_{Z_2} a_2 = 0$. Additionally, Y is Hamiltonian if and only if a_1 and a_2 vanish along the zero section $M \times \{0\}$. In this case, a Hamiltonian is given by $h(x,y) := \int_0^\infty a_2(x, e^{-t}y)\,\mathrm{d}\,t$. Hence, the first vertical Poisson cohomology of the Poisson bundle $(M \times N \overset{\mathrm{pr}_M}{\to} M, P)$ is*

$$H^1(M \times N, V, P) = C^\infty(M).[Z_1] \oplus C^\infty(M).[Z_2].$$

Proof. Observe from (5.70) that, for smooth functions $a_1, a_2 \in C^\infty(M \times N)$,

$$Y = a_1 Z_1 + a_2 Z_2 \tag{5.72}$$

is an infinitesimal Poisson automorphism of P if and only if $\mathrm{L}_{Z_1} a_1 + \mathrm{L}_{Z_2} a_2 = 0$. We claim that every vertical infinitesimal Poisson automorphism of P is of this form. Indeed, fix $Y \in \mathrm{Poiss}(M \times N, P) \cap \Gamma(V)$. Since Z_1, Z_2 are linearly independent on $M \times N_{\mathrm{reg}}$, there exist $b_1, b_2 \in C^\infty(M \times N_{\mathrm{reg}})$ such that $Y|_{M \times N_{\mathrm{reg}}} = b_1 Z_1 + b_2 Z_2$. Explicitly, $b_1 := \frac{1}{\|y\|^2}(y_1 Y_1 + y_2 Y_2)$ and $b_2 := \frac{1}{\|y\|^2}(-y_2 Y_1 + y_1 Y_2)$. We just need to show that b_1 and b_2 can be extended to some smooth functions a_1 and a_2 on $M \times N$. By (5.70), b_1 can be extended to the smooth function $a_1 := \frac{1}{2}\operatorname{div}^y(Y)$ on $M \times N$. Moreover, $Y_2 - a_1 y_2$ is a smooth function on $M \times N$ such that $Y_2 - a_1 y_2|_{M \times N_{\mathrm{reg}}} = b_2 y_1$. This implies that $Y_2 - a_1 y_2$ vanishes along the level set $y_1 = 0$ of $M \times N$. Hence, there exists $a_2 \in C^\infty(M \times N)$ such that $Y_2 - a_1 y_2 = a_2 y_1$. Hence, a_2 is a smooth function on $M \times N$ whose restriction to $M \times N_{\mathrm{reg}}$ is b_2. Finally, since such extensions are clearly unique, we conclude that every Poisson vector field Y admits a unique representation of the form (5.72), with $a_1, a_2 \in C^\infty(M \times N)$ satisfying $\mathrm{L}_{Z_1} a_1 + \mathrm{L}_{Z_2} a_2 = 0$.

Now, pick a Hamiltonian vector field $Y = P^\sharp \,\mathrm{d}\, h$, $h \in C^\infty(M \times N)$. By straightforward computations, the smooth functions a_1 and a_2 in (5.72) are given in this case by

$$a_1 = -\mathrm{L}_{Z_2} h \quad \text{and} \quad a_2 = \mathrm{L}_{Z_1} h. \tag{5.73}$$

In particular, a_1 and a_2 vanish on the symplectic leaf $M \times \{0\}$. We now see that the converse is also true: If $Y = a_1 Z_1 + a_2 Z_2 \in \mathrm{Poiss}(M \times N, P) \cap \Gamma(V)$ is such that $a_1(x,0) = a_2(x,0) = 0$, then $Y \in \mathrm{Ham}(M \times N, P)$. To see this, pick $Y = a_1 Z_1 + a_2 Z_2$ such that $\mathrm{L}_{Z_1} a_1 + \mathrm{L}_{Z_2} a_2 = 0$ with a_1, a_2 vanishing on $M \times \{0\}$. Define $h : M \times N \to \mathbb{R}$ by

$$h(x,y) := \int_0^\infty a_2(x, e^{-t}y)\,\mathrm{d}\,t. \tag{5.74}$$

Then, $h \in C^\infty(M \times N)$ and clearly satisfies $\mathrm{L}_{Z_1} h = a_2$. Moreover,

$$\mathrm{L}_{Z_1} a_1 = -\mathrm{L}_{Z_2} a_2 = -\mathrm{L}_{Z_2} \mathrm{L}_{Z_1} h = -\mathrm{L}_{Z_1} \mathrm{L}_{Z_2} h,$$

so $\mathrm{L}_{Z_1}(a_1 + \mathrm{L}_{Z_2} h) = 0$, which implies that $a_1 + \mathrm{L}_{Z_2} h$ is constant along the pr_M-fibers. By hypothesis, both a_1 and $\mathrm{L}_{Z_2} h$ vanish on $M \times \{0\}$. Hence, $a_1 + \mathrm{L}_{Z_2} h = 0$. Therefore, h is a solution of (5.73) and so is a Hamiltonian for Y. ■

We now describe the subspace $\mathcal{A}^1$. For $Y \in \mathrm{Poiss}(M \times N, P) \cap \Gamma(V)$, we have that $Y \in \mathcal{A}^1$ if and only if there exists a horizontal 1-form $\alpha \in \Gamma(V^\circ)$ such that $\partial^\gamma_{1,0} Y + \partial^P_{0,1}\alpha = 0$. Since γ is the trivial (flat) connection, the horizontal pr_M-projectable vector fields u are locally written in the form $u = u_i\frac{\partial}{\partial x_i}$. Therefore, the relation between Y and α reads $[u, Y] = -P^\sharp\,\mathrm{d}[\alpha(u)]$. Since Y is a vertical infinitesimal automorphism of P, there exists $a_1, a_2 \in C^\infty(M\times N)$ such that (5.72) holds and $Y = a_1Z_1 + a_2Z_2$. Then, $[u, Y] = (\mathrm{L}_u\, a_1)Z_1 + (\mathrm{L}_u\, a_2)Z_2$. Because of Proposition 5.30, $[u, Y]$ is Hamiltonian if and only if a_1 and a_2 are constant along the zero section $M\times\{0\}$. Hence,

$$\frac{\mathcal{A}^1}{\mathrm{Ham}(M\times N, P)} = \mathbb{R}.[Z_1]\oplus\mathbb{R}.[Z_2].$$

Observe also that the Hamiltonian function $\alpha(u)$ of $-[u, Y]$ can be given by the formula

$$\alpha(u(x,y)) = -\int_0^\infty (\mathrm{L}_u\, a_2)(x, e^{-t}y)\,\mathrm{d}\,t.$$

Furthermore, we have $\alpha = -\eth^\gamma_{1,0}h$, where h is given as in (5.74). This follows from the identity

$$(\mathrm{L}_u\, h)(x,y) = \int_0^\infty (\mathrm{L}_u\, a_2)(x, e^{-t}y)\,\mathrm{d}\,t \qquad \forall u.$$

Therefore, the flatness of γ implies $\eth^\gamma_{1,0}\alpha = -(\eth^\gamma_{1,0})^2h = -\mathrm{L}_{R^\gamma}\, h = 0$. Finally, we have $\eth^\sigma_{2,-1}Y = \mathrm{L}_Y\,\mathrm{pr}_M^*\,\omega = 0$. In consequence, $\rho_1(Y) = 0$ for all $Y\in\mathcal{A}^1$, so $\ker\rho_1 = \mathcal{A}^1$.

Theorem 5.31. *The first cohomology of the Dirac manifold $M\times N$ given by the product of a presymplectic manifold (M,ω) with the Poisson manifold $(N = \mathbb{R}^2_y, \Psi = \|y\|^2\frac{\partial}{\partial y_1}\wedge\frac{\partial}{\partial y_2})$ is*

$$H^1(M\times N, D)\cong H^1_{\mathrm{dR}}(M)\oplus H^1(N,\Psi).$$

Proof. Because of our above discussion, $H^1(\mathcal{N}^\bullet, \overline{\eth})\cong H^1_{\mathrm{dR}}(M)$, and $\frac{\ker\rho_1}{\mathrm{Ham}(M\times N,P)} = \mathbb{R}.[Z_1]\oplus\mathbb{R}.[Z_2]$. Therefore,

$$H^1(M\times N, D)\cong H^1_{\mathrm{dR}}(M)\oplus(\mathbb{R}.[Z_1]\oplus\mathbb{R}.[Z_2]).$$

Finally, the fact that $H^1(N,\Psi)\cong\mathbb{R}.[Z_1]\oplus\mathbb{R}.[Z_2]$ follows from Proposition 5.30 with M consisting of a single point (see also [49]). ■

5.8 Detailing the open book foliation case

The main goal of this section is to present a general scheme which allows to compute the first Poisson cohomology group of certain linear Poisson structures in $\mathbb{R}^n$. More precisely, we aim to study coordinate conditions for a vector field to be an infinitesimal Poisson automorphism.

General setup. Let $X := X_2\frac{\partial}{\partial x_2} + \cdots + X_n\frac{\partial}{\partial x_n}$ a *flat vector field* in $\mathbb{R}^n$, i.e., the coefficients $X_2,\ldots,X_n$ does not depend on the variable x_1:

$$\frac{\partial X_j}{\partial x_1} = 0, \qquad \forall j = 2,\ldots,n.$$

Now, consider the bivector field

$$\Pi := \frac{\partial}{\partial x_1}\wedge X.$$

Since X is a flat vector field, $[X, \frac{\partial}{\partial x_1}] = 0$. Therefore,

$$[\Pi,\Pi] = [\Pi, \tfrac{\partial}{\partial x_1}] \wedge X - \tfrac{\partial}{\partial x_1} \wedge [\Pi, X] = \tfrac{\partial}{\partial x_1} \wedge [X, \tfrac{\partial}{\partial x_1}] \wedge X - \tfrac{\partial}{\partial x_1} \wedge [\tfrac{\partial}{\partial x_1}, X] \wedge X = 0,$$

proving that Π is a linear Poisson structure in $\mathbb{R}^n$. Let us describe its infinitesimal Poisson automorphisms.

Let $Z = Z_1\frac{\partial}{\partial x_1} + \cdots + Z_n\frac{\partial}{\partial x_n}$ be a Poisson vector field in the Poisson manifold $(\mathbb{R}^n, \Pi)$. Then,

$$\begin{aligned}
0 = [Z,\Pi] &= [Z, \tfrac{\partial}{\partial x_1}] \wedge X + \tfrac{\partial}{\partial x_1} \wedge [Z, X] \\
&= \left[Z_i\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_1}\right] \wedge \left(X_j\frac{\partial}{\partial x_j}\right) + \frac{\partial}{\partial x_1} \wedge \left[Z_i\frac{\partial}{\partial x_i}, X_j\frac{\partial}{\partial x_j}\right] \\
&= -X_j\frac{\partial Z_i}{\partial x_1}\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_1} \wedge \left(Z_i\frac{\partial X_j}{\partial x_i}\frac{\partial}{\partial x_j} - X_j\frac{\partial Z_i}{\partial x_j}\frac{\partial}{\partial x_i}\right) \\
&= \left(X_l\frac{\partial Z_k}{\partial x_1} - X_k\frac{\partial Z_l}{\partial x_1}\right)\frac{\partial}{\partial x_l} \wedge \frac{\partial}{\partial x_k} \\
&\quad + \left(-X_k\frac{\partial Z_1}{\partial x_1} + Z_j\frac{\partial X_k}{\partial x_j} - X_j\frac{\partial Z_k}{\partial x_j}\right)\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_k}
\end{aligned}$$

where $i = 1, \ldots, n$, $j = 2, \ldots, n$, $l < k$. Therefore, the vector field Z is Poisson for Π if and only if

$$X_l\frac{\partial Z_k}{\partial x_1} - X_k\frac{\partial Z_l}{\partial x_1} = 0, \quad l, k = 2, \ldots, n, \tag{5.75}$$

$$-X_k\frac{\partial Z_1}{\partial x_1} + Z_j\frac{\partial X_k}{\partial x_j} = X_j\frac{\partial Z_k}{\partial x_j} \quad k = 2, \ldots, n. \tag{5.76}$$

Now, define $\Delta_{lk} := X_lZ_k - X_kZ_l$. Since $\frac{\partial X_i}{\partial x_1} = 0$, equation (5.75) is equivalent to

$$\frac{\partial \Delta_{lk}}{\partial x_1} = 0.$$

On the other hand, by applying (5.76) on the third step,

$$\begin{aligned}
\mathrm{L}_X\,\Delta_{lk} &= \mathrm{L}_X\,X_lZ_k - \mathrm{L}_X\,X_kZ_l + X_l\,\mathrm{L}_X\,Z_k - X_k\,\mathrm{L}_X\,Z_l \\
&= \mathrm{L}_X\,X_lZ_k - \mathrm{L}_X\,X_kZ_l + X_l\left(X_j\frac{\partial Z_k}{\partial x_j}\right) - X_k\left(X_j\frac{\partial Z_l}{\partial x_j}\right) \\
&= \mathrm{L}_X\,X_lZ_k - \mathrm{L}_X\,X_kZ_l + X_l\left(-X_k\frac{\partial Z_1}{\partial x_1} + Z_j\frac{\partial X_k}{\partial x_j}\right) - X_k\left(-X_l\frac{\partial Z_1}{\partial x_1} + Z_j\frac{\partial X_l}{\partial x_j}\right) \\
&= \mathrm{L}_X\,X_lZ_k - \mathrm{L}_X\,X_kZ_l + X_lZ_j\frac{\partial X_k}{\partial x_j} - X_kZ_j\frac{\partial X_l}{\partial x_j} \\
&= Z_kX_j\frac{\partial X_l}{\partial x_j} - Z_lX_j\frac{\partial X_k}{\partial x_j} + X_lZ_j\frac{\partial X_k}{\partial x_j} - X_kZ_j\frac{\partial X_l}{\partial x_j} \\
&= \Delta_{lj}\frac{\partial X_k}{\partial x_j} + \Delta_{jk}\frac{\partial X_l}{\partial x_j}.
\end{aligned}$$

Thus, the functions Δ_{lk} satisfy the following system of $\binom{n-1}{2}$ PDE,

$$\mathrm{L}_X\,\Delta_{lk} = \Delta_{lj}\frac{\partial X_k}{\partial x_j} + \Delta_{jk}\frac{\partial X_l}{\partial x_j}, \tag{5.77}$$

where the sum is taken over all indexes $j = 2, \ldots, n$ and $\Delta_{lk} \in C^\infty_{\mathbb{R}^n}$.

Diagonal vector fields an their associated PDE's. Let $X = d_2x_2\frac{\partial}{\partial x_2}+\ldots+d_nx_n\frac{\partial}{\partial x_n}$ be a linear diagonal vector field, with $d_i > 0$. Then, the system (5.77) takes the form

$$\mathrm{L}_X \Delta_{lk} = (d_k + d_l)\Delta_{lk},$$

which is disengaged. Let us study the solutions for $\Delta_{lk} \in C^\infty_{\mathbb{R}^{n-1}}$, where $\mathbb{R}^{n-1} = \{(x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$.

Lemma 5.32. *Consider the PDE*

$$d_2x_2\frac{\partial u}{\partial x_2} + \ldots + d_nx_n\frac{\partial u}{\partial x_n} = ru,$$

where $r \in \mathbb{R}$ is constant.

1. *If $r = 0$, then the only solutions are constants.*
2. *If u is a solution for the constant r, then $\frac{\partial u}{\partial x_j}$ is a solution for the constant $r - d_j$.*

Proof. Fix any $p = (p_2, \ldots, p_n) \in U$ and define

$$\alpha(t) := (e^{d_2t}p_2, \ldots, e^{d_nt}p_n)$$

Then by the chain rule

$$\begin{aligned}\frac{d}{dt}u(\alpha(t)) &= \sum_{i=2}^n \frac{\partial u}{\partial x_i}(\alpha(t))\frac{d}{dt}(e^{d_it}p_i) = \sum_{i=2}^n \frac{\partial u}{\partial x_i}(\alpha(t)) \cdot (d_ie^{d_it}p_i)\\ &= \sum_{i=2}^n d_ix_i(\alpha(t))\frac{\partial u}{\partial x_i}(\alpha(t)) = ru(\alpha(t)).\end{aligned}$$

Therefore, $u(\alpha(t)) = u(\alpha(0))e^{rt} = u(p)e^{rt}$.

1. If $r = 0$, then u is constant along the trajectory $\alpha(t)$. Since $d_i > 0$ for all $i = 2, \ldots, n$, then

   $$u(0) = \lim_{t\to-\infty} u(\alpha(t)) = u(\alpha(0)) = u(p),$$

   so u is constant since $p \in \mathbb{R}^{n-1}$ is arbitrary.

2. Since the partial derivatives commute,

   $$r\frac{\partial u}{\partial x_j} = \frac{\partial}{\partial x_j}\sum_{i=1}^n d_ix_i\frac{\partial u}{\partial x_i} = d_j\frac{\partial u}{\partial x_j} + \sum_{i=1}^n d_ix_i\frac{\partial}{\partial x_i}\frac{\partial u}{\partial x_j},$$

   which implies the desired result.

∎

Now take $d_2 = \ldots = d_n = 1$. In this case, we get

$$X = x_2\frac{\partial}{\partial x_2} + \ldots + x_n\frac{\partial}{\partial x_n},$$

which is the Euler vector field E in the variables $x_2, \ldots, x_n$. In this case, Δ_{lk} must be a solution of

$$\mathrm{L}_E \Delta_{lk} = 2\Delta_{lk}. \tag{5.78}$$

Lemma 5.33. *Let $u : U \subset \mathbb{R}^n \to \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$. Consider the Partial Differential Equation*

$$\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = ku. \tag{5.79}$$

1. *The solutions of* (5.79) *are homogenous functions of degree k.*
2. *If u is a solution of* (5.79)*, then $v = \frac{\partial u}{\partial x_j}$ is a solution of*
$$\sum_{i=1}^{n} x_i \frac{\partial v}{\partial x_i} = (k-1)v.$$
3. *If $U = \mathbb{R}^n$, then the solutions of* (5.79) *are the homogenous polynomials of degree k.*

Proof. By repeating arguments in Lemma 5.32, the solutions of the equation in (5.79) must satisfy $u(\alpha(t)) = u(p)e^{kt}$, where

$$\alpha(t) := e^t p = (e^t p_2, \dots, e^t p_n).$$

1. Note that for each $\lambda > 0$, $u(\lambda p) = u(\alpha(\ln(\lambda))) = u(p)e^{k\ln(\lambda)} = \lambda^k u(p)$, so u is an homogenous function of degree k. Conversely, if u is an homogeneous function of degree k, then the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = u(e^t p) - e^{kt}u(p)$ is identically zero. Thus,
$$0 = \frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(e^t p)\frac{d}{dt}e^t p_i - ke^{kt}u(p).$$
Evaluating at $t = 0$, we get the desired result.
2. This follows from Lemma 5.32.
3. We will prove this result by induction on the degree of homogeneity. If $k = 0$, Lemma 5.32 implies that u is constant, i.e., a polynomial of degree zero. Suppose that the only solutions of (5.79) are polynomials of degree k. If u is a solution of
$$\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = (k+1)u,$$
then, by the previous part, each of their partial derivatives $\frac{\partial u}{\partial x_j}$ are a solution of (5.79), i.e., are homogenous polynomials of degree k. This implies that u is a homogenous polynomial of degree $k+1$. By induction, we complete the proof.

■

Lemma 5.34. *The Partial Differential Equation*

$$x_2 \frac{\partial u}{\partial x_2} + \cdots + x_n \frac{\partial u}{\partial x_n} = k \tag{5.80}$$

has a solution defined on all $\mathbb{R}^{n-1}$ if and only if $k = 0$.

Proof. Clearly, the equation (5.80) has a solution if $k = 0$. Conversely, evaluating both sides of (5.80) at $p = 0$, we get

$$k = x_2|_0 \frac{\partial u}{\partial x_2}(0) + \cdots + x_n|_0 \frac{\partial u}{\partial x_n}(0) = 0.$$

■

Lemma 5.35. *Let $F \in C^{\infty}_{\mathbb{R}^{n-1}}$ a smooth function. The Partial Differential Equation*

$$x_2 \frac{\partial u}{\partial x_2} + \ldots + x_n \frac{\partial u}{\partial x_n} = F(x) \tag{5.81}$$

has a solution if and only if and only if $F(0) = 0$. In this case, the solution is unique up to adding constants.

Proof. For a fixed $x \in \mathbb{R}^{n-1}$, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} \frac{F(tx)-F(0)}{t} & \text{if } t \neq 0 \\ F'(0) & \text{if } t = 0. \end{cases}$$

Since F is smooth, so it is f. Hence, $u : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$u(x) = \int_0^1 \frac{F(tx) - F(0)}{t} dt$$

is smooth. Furthermore,

$$\frac{\partial u}{\partial x_i} = \int_0^1 \frac{\partial}{\partial x_i} \frac{F(tx) - F(0)}{t} dt = \int_0^1 \frac{\partial F}{\partial x_i}(tx) dt.$$

Hence,

$$\begin{aligned} x_2 \frac{\partial u}{\partial x_2} + \ldots + x_n \frac{\partial u}{\partial x_n} &= \int_0^1 \left(x_2 \frac{\partial F}{\partial x_2}(tx) + \ldots + x_n \frac{\partial F}{\partial x_n}(tx) \right) dt = \int_0^1 \frac{d}{dt} F(tx) dt \\ &= F(1 \cdot x) - F(0 \cdot x) = F(x) - F(0). \end{aligned}$$

We have shown that there exist a solution for the function $F(x) - F(0)$. Hence, in order to have a solution for (5.81), it must exist a solution for the constant function $k = F(0)$. By Lemma 5.34, this occurs if and only if $F(0) = 0$. ■

Remark 5.36. *Note that the previous construction allow us to establish the following result: If $f : \mathbb{R} \to \mathbb{R}$, then*

$$f(x) - f(0) = x \frac{d}{dx} \int_0^1 \frac{f(tx)}{t} dt.$$

Splitting of the Infinitesimal Automorphisms. Recall that in order to parameterize all the Poisson vector fields for the Poisson manifold $(\mathbb{R}^n, \Pi = \frac{\partial}{\partial x_1} \wedge E)$, we have to establish conditions for Z_k and Z_l in order to

$$x_l Z_k - x_k Z_l$$

be a solution of (5.78).

Lemma 5.37. *Fix $k, l = 2, \ldots, n$, and consider the map $M : C^\infty_{\mathbb{R}^{n-1}} \times C^\infty_{\mathbb{R}^{n-1}} \to C^\infty_{\mathbb{R}^{n-1}}$ given by*

$$M(Z, W) := x_l Z - x_k W.$$

Then,

1. *$M : C^\infty_{\mathbb{R}^n} \oplus C^\infty_{\mathbb{R}^n} \to C^\infty_{\mathbb{R}^n}$ is $C^\infty_{\mathbb{R}^n}$-linear.*
2. *$M(Z, W) = 0$ if and only if there exists $f_{kl} \in C^\infty_{\mathbb{R}^n}$ such that $Z = x_k f_{kl}$ and $W = x_l f_{kl}$.*
3. *If $M(g, h) = \Delta$, then every solution of the equation $M(Z, W) = \Delta$ has the form $Z = g + x_l f_{kl}$, $W = h + x_k f_{kl}$.*

Proof. Property 1 is clear, and it is easy to show that property 3 follows from 1 and 2. It remains to show property 2. Note that $M(Z, W) = 0$ if and only if

$$x_k Z = x_l W.$$

For each point p in the hyperplane $x_k = 0$, the right hand side of the last equation vanishes. Therefore, for each point p in the hyperplane $x_k = 0$ and not belonging to the line $x_k = x_l = 0$, it must be $Z(p) = 0$. By continuity, $Z(p) = 0$ for all p with $p_k = 0$. Similarly, $W(p) = 0$ if $p_l = 0$. By Remark 5.36, there exist smooth functions $f_k, f_l \in C^\infty_{\mathbb{R}^{n-1}}$ such that

$$Z = x_k f_k, \qquad W = x_l f_l.$$

Then,

$$x_k x_l (f_k - f_l) = x_l Z - x_k W = 0,$$

which means that $f_k = f_l$ in every point with $x_k x_l \neq 0$. Since this set is dense in $\mathbb{R}^n$, by continuity we conclude that $f_k = f_l$ in all $\mathbb{R}^n$. ∎

Now, we will prove the following result for Poisson vector fields:

Proposition 5.38. *Let Z be a Poisson vector field in $(\mathbb{R}^n, \Pi)$, where $\Pi = \frac{\partial}{\partial x_1} \wedge E$, where E is the Euler vector field in variables $x_2, \ldots, x_n$. There exist unique $z, f \in C^\infty_{\mathbb{R}^n}$ and $a_{ij} \in \mathbb{R}$, $i, j = 2, \ldots, n$ such that $a_{nn} = 0$ and*

$$Z = z \frac{\partial}{\partial x_1} + fE + (a_{ij} x_i) \frac{\partial}{\partial x_j}. \tag{5.82}$$

Furthermore, $z, f \in C^\infty_{\mathbb{R}^n}$ satisfy

$$\mathrm{L}_E f + \frac{\partial z}{\partial x_1} = 0. \tag{5.83}$$

Conversely, any vector field satisfying (5.82) *and* (5.83) *is Poisson for* Π.

Lemma 5.39. *If a vector field Z has the form* (5.82)*, then such representation is unique.*

Proof. Assume that there exist $z, f \in C^\infty_{\mathbb{R}^n}$ and $a_{ij} \in \mathbb{R}$, $i, j = 2, \dots, n$ such that $a_{nn} = 0$ and

$$z\frac{\partial}{\partial x_1} + fE + (a_{ij}x_i)\frac{\partial}{\partial x_j} = 0.$$

Then $z = 0$ and $a_{ij}x_i + fx_j = 0$ for each $j = 2, \dots, n$. We will show that f and each a_{ij} are zero. Fix $i \neq j$ and evaluate $a_{ij}x_i + fx_j = 0$ in $x_i = 1$ and $x_k = 0$, for all $k \neq i$, leading to $a_{ij} = 0$. This implies that $(a_{jj} + f)x_j = 0$ for each $j = 2, \dots, n-1$ and $fx_n = 0$. This means that f is zero outside the hyperplane $x_n = 0$. Since f is continuous, it follows that $f = 0$. Substituting in $(a_{jj} + f)x_j = 0$ with $x_j = 1$, we get $a_{jj} = 0$. It follows that if a vector field can be represented as in above, then its representation is unique, as desired. ∎

Proof of Proposition 5.38. Let $Z = Z_j\frac{\partial}{\partial x_j}$ the coordinate expression of a Poisson vector field Z. Denote $\Delta_{lk} := x_l Z_k - x_k Z_l$. Then, equation (5.75) reads $\frac{\partial \Delta_{lk}}{\partial x_1} = 0$, which means that Δ_{lk} only depends on the variables $x_2, \dots, x_n$. Now note that equations (5.76) imply

$$\mathrm{L}_E\, \Delta_{lk} = 2\Delta_{lk}.$$

Therefore, Δ_{lk} must be a $C^\infty_{\mathbb{R}^n}$-homogenous function of degree 2, i.e., $\Delta_{lk} = x_l Z_k - x_k Z_l$ is an homogenous polynomial of degree 2, due to Lemma 5.33. A particular solution for Z_k, Z_l is to take Z_k and Z_l linear. Then, by Lemma 5.37, in general there exist $f_{lk} \in C^\infty_{\mathbb{R}^3}$ such that

$$Z_k - x_k f_{lk}, \qquad\qquad Z_l - x_l f_{lk}$$

are linear functions. Furthermore, it can be shown that, for a fixed k, the functions f_{kl} and f_{kj} differ by a constant. Thus, the above linear functions can be chosen in order to have a fixed function $f_{lk} = f$ for any k, l. Also, we can modify this function f in order to have $Z_n = a_{n2}x_2 + \dots + a_{n,n-1}x_{n-1} + fx_n$. Taking $z = Z_1$, we get that

$$Z = z\frac{\partial}{\partial x_1} + fE + (a_{ij}x_i)\frac{\partial}{\partial x_j}$$

for some constants a_{ij}, with $a_{n,n} = 0$. Finally, a substitution of Z_k in (5.76) yields

$$-x_k\frac{\partial z}{\partial x_1} + Z_k = -X_k\frac{\partial Z_1}{\partial x_1} + Z_j\frac{\partial X_k}{\partial x_j} = X_j\frac{\partial Z_k}{\partial x_j} = Z_k + x_k\,\mathrm{L}_E\, f.$$

This implies that $\mathrm{L}_E\, f + \frac{\partial Z_1}{\partial x_1} = 0$ outside the plane $x_k = 0$, and hence, in all $\mathbb{R}^n$, completing the proof. ∎

First Poisson cohomology group

Lemma 5.40. *If Z is a Poisson vector field, then $Z(x_1, 0, 0) = Z(0, 0, 0)$.*

Proof. Since Z is a Poisson vector field, it can be represented as

$$Z = z\frac{\partial}{\partial x_1} + fE + (a_{ij}x_i)\frac{\partial}{\partial x_j},$$

where $\mathrm{L}_E\, f + \frac{\partial z}{\partial x_1} = 0$. Since E is a linear vector field in the variables $x_2, \dots, x_n$,

$$(\mathrm{L}_E\, f)(p_1, 0, \dots, 0) = 0,$$

so $\frac{\partial z}{\partial x_1}(p_1, 0, \dots, 0) = 0$ for all $p_1 \in \mathbb{R}$. Therefore, z must be constant along the x_1-axis. ∎

Calculus of the cohomology

Note that in the representation (5.82) for a Poisson vector field Z, the tangent part (in the regular domain) is $z\frac{\partial}{\partial x_1} + fE$ and the transversal part is the linear vector field $(a_{ij}x_i)\frac{\partial}{\partial x_j}$. Thus, in order to be Z a Hamiltonian vector field, we must have $a_{ij} = 0$. Furthermore, we will show that if $z(0) = 0$, then Z is Hamiltonian.

Theorem 5.41. *Let Z be a Poisson vector field in the Poisson manifold $(\mathbb{R}^n, \Pi)$ and consider its representation in the form (5.82). The vector field Z is Hamiltonian if and only if $z(0) = 0$ and $a_{ij} = 0$. Hence,*

$$H^1_\Pi(\mathbb{R}^n) = \mathbb{R}^{(n-1)^2}.$$

Proof. Recall that the Hamiltonian vector field X_h with Hamiltonian function h is

$$X_h = -(\mathrm{L}_E\, h)\frac{\partial}{\partial x_1} + \frac{\partial h}{\partial x_1}E,$$

and also note that this is its representation in the sense of Proposition 5.38. On the other hand, let

$$Z = z\frac{\partial}{\partial x_1} + fE + (a_{ij}x_i)\frac{\partial}{\partial x_j}$$

be a Poisson vector field. Since such representation is unique, it follows that Z is Hamiltonian if and only if $a_{ij} = 0$ and there exists $h \in C^\infty_{\mathbb{R}^n}$ such that $z = -\mathrm{L}_E\, h$ and $f = \frac{\partial h}{\partial x_1}$. In particular, Z has the form

$$Z = z\frac{\partial}{\partial x_1} + fE$$

and $Z(0) = z(0)\frac{\partial}{\partial x_1}$. By Lemmas 5.35 and 5.40, the equation $z = -\mathrm{L}_E\, H$ has a solution in H if and only if $z(0) = 0$. In this case, condition (5.83) implies

$$0 = \mathrm{L}_E\, f + \frac{\partial z}{\partial x_1} = \mathrm{L}_E\, f - \frac{\partial}{\partial x_1}\mathrm{L}_E\, H = \mathrm{L}_E\, f - \left(\left[\frac{\partial}{\partial x_1}, \mathrm{L}_E\right] + \mathrm{L}_E\frac{\partial}{\partial x_1}\right)H = \mathrm{L}_E\left(f - \frac{\partial H}{\partial x_1}\right).$$

Therefore, by Lemma 5.33, $f - \frac{\partial H}{\partial x_1}$ is a constant function in the variables $x_2, \ldots, x_n$, i.e., it only depends on the variable x_1:

$$f - \frac{\partial H}{\partial x_1} = k(x_1).$$

Now, define $h := H + \int_0^{x_1} k \,\mathrm{d}\, x_1$. Then,

$$\begin{aligned}
\mathrm{L}_E\, h &= \mathrm{L}_E\, H + \mathrm{L}_E \int_0^{x_1} k \,\mathrm{d}\, x_1 = -z + 0 = -z,\\
\frac{\partial h}{\partial x_1} &= \frac{\partial H}{\partial x_1} + k(x_1) = f,
\end{aligned}$$

proving that h is the Hamiltonian for the Poisson vector field Z. As consequence, we have that every Poisson cohomology class is uniquely determined by the value of each a_{ij} and $z(0)$. Therefore,

$$H^1_\Pi(\mathbb{R}^n) = \mathbb{R}^{(n-1)^2}.$$

∎

Chapter 6

The Modular Class of Poisson Structures on Foliated Manifolds

In this chapter we study the behavior of the modular class of an orientable Poisson manifold and formulate some unimodularity criteria in the semilocal context, around a (singular) symplectic leaf. Our results generalize some known unimodularity criteria for regular Poisson manifolds related to the notion of the Reeb class. In particular, we show that the unimodularity of the transverse Poisson structure of the leaf is a necessary condition for the semilocal unimodular property. Our main tool is an explicit formula for a bigraded decomposition of modular vector fields of a coupling Poisson structure on a foliated manifold. Moreover, we also exploit the notion of the modular class of a Poisson foliation and its relationship with the Reeb class.

This chapter is organized as follows. In Section 6.1 we review some basic notions and facts about foliations. In Section 6.2 we study the relationship between the modular class of a Poisson foliation and the Reeb class. The modular vector fields of coupling Poisson structures are described in Section 6.3 by using bigraded calculus on foliated manifolds. Section 6.4 is devoted to the study of the behavior of the unimodularity property under gauge transformations. In Section 6.5 we derive some unimodularity criteria for coupling Poisson structures and find cohomological obstructions to the unimodularity. In Section 6.6 we examine the general unimodularity criteria for compatible Poisson structures on flat Poisson foliations. Here the cohomological obstructions take values on the foliated de Rham-Casimir complex. In Section 6.7 we apply the above results to describe the unimodularity in a neighborhood of a symplectic leaf. Finally, in Section 6.8, we prove the well-definiteness of the notion of generalized Reeb class of a symplectic leaf.

The contents of this chapter, with exception of Section 6.8, were published in [55].

6.1 Preliminaries: orientable foliations

We start by recalling some definitions and known facts about calculus on foliated manifolds. For details, we refer to [31, 35, 65].

Foliated de Rham differential. Let $\mathcal{V}$ be a regular foliation on M. Denote by $V := T\mathcal{V}$ the tangent bundle of $\mathcal{V}$. There exists a derivation $\mathrm{d}_{\mathcal{V}} \in \mathrm{Der}^1_{\mathbb{R}}\,\Gamma(\wedge^\bullet V^*)$ of degree 1 which is a coboundary operator, $\mathrm{d}_{\mathcal{V}}^2 = 0$, called the *foliated exterior derivative.* Notice that the foliated de Rham complex $(\Gamma(\wedge^\bullet V^*), \mathrm{d}_{\mathcal{V}})$ is just the cochain complex of the Lie algebroid V associated to the foliation $\mathcal{V}$. The cohomology of the foliated de Rham complex of $\mathcal{V}$ will be denoted by $H^\bullet_{\mathrm{dR}}(\mathcal{V})$.

Fix a normal distribution $H \subset TM$ of the foliation $\mathcal{V}$,

$$TM = H \oplus V. \tag{6.1}$$

Then, the vector-valued 1-form $\gamma \in \Gamma(T^*M \otimes V)$, defined as the projection $\mathrm{pr}_V : TM \to V$ along H in (6.1), is said to be a *connection form* on the foliated manifold $(M, \mathcal{V})$. Conversely, every vector-valued 1-form $\gamma \in \Gamma(T^*M \otimes V)$ with $\gamma|_V = \mathrm{Id}_V$ induces the normal bundle $H := \ker \gamma$ of $\mathcal{V}$. Then, the curvature form $R^\gamma \in \Gamma(\wedge^2 T^*M \otimes V)$ of the connection is given by [35]

$$R^\gamma(X, Y) := \gamma[(\mathrm{Id}_{TM} - \gamma)X, (\mathrm{Id}_{TM} - \gamma)Y] \qquad \forall X, Y \in \Gamma(TM)$$

and controls the integrability of the normal bundle H. The connection γ is said to be *flat* if $R^\gamma = 0$.

Splitting (6.1) induces H-dependent bigradings of the exterior algebras of multivector fields and differential forms on M:

$$\Gamma(\wedge^\bullet TM) = \bigoplus_{p,q\in\mathbb{Z}} \Gamma(\wedge^{p,q} TM), \qquad \Gamma(\wedge^\bullet T^*M) = \bigoplus_{p,q\in\mathbb{Z}} \Gamma(\wedge^{p,q} T^*M). \tag{6.2}$$

Here, $\wedge^{p,q} TM := \wedge^p H \otimes \wedge^q V$ and $\wedge^{p,q} T^*M := \wedge^p V^\circ \otimes \wedge^q H^\circ$, where $H^\circ := \mathrm{Ann}(H)$ and $V^\circ := \mathrm{Ann}(V)$ denote the annihilators of H and V, respectively. For a multivector field A, the term of bidegree (p, q) in decomposition (6.2) is denoted by $A_{p,q}$. We follow same notation for differential forms. Moreover, we have a bigraded decomposition for any linear operator on these exterior algebras. In particular, the exterior differential splits as $\mathrm{d} = \mathrm{d}^\gamma_{1,0} + \mathrm{d}^\gamma_{2,-1} + \mathrm{d}^\gamma_{0,1}$, where $\mathrm{d}^\gamma_{1,0}$ is the *covariant exterior derivative* of γ and $\mathrm{d}^\gamma_{2,-1} = -\,\mathrm{i}_{R^\gamma}$. Furthermore,

$$(\mathrm{d}^\gamma_{1,0})^2 = \mathrm{L}_{R^\gamma} \tag{6.3}$$

and $(\mathrm{d}^\gamma_{0,1})^2 = 0$ (for the definition of the Lie derivative L_{R^γ}, see [35]). It is clear that the canonical inclusion of the leaves of $\mathcal{V}$ in M induces a cochain complex isomorphism,

$$(\Gamma(\wedge^\bullet V^*), \mathrm{d}_{\mathcal{V}}) \cong \left(\Gamma(\wedge^\bullet H^\circ), \mathrm{d}^\gamma_{0,1}\right). \tag{6.4}$$

For each $\mu \in \Gamma(\wedge^\bullet V^*)$, we will denote by $\mu_\gamma \in \Gamma(\wedge^\bullet H^\circ)$ the corresponding element under the above isomorphism. We use the same notation for cohomology classes.

We will denote by

$$\mathrm{aut}(M, \mathcal{V}) := \{X \in \Gamma(TM) \mid [X, \Gamma(V)] \subset \Gamma(V)\}$$

the Lie subalgebra of *$\mathcal{V}$-projectable vector fields*. For each $\mu \in \Gamma(\wedge^\bullet V^*)$ and $X \in \mathrm{aut}(M, \mathcal{V})$, the Lie derivative $\mathrm{L}_X \mu \in \Gamma(\wedge^\bullet V^*)$ is well-defined by the standard formula.

Divergence 1-form. Suppose that the foliation $\mathcal{V}$ on M is *orientable*, that is, there exists a nowhere vanishing element $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$, called a *leaf-wise volume form* of $\mathcal{V}$. Therefore, the restriction of τ to each leaf L of $\mathcal{V}$ gives a volume form on L. For each $X \in \mathrm{aut}(M, \mathcal{V})$, the divergence $\mathrm{div}^\tau(X) \in C^\infty(M)$ with respect to τ is defined by the relation $\mathrm{L}_X \tau = \mathrm{div}^\tau(X)\tau$.

Fix a connection form γ on $(M, \mathcal{V})$ associated to a normal bundle H of $\mathcal{V}$. Then, one can think of a leaf-wise volume form $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$ as a differential form $\tau_\gamma \in \Gamma(\wedge^{\mathrm{top}} H^\circ)$ vanishing only on the sections of H. Recall that H° and V^* are of the same rank k. The divergence is given by the formula

$$\mathrm{div}^\tau(X)\tau_\gamma = (\mathrm{L}_X \tau_\gamma)_{0,k} \tag{6.5}$$

for any $X \in \mathrm{aut}(M, \mathcal{V})$. Here, the bigraded decomposition of the k-form $\mathrm{L}_X \tau_\gamma$ consists of the terms of bidegree $(0, k)$ and $(1, k-1)$.

Now, we observe that there exists a unique 1-form vanishing on vector fields tangent to $\mathcal{V}$, $\theta^\gamma_\tau \in \Gamma(V^\circ)$, and such that

$$\mathrm{d}^\gamma_{1,0}\,\tau_\gamma = \theta^\gamma_\tau \wedge \tau_\gamma. \tag{6.6}$$

Moreover, θ^γ_τ is related with the divergence by the condition

$$\theta^\gamma_\tau(X) = \mathrm{div}^\tau(X) \qquad \forall X \in \Gamma(H) \cap \mathrm{aut}(M, \mathcal{V}). \tag{6.7}$$

Therefore, the 1-form $\theta^\gamma_\tau \in \Gamma(V^\circ)$ can be called the *divergence form* associated to the pair (τ, γ). By using (6.6), (6.3), and (6.5), one can derive the following useful relations

$$\mathrm{d}^\gamma_{1,0}\,\theta^\gamma_\tau(X_1, X_2) = \mathrm{div}^\tau(R^\gamma(X_1, X_2)), \tag{6.8}$$

$$\theta^\gamma_{f\tau} = \theta^\gamma_\tau + \mathrm{d}^\gamma_{1,0} \ln |f|, \tag{6.9}$$

for all $X_1, X_2 \in \Gamma(H) \cap \mathrm{aut}(M, \mathcal{V})$ and each nowhere vanishing $f \in C^\infty(M)$.

The Reeb class. Let $\mathcal{V}$ be a regular foliation of M. Consider the tangent bundle $V = T\mathcal{V}$ and its annihilator V°. We say that the foliation $\mathcal{V}$ is *transversally orientable* if there exists a nowhere vanishing element $\varsigma \in \Gamma(\wedge^{\mathrm{top}} V^\circ)$. In this case, we say that ς is a *transversal volume element* of $\mathcal{V}$. In particular, we have $V = \{X \in TM \mid \mathrm{i}_X \varsigma = 0\}$.

It follows from the identity $\mathrm{i}_{[X,Y]} = [\mathrm{L}_X, \mathrm{i}_Y]$ $\forall X, Y \in \Gamma(V)$ that the Lie derivative along every $\mathcal{V}$-tangent vector field preserves the space of sections $\Gamma(\wedge^{\mathrm{top}} V^\circ)$. As a consequence, for each transversal volume element ς of $\mathcal{V}$, there exists a unique foliated 1-form $\lambda_\varsigma \in \Gamma(V^*)$ defined by the relation

$$\mathrm{L}_X\,\varsigma = \lambda_\varsigma(X)\varsigma, \qquad \forall X \in \Gamma(V). \tag{6.10}$$

Then, λ_ς is a closed foliated 1-form, $\mathrm{d}_\mathcal{V}\,\lambda_\varsigma = 0$. Moreover, by the standard arguments, the $\mathrm{d}_\mathcal{V}$-cohomology class (foliated de Rham cohomology class)

$$\mathrm{Mod}(\mathcal{V}) := [\lambda_\varsigma] \in H^1_{\mathrm{dR}}(\mathcal{V}), \tag{6.11}$$

is independent of the choice of a transversal volume element ς and called the *Reeb class* of $\mathcal{V}$ (see, for example, [31, Section 2], [32, Section 1], [1, Section 2]). The Reeb class is an obstruction to the existence of a transversal volume element of $\mathcal{V}$ which is invariant under the flow of any vector field tangent to the foliation. Alternatively, the vanishing of $\mathrm{Mod}(\mathcal{V})$ is equivalent to the existence of a closed transversal volume element of $\mathcal{V}$ [31].

Example 6.1. Let $\pi : M \to S$ be a fiber bundle over an orientable base S. Consider the foliation $\mathcal{V} := \{\pi^{-1}(x)\}_{x \in S}$ on M given by the surjective submersion π, called *simple foliation.* Then, $\mathcal{V}$ is a transversally orientable foliation with trivial Reeb class. Indeed, given a volume form ς_0 on the base S, we get the transversal volume element $\varsigma = \pi^*\varsigma_0$ of $\mathcal{V}$. It is clear that ς is closed on M and hence $\mathrm{Mod}(\mathcal{V}) = 0$. ▼

Pick a connection γ on $(M, \mathcal{V})$ associated to a normal bundle H of $\mathcal{V}$. For each transversal volume element ς of $\mathcal{V}$, there exists a 1-form $\lambda^\gamma_\varsigma \in \Gamma(H^\circ)$ uniquely defined by the relation

$$\mathrm{d}^\gamma_{0,1}\,\varsigma = \lambda^\gamma_\varsigma \wedge \varsigma. \tag{6.12}$$

From here, taking into account the bidegrees of λ_ς^γ and ς, we conclude that $\mathrm{d}_{0,1}^\gamma \lambda_\varsigma^\gamma = 0$ and hence λ_ς^γ is a 1-cocycle of $\mathrm{d}_{0,1}^\gamma$. Then, under the isomorphism (6.4), the Reeb class of the foliation $\mathcal{V}$ equals the $\mathrm{d}_{0,1}^\gamma$-cohomology class of λ_ς^γ,

$$\mathrm{Mod}(\mathcal{V})_\gamma = [\lambda_\varsigma^\gamma]. \tag{6.13}$$

Indeed, this is consequence of the following computation for all $X \in \Gamma(V)$:

$$\lambda_\varsigma(X)\varsigma = \mathrm{L}_X \varsigma = \mathrm{d}\,\mathrm{i}_X \varsigma + \mathrm{i}_X \mathrm{d}\,\varsigma = \mathrm{i}_X \mathrm{d}_{0,1}^\gamma \varsigma = \lambda_\varsigma^\gamma(X)\varsigma.$$

Observe that in the flat case, $R^\gamma = 0$, each leaf-wise volume form $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$ of $\mathcal{V}$ induces the transversal volume element τ_γ of the integral foliation $\mathcal{H}$ of H. Furthermore, $\partial_{\mathcal{H}} := \mathrm{d}_{1,0}^\gamma$ is the corresponding foliated exterior derivative and, by (6.8), θ_τ^γ is a 1-cocycle of $\partial_{\mathcal{H}}$. Then, taking into account (6.13), we conclude that the $\partial_{\mathcal{H}}$-cohomology class of θ_τ^γ coincides with the Reeb class of $\mathcal{H}$.

To end this section we make the following remarks on the different interpretations of the Reeb class.

Remark 6.2. *The Reeb class is related to some characteristic classes of representations of the Lie algebroid V associated to the foliation $\mathcal{V}$ [22, 38]. First, note that the Lie derivative along $\mathcal{V}$-tangent vector fields gives a representation D on the line bundle $\wedge^{\mathrm{top}} V^\circ$. By* (6.10) *and* (6.11)*, the Reeb class is just the characteristic class of this representation,* $\mathrm{Mod}(\mathcal{V}) = \mathrm{Char}(\wedge^{\mathrm{top}} V^\circ)$*. On the other hand, the Reeb class can be expressed in terms of the Bott connection ∇^{Bott} on the normal bundle $E := \frac{TM}{V}$ of $\mathcal{V}$ [83]. Under the natural identification $V^\circ \cong E^*$, the dual of the representation ∇^{Bott} on $\wedge^{\mathrm{top}} E$ coincides with D. Then,* $\mathrm{Char}(\wedge^{\mathrm{top}} E) = -\mathrm{Mod}(\mathcal{V})$*. Finally, we observe that the Reeb class coincides with the modular class of the Lie algebroid V [22, 36, 38],* $\mathrm{Mod}(\mathcal{V}) = \mathrm{Char}(\wedge^{\mathrm{top}} V \otimes \wedge^{\mathrm{top}} T^*M)$*.*

6.2 The modular class of a Poisson foliation

In this section, we describe the relationship between the modular class of a leaf-tangent Poisson structure on a foliated manifold and the Reeb class.

First, let us recall the definitions and some properties of modular vector fields and the modular class of a Poisson manifold [41, 83]. Let (M, Π) be an orientable Poisson manifold with Poisson bivector field Π on M. Denote by $\Pi^\sharp : T^*M \to TM$ the vector bundle morphism given by $\langle \beta, \Pi^\sharp \alpha \rangle := \Pi(\alpha, \beta)$. Let $\mathrm{Poiss}(M, \Pi) := \{X \in \Gamma(TM) \mid L_X \Pi = 0\}$ and $\mathrm{Ham}(M, \Pi) := \{\Pi^\sharp \mathrm{d} f \mid f \in C^\infty(M)\}$ be the Lie algebras of Poisson and Hamiltonian vector fields on (M, Π), respectively. Then, the first Poisson cohomology is $H^1_\Pi(M) = \mathrm{Poiss}(M, \Pi)/\,\mathrm{Ham}(M, \Pi)$.

Given a volume form Ω of M, one can define a derivation Z_Π^Ω of $C^\infty(M)$ by the formula $Z_\Pi^\Omega(f) := \mathrm{div}^\Omega(\Pi^\sharp \mathrm{d} f)$, where div^Ω is the divergence operator. The vector field Z_Π^Ω is a Poisson vector field of Π, called the *modular vector field* [41, 83] of the oriented Poisson manifold (M, Π, Ω).

In terms of the interior product, the modular vector field can be also defined by $-\mathrm{i}_{Z_\Pi^\Omega} \Omega = \mathrm{d}\,\mathrm{i}_\Pi \Omega$. Here, i denotes the insertion operator which on decomposable multivector fields is given by $\mathrm{i}_{X_1 \wedge \cdots \wedge X_k} = \mathrm{i}_{X_1} \circ \cdots \circ \mathrm{i}_{X_k}$ $\forall X_i \in \Gamma(TM)$. Furthermore, if Ω' is another volume form on M, then $Z_\Pi^{\Omega'} = Z_\Pi^\Omega - \Pi^\sharp \mathrm{d} \ln |f|$, where $f = \Omega'/\Omega$. Hence, the Poisson cohomology class of Z_Π^Ω is independent of the choice of Ω. Therefore, $\mathrm{Mod}(M, \Pi) := [Z_\Pi^\Omega] \in H^1_\Pi(M)$ is an intrinsic Poisson cohomology class called the *modular class* [83] of the orientable Poisson manifold (M, Π). A Poisson manifold with vanishing modular class is said to be *unimodular*. The modular class is an obstruction to the existence of a volume form which is invariant with respect to all Hamiltonian vector fields.

As an example, consider the 3-dimensional oriented linear Poisson manifold $(\mathbb{R}^3_{(x,y,z)}, \Pi, \Omega)$, where $\Pi = \frac{1}{2}\frac{\partial}{\partial z} \wedge \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$ and Ω is the Euclidean volume form. Then, $Z^{\Omega}_{\Pi} = \frac{\partial}{\partial z}$ cannot be a Hamiltonian vector field since it is non-zero at 0. Moreover, on the regular domain $N_{\mathrm{reg}} := \mathbb{R}^3 - \{z\text{-axis}\}$, $\frac{\partial}{\partial z}$ admits the Hamiltonian $-\ln(x^2+y^2)$. Thus, even though $\mathrm{Mod}(N_{\mathrm{reg}}, \Pi|_{N_{\mathrm{reg}}}) = 0$, the Poisson manifold $(\mathbb{R}^3_{(x,y,z)}, \Pi)$ is not unimodular.

Poisson foliations and orientability. A *Poisson foliation* consists of a triple $(M, \mathcal{V}, P)$, where $\mathcal{V}$ is a regular foliation on a manifold M endowed with a leaf-tangent Poisson bivector field $P \in \Gamma(\wedge^2 V), V = T\mathcal{V}$. It is clear that the characteristic distribution of P belongs to V, $P^\sharp(T^*M) \subset V$, and hence each leaf L of $\mathcal{V}$ inherits from P a unique Poisson structure P_L such that the inclusion $\iota_L : L \hookrightarrow M$ is a Poisson map. Therefore, M is foliated by the Poisson manifolds (L, P_L). Denote by $\mathrm{Poiss}_{\mathcal{V}}(M,P) := \Gamma(V) \cap \mathrm{Poiss}(M,P)$ the Lie algebra of all Poisson vector fields of P tangent to the foliation $\mathcal{V}$. It is clear that, for every $Z \in \mathrm{Poiss}_{\mathcal{V}}(M,P)$, the restriction to a given leaf L of $\mathcal{V}$ is a Poisson vector field of P_L, $Z|_L \in \mathrm{Poiss}(L, P_L)$. Note also that the morphism $P^\sharp : \Gamma(V^*) \to \Gamma(V)$ associated to the leaf-tangent Poisson structure P induces a linear mapping in cohomology $(P^\sharp)^* : H^1_{\mathrm{dR}}(\mathcal{V}) \to \frac{\mathrm{Poiss}_{\mathcal{V}}(M,P)}{\mathrm{Ham}(M,P)} \subset H^1_P(M)$ by $(P^\sharp)^*[\mu] := [P^\sharp \mu]$. We call the quotient $\frac{\mathrm{Poiss}_{\mathcal{V}}(M,P)}{\mathrm{Ham}(M,P)}$ the *first cohomology of the Poisson foliation* $(M, \mathcal{V}, P)$, which is just the first cohomology of the Lie algebroid $(V^*, \iota \circ P^\sharp, \{,\}_P)$. Here, $\iota : V \hookrightarrow TM$ is the inclusion map, and $\{,\}_P$ denotes the bracket of foliated 1-forms induced by P [36].

Suppose that $\mathcal{V}$ is orientable and fix a leaf-wise volume form $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$. The *modular vector field of the Poisson foliation* $(M, \mathcal{V}, P)$ with respect to τ is the leaf-tangent vector field $Z^\tau_P \in \Gamma(V)$ defined by the equality

$$\mathrm{i}_{Z^\tau_P}\, \tau = -\, \mathrm{d}_{\mathcal{V}}\, \mathrm{i}_P\, \tau. \tag{6.14}$$

It follows from (6.14) that $Z^\tau_P \in \mathrm{Poiss}(M,P)$ and $Z^{f\tau}_P = Z^\tau_P - P^\sharp\, \mathrm{d}\, f$ for all nowhere vanishing $f \in C^\infty(M)$. Hence, there is a well-defined cohomology class of the Poisson foliation $(M, \mathcal{V}, P)$

$$\mathrm{Mod}(M, \mathcal{V}, P) := [Z^\tau_P] \in \frac{\mathrm{Poiss}_{\mathcal{V}}(M,P)}{\mathrm{Ham}(M,P)}$$

which can be called the *modular class* of the Poisson foliation $(M, \mathcal{V}, P)$, or shortly, the *foliated modular class.*

It is clear that in the case when a Poisson foliation $\mathcal{V} = \{M\}$ consists of a single leaf, the foliated modular class of $(M, \mathcal{V}, P)$ just coincides with the modular class of the Poisson manifold (M, P).

Note that the modular class of a Poisson foliation $(M, \mathcal{V}, P)$ can be viewed as a particular case of the more general notion of the modular class of the corresponding triangular Lie bi-algebroid (V, P), $P \in \Gamma(\wedge^2 V)$ [36, 38].

The Poisson foliation $(M, \mathcal{V}, P)$ is said to be *unimodular* if $\mathrm{Mod}(M, \mathcal{V}, P) = 0$. Since the foliated differential $\mathrm{d}_{\mathcal{V}}$ and the leaf-wise volume form τ restrict to the exterior differential and a volume form $\tau_L = \iota^*_L \tau$ on each leaf L of $\mathcal{V}$, we conclude from (6.14) that the restriction of the modular vector field Z^τ_P to the leaf L is the modular vector field with respect to τ_L of the Poisson structure P_L, $Z^\tau_P|_L = Z^{\tau_L}_{P_L}$. Therefore, the unimodularity of the Poisson foliation implies the unimodularity of each leaf. But the converse is not necessarily true.

Here are some useful properties of the modular vector field of the Poisson foliation $(M, \mathcal{V}, P)$. Note that, for all $X \in \mathrm{aut}(M, \mathcal{V})$, we have $[X, \Gamma(\wedge^\bullet V)] \subset \Gamma(\wedge^\bullet V)$, where $[\cdot,\cdot]$ denotes the Schouten-Nijenhuis bracket for multivector fields [21]. By definition (6.14), and from the standard commuting relations between the operators L_X, $\mathrm{d}_{\mathcal{V}}$, and i_A, $A \in \Gamma(\wedge^\bullet V)$, we derive the following properties of the modular vector field Z_P^τ

$$\mathrm{L}_{Z_P^\tau} f = \operatorname{div}^\tau(P^\sharp \,\mathrm{d} f), \tag{6.15}$$

$$[Z_P^\tau, X] = P^\sharp \,\mathrm{d}(\operatorname{div}^\tau(X)), \tag{6.16}$$

for any $f \in C^\infty(M)$ and $X \in \operatorname{aut}(M, \mathcal{V}) \cap \operatorname{Poiss}(M, P)$.

Furthermore, given a connection form $\gamma \in \Gamma(T^*M \otimes V)$ on $(M, \mathcal{V})$, the modular vector field of $(M, \mathcal{V}, P)$ relative to $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$ is determined by

$$-\,\mathrm{i}_{Z_P^\tau} \tau_\gamma = \mathrm{d}_{0,1}^\gamma \,\mathrm{i}_P \,\tau_\gamma. \tag{6.17}$$

If, in addition to the orientability of $\mathcal{V}$, the manifold M is orientable (or, equivalently, $\mathcal{V}$ is transversally orientable), then we have a relation between the modular class of the Poisson structure P on M, the modular class of the Poisson foliation $(M, \mathcal{V}, P)$ and the Reeb class of $\mathcal{V}$.

Proposition 6.3. *Let $(M, \mathcal{V}, P)$ be an orientable and transversally orientable Poisson foliation. Then, the modular class* $\operatorname{Mod}(M, P)$ *of the Poisson manifold $P \in \Gamma(\wedge^2 V)$ is related to the foliated modular class* $\operatorname{Mod}(M, \mathcal{V}, P)$ *and the Reeb class* $\operatorname{Mod}(\mathcal{V})$ *of $\mathcal{V}$ by the formula*

$$\operatorname{Mod}(M, P) = \operatorname{Mod}(M, \mathcal{V}, P) - (P^\sharp)^* \operatorname{Mod}(\mathcal{V}). \tag{6.18}$$

Proof. Let $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$ be a leaf-wise volume form and $\varsigma \in \Gamma(\wedge^s V^\circ)$ a transversal volume element of $\mathcal{V}$, $s = \operatorname{codim} \mathcal{V}$. Pick a connection form γ on $(M, \mathcal{V})$ associated to a normal bundle H of $\mathcal{V}$. Then, $\Omega := \varsigma \wedge \tau_\gamma$ is a volume form on M. Let Z_P^Ω and Z_P^τ be the modular vector fields of (M, P) and $(M, \mathcal{V}, P)$ with respect to the volume forms Ω and τ, respectively. Consider also the 1-form $\lambda_\varsigma^\gamma \in \Gamma(H^\circ)$ given by (6.12). We claim that

$$Z_P^\Omega = Z_P^\tau - P^\sharp \lambda_\varsigma^\gamma. \tag{6.19}$$

Indeed, by bigrading arguments and equality (6.17), we get

$$\begin{aligned} -\,\mathrm{i}_{Z_P^\Omega} \Omega &= \mathrm{d}\,\mathrm{i}_P \,\Omega = \mathrm{d}_{0,1}^\gamma (\varsigma \wedge \mathrm{i}_P \,\tau_\gamma) = \mathrm{d}_{0,1}^\gamma \,\varsigma \wedge \mathrm{i}_P \,\tau_\gamma + (-1)^s \varsigma \wedge \mathrm{d}_{0,1}^\gamma \,\mathrm{i}_P \,\tau_\gamma \\ &= \lambda_\varsigma^\gamma \wedge \varsigma \wedge \mathrm{i}_P \,\tau_\gamma - (-1)^s \varsigma \wedge \mathrm{i}_{Z_P^\tau} \tau_\gamma = (-1)^s \varsigma \wedge \mathrm{i}_{\mathrm{i}_{\lambda_\varsigma^\gamma} P} \tau_\gamma - (-1)^s \varsigma \wedge \mathrm{i}_{Z_P^\tau} \tau_\gamma \\ &= \mathrm{i}_{P^\sharp \lambda_\varsigma^\gamma} \Omega - \mathrm{i}_{Z_P^\tau} \Omega. \end{aligned}$$

Here we have applied, on the second and fifth steps, the identity

$$\mathrm{i}_A(\alpha \wedge \beta) = (-1)^{|A|} (\alpha \wedge \mathrm{i}_A \,\beta - \mathrm{i}_{\mathrm{i}_\alpha A} \,\beta), \tag{6.20}$$

valid for all $\alpha \in \Gamma(T^*M), \beta \in \Gamma(\wedge^\bullet T^*M)$ and $A \in \Gamma(\wedge^\bullet TM)$. Thus, we have proved (6.19), which implies (6.18). ■

The following corollary to Proposition 6.3 gives us a unimodularity criterion for a class of Poisson foliations coming from fibrations.

Corollary 6.4. *Let $(M \xrightarrow{\pi} S, P)$ be a locally trivial Poisson fiber bundle. Suppose that the total space M and the base S are orientable. If the typical fiber F is a unimodular Poisson manifold, then* $\operatorname{Mod}(M, P) = 0$.

Proof. Consider the regular foliation $\mathcal{V} := \{\pi^{-1}(x)\}_{x\in S}$ on M associated to the projection π. As explained in Example 6.1, the orientability of the base implies $\operatorname{Mod}(\mathcal{V}) = 0$. Then, by (6.18), it suffices to show that $\operatorname{Mod}(M, \mathcal{V}, P) = 0$. Fix a nowhere vanishing top section $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$, where $V := \ker d\pi$ is the vertical bundle, and a family of trivializations $M_i := \pi^{-1}(U_i) \cong U_i \times F$ over open sets U_i which cover S. By the unimodularity hypothesis for F, one can equip each trivial Poisson bundle $(\pi_i : M_i \to U_i, P_i := P|_{M_i})$ with a leaf-wise volume form of positive orientation $\tau_i \in \Gamma(\wedge^{\mathrm{top}} V^*|_{M_i})$ such that the corresponding modular vector field is zero, $Z_{P_i}^{\tau_i} = 0$. From here and the partition of unity argument, we conclude that there exists a global leaf-wise volume form τ_0 of $\mathcal{V}$ such that $\mathrm{L}_{P^\sharp \mathrm{d} f}\, \tau_0 = 0$ for all $f \in C^\infty(M)$. ■

In the regular case, as a consequence of Proposition 6.3, we recover the result due to [83, 1] which says that the modular class of an orientable regular Poisson manifold is determined by the Reeb class of its symplectic foliation. Indeed, suppose that the Poisson manifold (M, P) is regular with $\operatorname{rank} P = 2s$. Let $\mathcal{V} = \mathcal{S}$ be the symplectic foliation of P equipped with the leaf-wise symplectic form ω. Then, the canonical leaf-wise volume form $\tau = \wedge^s \omega$ of the symplectic foliation is such that the modular vector field of the Poisson foliation $(M, \mathcal{S}, P)$ is zero, $Z_P^\tau = 0$. If, in addition, M is orientable, then the symplectic foliation is transversally orientable. Therefore, in this case, formula (6.18) reads $\operatorname{Mod}(M, P) = -(P^\sharp)^* \operatorname{Mod}(\mathcal{S})$.

6.3 Modular vector fields of coupling Poisson structures

Let $\mathcal{V}$ be a regular foliation of the smooth manifold M. Consider the tangent bundle $V = T\mathcal{V}$ and its annihilator $V^\circ \subset T^*M$.

Suppose we are given a $\mathcal{V}$-*coupling Poisson structure* [77, 65] $\Pi \in \Gamma(\wedge^2 TM)$, that is, a Poisson bivector on M such that

$$H := \Pi^\sharp(V^\circ) \tag{6.21}$$

is a normal bundle of the foliation,

$$TM = H \oplus V. \tag{6.22}$$

Then, the bigraded decomposition of Π with respect to (6.22) is of the form $\Pi = \Pi_{2,0} + \Pi_{0,2}$, where $\Pi_H := \Pi_{2,0} \in \Gamma(\wedge^2 H)$ is a bivector field of constant rank, with $\operatorname{rank} \Pi_H = \operatorname{rank} H$, and $\Pi_{0,2} \in \Gamma(\wedge^2 V)$ is a Poisson tensor on M tangent to the foliation $\mathcal{V}$. Therefore, we can associate to the $\mathcal{V}$-coupling Poisson structure Π the Poisson foliation $(M, \mathcal{V}, P := \Pi_{0,2})$. Notice that the characteristic distribution of Π splits as $\Pi^\sharp(T^*M) = H \oplus P^\sharp(V^*)$. Hence, $\operatorname{rank} \Pi = \operatorname{rank} H + \operatorname{rank} P$, so the set of singular points of Π and P coincide.

Moreover, the restriction $\Pi_H^\sharp|_{V^\circ} : V^\circ \to H$ is a vector bundle isomorphism and hence one can define an H-nondegenerated 2-form $\sigma \in \Gamma(\wedge^2 V^\circ)$, called the *coupling form*, by

$$\sigma^\flat|_H := -(\Pi_H^\sharp|_{V^\circ})^{-1} : H \to V^\circ. \tag{6.23}$$

Let γ be the connection form on $(M, \mathcal{V})$ associated to the normal bundle H in (6.21). Then, the *geometric data* (P, γ, σ) associated to the coupling Poisson tensor Π satisfy the following structure equations [77, 65, 78]

$$[X,P]=0 \qquad \forall X\in\Gamma(H)\cap\operatorname{aut}(M,\mathcal{V}), \tag{6.24}$$
$$R^\gamma(X,Y)=-P^\sharp\,\mathrm{d}[\sigma(X,Y)] \qquad \forall X,Y\in\Gamma(H)\cap\operatorname{aut}(M,\mathcal{V}), \tag{6.25}$$
$$\mathrm{d}^\gamma_{1,0}\,\sigma=0. \tag{6.26}$$

In particular, the first equation means that γ is a *Poisson connection* on $(M,\mathcal{V},P)$. Moreover, by the H-nondegeneracy property of the coupling form σ, the foliation $\mathcal{V}$ admits a canonical transversal volume element given by l times the product of σ, $\sigma^l:=\sigma\wedge\cdots\wedge\sigma\in\Gamma(\wedge^{\mathrm{top}}V^\circ)$, where $2l:=\operatorname{rank}V^\circ=\operatorname{rank}H$.

Now, assume that $\mathcal{V}$ is an orientable foliation. Then, one can associate to each leaf-wise volume form $\tau\in\Gamma(\wedge^{\mathrm{top}}V^*)$ of $\mathcal{V}$ a volume form Ω of M by $\Omega:=\sigma^l\wedge\tau_\gamma$. Moreover, recall that τ gives rise to the divergence 1-form $\theta^\gamma_\tau\in\Gamma(V^\circ)$, defined by (6.6), and the modular vector field Z^τ_P of the Poisson foliation $(M,\mathcal{V},P)$, introduced in (6.14). We describe the modular vector fields of coupling Poisson structures in terms of these objects.

Proposition 6.5. *Let Π be a coupling Poisson structure on the orientable foliated manifold $(M,\mathcal{V})$ associated to geometric data (P,γ,σ). Fix a leaf-wise volume form τ of $\mathcal{V}$ and consider the volume form $\Omega:=\sigma^l\wedge\tau_\gamma$ of M. If $Z:=Z^\Omega_\Pi$ is the corresponding modular vector field, then the bigraded components of Z relative to the splitting* (6.22) *are given by*

$$Z_{1,0}=-\Pi^\sharp(\theta^\gamma_\tau), \qquad Z_{0,1}=Z^\tau_P. \tag{6.27}$$

Proof. By the definition of the modular vector field Z and using the bigraded decompositions of d and Π, we have

$$\begin{aligned}-\mathrm{i}_Z\,\Omega&=\mathrm{d}\,\mathrm{i}_\Pi\,\Omega=\mathrm{d}^\gamma_{1,0}\,\mathrm{i}_{\Pi_H}\,\Omega+\mathrm{d}^\gamma_{0,1}\,\mathrm{i}_P\,\Omega+\mathrm{d}^\gamma_{2,-1}\,\mathrm{i}_{\Pi_H}\,\Omega\\&=\mathrm{d}^\gamma_{1,0}\,\mathrm{i}_{\Pi_H}\,\sigma^l\wedge\tau_\gamma+\mathrm{i}_{\Pi_H}\,\sigma^l\wedge\mathrm{d}^\gamma_{1,0}\,\tau_\gamma+\mathrm{d}^\gamma_{0,1}\,\sigma^l\wedge\mathrm{i}_P\,\tau_\gamma\\&\qquad+\sigma^l\wedge\mathrm{d}^\gamma_{0,1}\,\mathrm{i}_P\,\tau_\gamma-\mathrm{i}_{\mathrm{i}_{\Pi_H}R^\gamma}\,\Omega.\end{aligned} \tag{6.28}$$

It follows from (6.23) that $\mathrm{i}_{\Pi_H}\,\sigma^l=-l\sigma^{l-1}$. This together with (6.26) implies $\mathrm{d}^\gamma_{1,0}\,\mathrm{i}_{\Pi_H}\,\sigma^l=0$. On the other hand, there exists a 1-form $\Lambda\in\Gamma(H^\circ)$ satisfying the relation $\mathrm{d}^\gamma_{0,1}\,\sigma^l=\Lambda\wedge\sigma^l$. Then, from (6.28), by using (6.17), (6.6) and (6.20), we get

$$\begin{aligned}-\mathrm{i}_Z\,\Omega&=\theta^\gamma_\tau\wedge\mathrm{i}_{\Pi_H}\,\sigma^l\wedge\tau_\gamma+\Lambda\wedge\sigma^l\wedge\mathrm{i}_P\,\tau_\gamma-\sigma^l\wedge\mathrm{i}_{Z^\tau_P}\,\tau_\gamma-\mathrm{i}_{\mathrm{i}_{\Pi_H}R^\gamma}\,\Omega\\&=\mathrm{i}_{\Pi^\sharp_H(\theta^\gamma_\tau)}\,\Omega+\mathrm{i}_{P^\sharp\Lambda}\,\Omega-\mathrm{i}_{Z^\tau_P}\,\Omega-\mathrm{i}_{\mathrm{i}_{\Pi_H}R^\gamma}\,\Omega.\end{aligned}$$

It is left to show

$$\mathrm{i}_{\Pi_H}\,R^\gamma=P^\sharp\Lambda. \tag{6.29}$$

Consider the $2l$-vector field given by l times the product of Π_H, $\Pi^l_H:=\Pi_H\wedge\cdots\wedge\Pi_H$. Using again identities (6.20), (6.26) and the bigrading argument, we evaluate

$$\begin{aligned}(\mathrm{i}_{\Pi^l_H}\,\sigma^l)\Lambda&=\mathrm{i}_{\Pi^l_H}(\Lambda\wedge\sigma^l)=\mathrm{i}_{\Pi^l_H}(\mathrm{d}\,\sigma^l)=\mathrm{i}_{\Pi^l_H}(l\,\mathrm{d}\,\sigma\wedge\sigma^{l-1})\\&=-\,\mathrm{i}_{\Pi^l_H}(\mathrm{d}\,\sigma\wedge\mathrm{i}_{\Pi_H}\,\sigma^l)=-(\mathrm{i}_{\Pi^l_H}\,\sigma^l)\,\mathrm{i}_{\Pi_H}\,\mathrm{d}\,\sigma.\end{aligned}$$

From here and taking into account that $\mathrm{i}_{\Pi_H^l}\,\sigma^l \neq 0$, we conclude $\Lambda = -\,\mathrm{i}_{\Pi_H}\,\mathrm{d}\,\sigma$. On the other hand, the curvature identity (6.25) implies the equality $\mathrm{i}_{\Pi_H}\,R^\gamma = -P^\sharp\,\mathrm{i}_{\Pi_H}\,\mathrm{d}\,\sigma$ which together with the above representation for Λ proves (6.29). ■

As mentioned above, the set of singular points of the coupling Poisson structure Π coincides with the set of singular points of its leaf-tangent part $P = \Pi_{0,2}$. From the relations (6.27), we derive the following information on the behavior of the modular vector fields of Π at the singular points.

Corollary 6.6. *A modular vector field of the Poisson manifold* (M, Π) *is tangent to the symplectic foliation of* Π *at a point* $x \in M$ *if and only if a modular vector field* $Z_P^\tau \in \Gamma(V)$ *of the Poisson foliation* $(M, \mathcal{V}, P)$ *is tangent to the symplectic foliation of* P *at* x*. In particular, this is true if* x *is a regular point of* P*.*

Remark 6.7. *More generally, for a Poisson submanifold* N *of a Poisson manifold* (M, Π)*, one can introduce the notion of a relative modular class of* N *[11]. If this class vanishes, then the modular vector field of* (M, Π) *is tangent to* N*. In particular, this criterion can be applied when* N *is a symplectic leaf.*

Notice that the 1-form $\Lambda \in \Gamma(H^\circ)$, arising in (6.29), just coincides with the 1-form λ_ς^γ defined by (6.12) for $\varsigma = \sigma^l$, whose $\mathrm{d}_{0,1}^\gamma$-cohomology class gives the Reeb class of the foliation $\mathcal{V}$. Moreover, by the curvature relation (6.29) and Proposition 6.3, we conclude that if γ is flat, $R^\gamma = 0$, then

$$\mathrm{Mod}(M, P) = \mathrm{Mod}(M, \mathcal{V}, P). \tag{6.30}$$

Now, let us consider the Lie algebra $\mathrm{Poiss}_{\mathcal{V}}(M, P)$ of all $\mathcal{V}$-tangent Poisson vector fields of P. Then, the projection $\gamma : TM \to V$ along H in decomposition (6.22) induces the linear mapping [71]

$$\gamma^* : H^1_\Pi(M) \to \frac{\mathrm{Poiss}_{\mathcal{V}}(M, P)}{\mathrm{Ham}(M, P)} \subset H^1_P(M) \tag{6.31}$$

from the first Poisson cohomology of (M, Π) to the first cohomology of the Poisson foliation $(M, \mathcal{V}, P)$. As a consequence of Proposition 6.5, this map is natural with respect to the modular classes.

Corollary 6.8. *The quotient map* (6.31) *takes the modular class of the Poisson manifold* (M, Π) *to the modular class of the Poisson foliation* $(M, \mathcal{V}, P)$*,* $\gamma^*(\mathrm{Mod}(M, \Pi)) = \mathrm{Mod}(M, \mathcal{V}, P)$*.*

6.4 Gauge transformations

As we already mentioned above, according to [83, 1], the unimodularity of an orientable regular Poisson manifold (M, Π) is equivalent to the triviality of the Reeb class of the characteristic (symplectic) foliation $\mathcal{S}$ of Π. In other words, this means that the unimodularity property is independent of the leaf-wise symplectic structure on $\mathcal{S}$ in the following sense: if $\widetilde{\Pi}$ is another regular Poisson structure on M which has the same characteristic foliation $\mathcal{S}$, then the unimodularity of Π implies $\mathrm{Mod}(M, \widetilde{\Pi}) = 0$. But this fact is no longer true in the singular case.

For example, let us consider on $\mathbb{R}^3$, with coordinate functions (x_1, x_2, x_3), the linear Poisson structure $\Pi = \frac{1}{2}\epsilon_{ijk} x_i \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}$ associated to the Lie algebra $\mathfrak{so}(3)$. Here ϵ_{ijk} are the *Levi-Civita symbols.* We are using the Einstein summation convention. Consider also the homogeneous Poisson structure $\widetilde{\Pi} = f\Pi$, where $f(x_1, x_2, x_3) := x_1^4 + x_2^4 + x_3^4$. It is clear that the characteristic foliations of these structures coincide. Computing the corresponding modular vector fields with respect to the Euclidean volume form Ω in $\mathbb{R}^3$, we get $Z_\Pi^\Omega \equiv 0$ and

$$Z_{\widetilde{\Pi}}^\Omega = 2\epsilon_{ijk}\left(x_i^3 x_j - x_i x_j^3\right)\frac{\partial}{\partial x_k} \neq 0.$$

This shows that Π is unimodular, while $\widetilde{\Pi}$ is not, even though they have the same characteristic foliation.

On the other hand, there exists an equivalence relation for (possibly singular) Poisson structures, called the *gauge equivalence* [79], which preserves the unimodularity property.

Let (M, Π) be a Poisson manifold. Suppose we are given a closed 2-form B on M such that the endomorphism

$$(\mathrm{Id} - B^\flat \circ \Pi^\sharp) : T^*M \to T^*M \tag{6.32}$$

is invertible. Then, there exists a Poisson bivector field $\widetilde{\Pi}$ on M defined by the relation $\widetilde{\Pi}^\sharp = \Pi^\sharp \circ (\mathrm{Id} - B^\flat \circ \Pi^\sharp)^{-1}$ and represents the result of Π under the gauge transformation induced by B [79, 9]. In this case, we say that $\widetilde{\Pi}$ is *gauge equivalent* to Π. The gauge transformation modifies only the leaf-wise symplectic form of Π by means of the pull-back of the closed 2-form B, preserving the characteristic foliation. Furthermore, gauge transformations preserve the unimodularity property.

Proposition 6.9. *If Π and $\widetilde{\Pi}$ are gauge equivalent Poisson structures on M, then*

$$\mathrm{Mod}(M, \Pi) = 0 \iff \mathrm{Mod}(M, \widetilde{\Pi}) = 0. \tag{6.33}$$

Proof. The modular class $\mathrm{Mod}(M, \Pi)$ of the orientable Poisson manifold (M, Π) is one-half the modular class of the cotangent bundle T^*M of M with the Lie algebroid structure defined by Π [22]. As is known [79], the map (6.32) induced by the gauge transformation is an isomorphism between the cotangent Lie algebroids associated to Π and $\widetilde{\Pi}$. This proves the statement. ■

6.5 Unimodularity criteria

Assume that on the orientable foliated manifold $(M, \mathcal{V})$, we are given a $\mathcal{V}$-coupling Poisson structure Π associated to geometric data (P, γ, σ). Our point is to formulate some conditions for the unimodularity of Π in terms of the geometric data.

The following fact is a direct consequence of Corollary 6.8.

Lemma 6.10. *The unimodularity of the coupling Poisson structure Π implies the unimodularity of the Poisson foliation $(M, \mathcal{V}, P)$.*

Therefore, a necessary condition for vanishing of the modular class of Π is $\mathrm{Mod}(M, \mathcal{V}, P) = 0$. Moreover, it follows from Proposition 6.3 that the unimodularity of Π implies the unimodularity of the leaf-tangent Poisson structure P in the case when the Reeb class of the foliation $\mathcal{V}$ is trivial.

The next criterion follows from Proposition 6.5 and the following well-known fact [83]: a Poisson manifold is unimodular if and only if the modular vector field is zero with respect to a certain volume form.

Lemma 6.11. *The $\mathcal{V}$-coupling Poisson structure Π is unimodular if and only if there exists a leaf-wise volume form $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$, $V = T\mathcal{V}$, such that*

$$Z_P^\tau = 0 \qquad \text{and} \qquad \mathrm{d}_{1,0}^\gamma \tau_\gamma = 0.$$

It follows that the unimodularity of Π is independent of the coupling form σ. In other words, the mapping

$$(P, \gamma, \sigma) \mapsto (P, \gamma, \widetilde{\sigma})$$

is a foliation-preserving transformation which do not alter the unimodularity property, provided that $\widetilde{\sigma}$ satisfies the nondegeneracy condition and the structure equations (6.25), (6.26). This is also a "singular" analog of the fact that, for a regular Poisson manifold, the unimodularity is independent of the leaf-wise symplectic form.

Now let us describe a special class of gauge transformations which preserve the coupling Poisson structures and naturally appear in the context of the averaging method [68]. Consider the case when the gauge form B is exact with a primitive μ vanishing along the leaves of the foliation $\mathcal{V}$:

$$B = -\,\mathrm{d}\,\mu, \quad \mu \in \Gamma(V^\circ). \tag{6.34}$$

Then, assuming that the map (6.32) is invertible, one can show [68] that the Poisson structure $\widetilde{\Pi}$ resulting of the gauge transformation of Π is again $\mathcal{V}$-coupling. Furthermore, if $(\widetilde{P}, \widetilde{\gamma}, \widetilde{\sigma})$ is the geometric data associated to $\widetilde{\Pi}$, then $\widetilde{P} = P$ and $\widetilde{\gamma}$ is related to γ by

$$\gamma(X) - \widetilde{\gamma}(X) = P^\sharp\,\mathrm{d}[\mu(X)] \qquad \forall X \in \mathrm{aut}(M). \tag{6.35}$$

Fix a nowhere vanishing section $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$ and let us look at the corresponding divergence forms $\theta_\tau^{\widetilde{\gamma}}$ and θ_τ^γ. By relations (6.7) and (6.35), for every $X \in \mathrm{aut}(M, \mathcal{V})$, we have

$$\theta_\tau^{\widetilde{\gamma}}(X) - \theta_\tau^\gamma(X) = \mathrm{div}^\tau(P^\sharp\,\mathrm{d}\,\mu(X)) = \mathrm{L}_{Z_P^\tau}(\mu(X)). \tag{6.36}$$

Here, we used the identity (6.15). Formulas (6.27), (6.36), give the transition rule for the modular vector fields of Π and $\widetilde{\Pi}$.

Next, if Π is unimodular, then by Lemma 6.11 we can choose a leaf-wise volume form τ of $\mathcal{V}$ such that $Z_P^\tau = 0$ and $\theta_\tau^\gamma = 0$. In this case, we have $\theta_\tau^{\widetilde{\gamma}} = \theta_\tau^\gamma = 0$. Hence by Proposition 6.5, if the modular vector field of Π with respect to the volume form $\Omega = \sigma^l \wedge \tau_\gamma$ is zero, then the modular vector field of $\widetilde{\Pi}$ with respect to $\widetilde{\Omega} = \sigma^l \wedge \tau_{\widetilde{\gamma}}$ is also zero.

Cohomological obstructions to the unimodularity. By Lemma 6.10, a necessary condition for the unimodularity of the $\mathcal{V}$-coupling Poisson structure Π on $(M, \mathcal{V})$ is the unimodularity of the Poisson foliation $(M, \mathcal{V}, P)$. We will show that this condition is not sufficient, since there exists a cohomological obstruction to the unimodularity of Π.

Consider the Poisson foliation $(M, \mathcal{V}, P, \gamma)$ equipped with the Poisson connection γ corresponding to the normal bundle H in (6.21). Then, one can associate to this setup the following cochain complex $(\mathcal{C}^\bullet, \overline{\mathrm{d}}^\gamma)$, where the subspaces $\mathcal{C}^p \subset \Gamma(\wedge^p V^\circ)$ are defined by

$$\mathcal{C}^p := \{\beta \in \Gamma(\wedge^p V^\circ) \mid \mathrm{i}_{X_1} \cdots \mathrm{i}_{X_p}\,\beta \in \mathrm{Casim}(M, P),\ \forall\ X_i \in \mathrm{aut}(M, \mathcal{V})\} \tag{6.37}$$

and $\overline{\mathrm{d}}^{\gamma} := \mathrm{d}^{\gamma}_{1,0}|_{\mathcal{C}^{\bullet}}$ is the restriction of $\mathrm{d}^{\gamma}_{1,0}$ to $\mathcal{C}^{\bullet}$. Therefore, $\mathcal{C}^{p}$ consists of p-forms on M vanishing along the leaves of $\mathcal{V}$ and taking values in the space of Casimir functions of P on the projectable vector fields.

There exists the following short exact sequence [71]:

$$0 \to H^{1}_{\overline{\mathrm{d}}^{\gamma}} \xrightarrow{(\Pi^{\sharp}_{H})^{*}} H^{1}_{\Pi}(M) \xrightarrow{\gamma^{*}} \frac{\ker\rho}{\mathrm{Ham}(M,P)} \to 0, \tag{6.38}$$

where $\rho : \mathcal{A}^{\gamma} \to H^{2}_{\overline{\mathrm{d}}^{\gamma}}$ is a morphism from a Lie subalgebra $\mathcal{A}^{\gamma} \subset \mathrm{Poiss}_{\mathcal{V}}(M,P)$, associated with the pair (P,γ), to the second cohomology space of $(\mathcal{C}^{\bullet}, \overline{\mathrm{d}}^{\gamma})$.

According to Corollary 6.8 and (6.38), if

$$\mathrm{Mod}(M,\mathcal{V},P) = 0, \tag{6.39}$$

then there exists a unique cohomology class in $H^{1}_{\overline{\mathrm{d}}^{\gamma}}$ such that its image under $-(\Pi^{\sharp}_{H})^{*}$ is $\mathrm{Mod}(M,\Pi)$. This cohomology class can be described as follows.

Theorem 6.12. *Let M be an orientable manifold, and $\mathcal{V}$ an orientable foliation on M. Suppose that the $\mathcal{V}$-coupling Poisson structure Π on M satisfies* (6.39). *Fix a leaf-wise volume form $\tau \in \Gamma(\wedge^{\mathrm{top}}V^{*})$ of $\mathcal{V}$ such that $Z^{\tau}_{P} = 0$ and consider the Poisson connection γ associated to H in* (6.21). *Then, the corresponding divergence form θ^{γ}_{τ} in* (6.6) *is a 1-cocycle of the cochain complex $(\mathcal{C}^{\bullet}, \overline{\mathrm{d}}^{\gamma})$, $\theta^{\gamma}_{\tau} \in \mathcal{C}^{1}$ and $\overline{\mathrm{d}}^{\gamma}\theta^{\gamma}_{\tau} = 0$. Furthermore, the $\overline{\mathrm{d}}^{\gamma}$-cohomology class of θ^{γ}_{τ} is independent of the choice of τ and related with the modular class of Π by $\mathrm{Mod}(M,\Pi) = -(\Pi^{\sharp}_{H})^{*}[\theta^{\gamma}_{\tau}]$.*

Proof. By (6.24), every projectable section $X \in \Gamma(H) \cap \mathrm{aut}(M,\mathcal{V})$ is a Poisson vector field of P. Then, by using the condition $Z^{\tau}_{P} = 0$, properties (6.7) and (6.16), we get

$$P^{\sharp}\,\mathrm{d}[\theta^{\gamma}_{\tau}(X)] = P^{\sharp}\,\mathrm{d}[\mathrm{div}^{\tau}(X)] = [Z^{\tau}_{P}, X] = 0.$$

Therefore, $\theta^{\gamma}_{\tau}(X) \in \mathrm{Casim}(M,P)$ $\forall X \in \Gamma(H) \cap \mathrm{aut}(M,\mathcal{V})$ and hence $\theta^{\gamma}_{\tau} \in \mathcal{C}^{1}$. Moreover, relations (6.8), (6.25) and (6.15) imply that θ^{γ}_{τ} is $\overline{\mathrm{d}}^{\gamma}$-closed. Indeed, for all $X_1, X_2 \in \mathrm{aut}(M,\mathcal{V})$,

$$\begin{aligned}(\overline{\mathrm{d}}^{\gamma}\theta^{\gamma}_{\tau})(X_1,X_2) &= \mathrm{div}^{\tau}(R^{\gamma}(X_1,X_2)) = -\,\mathrm{div}^{\tau}(P^{\sharp}\,\mathrm{d}[\sigma(X_1,X_2)]) \\ &= -L_{Z^{\tau}_{P}}[\sigma(X_1,X_2)] = 0.\end{aligned}$$

Note that any two leaf-wise volume forms for which the modular vector fields of the Poisson foliation $(M,\mathcal{V},P)$ vanish are related by multiplication of a Casimir function. Thus, it follows from the transition rule (6.9) that $[\theta^{\gamma}_{\tau}] \in H^{1}_{\overline{\mathrm{d}}^{\gamma}}$ is independent on the choice of τ. Finally, $\mathrm{Mod}(M,\Pi) = -(\Pi^{\sharp}_{H})^{*}[\theta^{\gamma}_{\tau}]$ follows from (6.27). ■

Corollary 6.13. *If the Poisson foliation $(M,\mathcal{V},P)$ associated to the $\mathcal{V}$-coupling Poisson structure Π is unimodular, then the unimodularity of Π is equivalent to the triviality of the $\overline{\mathrm{d}}^{\gamma}$-cohomology class of θ^{γ}_{τ}, that is,* $\mathrm{Mod}(M,\Pi) = 0 \Longleftrightarrow [\theta^{\gamma}_{\tau}] = 0$.

Example 6.14. Consider the particular case when the leaf-tangent Poisson structure P is trivial, $P = 0$. Then, the coupling Poisson structure Π is regular since its characteristic distribution coincides with the normal bundle H. Moreover, $(\mathcal{C}^{\bullet}, \overline{\mathrm{d}}^{\gamma})$ identifies with the foliated de Rham complex of the symplectic foliation $\mathcal{S}$ of Π. In particular, the cohomology class $[\theta^{\gamma}_{\tau}]$ coincides with the Reeb class $\mathrm{Mod}(\mathcal{S})$. ▼

Note that the coupling Poisson structure Π with $P = 0$ can be characterized as a regular Poisson structure whose symplectic foliation $\mathcal{S}$ admits a transversal foliation $\mathcal{V}$, $TM = T\mathcal{S} \oplus T\mathcal{V}$. So, in this case, the unimodularity criterion of Corollary 6.13 recovers the results due to [83, 1].

6.6 Flat Poisson foliations

Suppose we start with a Poisson foliation $(M, \mathcal{V}, P)$ consisting of a regular foliation $\mathcal{V}$ on M and a leaf-tangent Poisson structure $P \in \Gamma(\wedge^2 V)$. Suppose we are also given a regular foliation $\mathcal{F}$ on M with properties: the tangent bundle $\mathbb{F} := T\mathcal{F}$ is complementary to $V = T\mathcal{V}$, $TM = \mathbb{F} \oplus V$, and every $\mathcal{V}$-projectable section Z of $\mathbb{F}$ is a Poisson vector field on (M, P),

$$Z \in \Gamma_{\mathrm{pr}}(\mathbb{F}) \implies L_Z P = 0. \tag{6.40}$$

In other words, there is a flat Poisson connection $\gamma_0 \in \Gamma(T^*M \otimes V)$ on $(M, \mathcal{V}, P)$ associated to the tangent bundle of $\mathcal{F}$, $\mathbb{F} = \ker \gamma_0$, and hence $\gamma_0 : TM \to V$ is the projection along $\mathbb{F}$.

Let us associate to the *flat Poisson foliation* $(M, \mathcal{V}, P, \mathcal{F})$ the following objects. According to the dual splitting $T^*M = V^\circ \oplus \mathbb{F}^\circ$, we have the bigrading of differential forms on M and the bigraded decomposition of the exterior differential on M: $\mathrm{d} = \partial_{\mathcal{F}} + \partial_{\mathcal{V}}$, where $\partial_{\mathcal{F}} := \mathrm{d}^{\gamma_0}_{1,0}$ and $\partial_{\mathcal{V}} := \mathrm{d}^{\gamma_0}_{0,1}$ are the coboundary operators on $\Gamma(\wedge^\bullet T^*M)$ associated to the foliated differentials $\mathrm{d}_{\mathcal{F}}$ and $\mathrm{d}_{\mathcal{V}}$. So, $\partial_{\mathcal{F}}^2 = 0$, $\partial_{\mathcal{V}}^2 = 0$ and $\partial_{\mathcal{F}}\partial_{\mathcal{V}} + \partial_{\mathcal{V}}\partial_{\mathcal{F}} = 0$.

Consider the subspaces $\mathcal{C}^p$ defined in (6.37). In particular, $\mathcal{C}^\circ = \mathrm{Casim}(M, P)$. Furthermore, because of (6.40), $\mathcal{C}^\bullet := \bigoplus_{p \in \mathbb{Z}} \mathcal{C}^p$ is a $\partial_{\mathcal{F}}$-invariant subspace of $\Gamma(\wedge^\bullet T^*M)$ and hence the restriction $\overline{\partial}_{\mathcal{F}} := \partial_{\mathcal{F}}|_{\mathcal{C}^\bullet}$ is a well-defined coboundary operator. This gives rise to a cochain subcomplex $(\mathcal{C}^\bullet, \overline{\partial}_{\mathcal{F}})$ of $(\Gamma(\wedge^\bullet V^\circ), \partial_{\mathcal{F}})$ attributed to the flat Poisson foliation which will be called the *foliated de Rham-Casimir complex* [71]. The corresponding cohomology space will be denoted by $H^\bullet_{\overline{\partial}_{\mathcal{F}}}$.

We have the following useful property [71].

Lemma 6.15. *The natural homomorphism from $H^1_{\overline{\partial}_{\mathcal{F}}}$ to the first foliated de Rham cohomology $H^1_{\mathrm{dR}}(\mathcal{F})$ is injective if and only if*

$$\partial_{\mathcal{F}}(\mathrm{Casim}(M, P)) = \partial_{\mathcal{F}}(C^\infty(M)) \cap \mathcal{C}^1. \tag{6.41}$$

We say that a $\mathcal{V}$-coupling Poisson structure Π on the flat Poisson foliation $(M, \mathcal{V}, P, \mathcal{F})$ is *compatible* if $\Pi_{0,2} = P$ and the Poisson connection γ induced by the normal subbundle $H = \Pi^\sharp(T^*M)$ satisfies the condition

$$\gamma_0(X) - \gamma(X) \text{ is tangent to } P^\sharp(T^*M), \quad \forall X \in \Gamma(TM).$$

This compatibility condition implies that $\overline{\mathrm{d}}^\gamma = \overline{\partial}_{\mathcal{F}}$ and hence the cochain complex $(\mathcal{C}^\bullet, \overline{\mathrm{d}}^\gamma)$ associated to Π coincides with the foliated de Rham-Casimir complex. Also, we say that Π is *strongly compatible* if there exists $\mu \in \Gamma(V^\circ)$ such that γ and γ_0 are related by (6.35).

First, we formulate a unimodularity criterion for the class of strongly compatible Poisson structures which involves the injectivity condition (6.41).

Theorem 6.16. *Let $(M, \mathcal{V}, P, \mathcal{F})$ be a flat Poisson foliation and Π a strongly compatible coupling Poisson structure. If Π is unimodular then*

$$\mathrm{Mod}(\mathcal{F}) = 0 \qquad \textit{and} \qquad \mathrm{Mod}(M, \mathcal{V}, P) = 0. \tag{6.42}$$

Conversely, under the injectivity condition (6.41), *the unimodularity of* (M, Π) *is equivalent to* (6.42).

Proof. Since Π is compatible, we have $\overline{\mathrm{d}}^{\gamma} = \overline{\partial}_{\mathcal{F}}$, so $H^1_{\overline{\mathrm{d}}^\gamma} = H^1_{\overline{\partial}_{\mathcal{F}}}$. Moreover, if $\mathrm{Mod}(M,\mathcal{V},P)=0$, then the cohomology classes $[\theta^{\gamma}_{\tau}] \in H^1_{\overline{\mathrm{d}}^\gamma}$ and $[\theta^{\gamma_0}_{\tau}] \in H^1_{\overline{\partial}_{\mathcal{F}}}$ of the divergence 1-forms also coincide. Indeed, by the strong compatibility, formula (6.36) holds, so condition $\mathrm{Mod}(M,\mathcal{V},P)=0$ implies $[\theta^{\gamma}_{\tau}] = [\theta^{\gamma_0}_{\tau}]$. On the other hand, as shown in Section 6.1, the $\partial_{\mathcal{F}}$-cohomology class of $\theta^{\gamma_0}_{\tau}$ is the Reeb class $\mathrm{Mod}(\mathcal{F}) \in H^1_{\mathrm{dR}}(\mathcal{F})$. In other words, the image of $[\theta^{\gamma_0}_{\tau}]$ under the morphism in Lemma 6.15 is $\mathrm{Mod}(\mathcal{F})$. Finally, recall that by Corollary 6.13, the unimodularity of (M,Π) is equivalent to $\mathrm{Mod}(M,\mathcal{V},P)=0$ and $[\theta^{\gamma}_{\tau}]=0$. By our above discussion, this implies $\mathrm{Mod}(\mathcal{F})=0$. Conversely, under the injectivity condition (6.41), equation (6.42) implies $[\theta^{\gamma}_{\tau}] = [\theta^{\gamma_0}_{\tau}] = 0$. By Corollary 6.13, the proof is complete. ∎

We have also the following unimodularity criterion in the case when the first cohomology of the foliated de Rham-Casimir complex is trivial.

Theorem 6.17. *Let Π be a compatible coupling Poisson structure on the flat Poisson foliation $(M,\mathcal{V},P,\mathcal{F})$. If*

$$H^1_{\overline{\partial}_{\mathcal{F}}} = \{0\}, \tag{6.43}$$

then (M,Π) is unimodular if and only if $\mathrm{Mod}(M,\mathcal{V},P)=0$.

Proof. The compatibility condition implies $H^1_{\overline{\mathrm{d}}^\gamma} = H^1_{\overline{\partial}_{\mathcal{F}}}$. Thus, the short exact sequence (6.38) reads

$$0 \to H^1_{\overline{\partial}_{\mathcal{F}}} \xrightarrow{(\Pi^\sharp_H)^*} H^1_{\Pi}(M) \xrightarrow{\gamma^*} \frac{\ker\rho}{\mathrm{Ham}(M,P)} \to 0.$$

Hence, under condition (6.43), the projection γ^* is an isomorphism. Moreover, by Corollary 6.8, γ^* maps $\mathrm{Mod}(M,\Pi)$ to $\mathrm{Mod}(M,\mathcal{V},P)$. ∎

Now let us discuss some realizations of conditions (6.41), (6.43). Consider the space $\mathrm{Ham}(M,P)$ of Hamiltonian vector fields of the $\mathcal{V}$-tangent Poisson structure P. Then one can introduce the following two subspaces of $\mathrm{Ham}(M,P)$ depending on the foliation $\mathcal{F}$. Let $\mathrm{Ham}_{\mathcal{F}}(M,P) := \{P^\sharp \mathrm{d} f \mid f \in C^\infty(M), \partial_{\mathcal{F}} f = 0\}$ be the Lie algebra of all Hamiltonian vector fields of $\mathcal{F}$-projectable functions, and $\mathrm{Ham}_0(M,P) := \{Y \in \mathrm{Ham}(M,P) \mid [Y,\Gamma_{\mathrm{pr}}(\mathbb{F})]=0\}$ the Lie algebra of $\mathcal{F}$-projectable Hamiltonian vector fields. It follows from $\Gamma_{\mathrm{pr}}(\mathbb{F}) \subset \mathrm{Poiss}(M,P)$ that $\mathrm{Ham}_{\mathcal{F}}(M,P) \subseteq \mathrm{Ham}_0(M,P)$. Then, we have the following fact [71]: injectivity condition (6.41) holds if and only if $\mathrm{Ham}_{\mathcal{F}}(M,P) = \mathrm{Ham}_0(M,P)$. This condition together with the assumption $H^1_{\mathrm{dR}}(\mathcal{F}) = \{0\}$ on the triviality of the first foliated de Rham cohomology of $(M,\mathcal{F})$ implies (6.43).

Moreover, we have the following realization of condition (6.43) in the case of a flat Poisson fibration. Suppose we have a transversal bi-fibration $N \overset{\nu}{\leftarrow} M \overset{\pi}{\rightarrow} S$,

$$TM = \ker d\nu \oplus \ker d\pi.$$

Let $\mathcal{F} = \{\nu^{-1}(\xi)\}_{\xi \in N}$ and $\mathcal{V} = \{\pi^{-1}(q)\}_{q\in S}$ be the regular foliations of M defined by the fibers of the submersions ν and π, respectively. So, $\mathbb{F} = T\mathcal{F} = \ker d\nu$ and $V = T\mathcal{V} = \ker d\pi$. Assume also that we are given a Poisson tensor $P \in \Gamma(\wedge^2 V)$ such that the triple $(M \overset{\pi}{\to} S, P, \mathcal{F})$ is a flat Poisson fibration, that is, $\Gamma_{\mathrm{pr}}(\mathbb{F}) \subset \mathrm{Poiss}(M,P)$. Then, there exists a unique Poisson structure Ψ on N such that the projection $\nu : (M,P) \to (N,\Psi)$ is a Poisson map. One can show [71] that condition (6.43) holds if

$$H^1_{\mathrm{dR}}(\mathcal{F}) = \{0\} \text{ and } H^1_{\Psi}(N) = \{0\}.$$

Notice that the last condition implies (6.41).

We conclude this section by constructing a class of unimodular compatible Poisson structures.

Flat coupling Poisson structures. Let $(M, \mathcal{V}, P, \mathcal{F})$ be a flat Poisson foliation. Suppose we are given a $\overline{\partial}_{\mathcal{F}}$-closed, $\mathbb{F}$-nondegenerated 2-form $\sigma_0 \in \mathcal{C}^2$, that is, $\overline{\partial}_{\mathcal{F}}\sigma_0 = 0$ and $\sigma_0^\flat|_{\mathbb{F}} : \mathbb{F} \to V^\circ$ is an isomorphism. Then, one can define a coupling Poisson structure associated with the geometric data (P, γ_0, σ_0):

$$\Pi_{\text{flat}} = \Pi_F + P, \tag{6.44}$$

where $\Pi_F \in \Gamma(\wedge^2\mathbb{F})$ is a bivector field defined by the condition that the restriction $(\Pi_F)^\sharp|_{V^\circ}$ equals to the inverse of $-\sigma_0^\flat|_{\mathbb{F}}$. In this case, Π_F is a regular Poisson tensor which together with P forms a Poisson pair. Since the symplectic foliation of Π_F is just $\mathcal{F}$, it is clear that Π_{flat} is a compatible Poisson structure and $\operatorname{Mod}(M, \Pi_F) = -\Pi_F^\sharp(\operatorname{Mod}(\mathcal{F}))$. Assuming that $\mathcal{V}$ is orientable and equipped with a nowhere vanishing section $\tau \in \Gamma(\wedge^{\text{top}}V^*)$, we define a volume form as $\Omega_0 = \sigma_0^l \wedge \tau_{\gamma_0}$, $2l := \operatorname{rank}\mathbb{F}$. Then, the modular vector field of Π_{flat} relative to Ω_0 is represented as $Z^{\Omega_0}_{\Pi_{\text{flat}}} = Z^{\Omega_0}_{\Pi_F} + Z^{\Omega_0}_{P}$. Under the injectivity condition (6.41), we conclude from (6.30) and Theorem 6.16 that Π_{flat} is unimodular if and only if $\operatorname{Mod}(\mathcal{F}) = 0$ and $\operatorname{Mod}(M, P) = 0$. In this case, according to Proposition 6.9, a gauge transformation (6.32), (6.34) modifies Π_{flat} preserving the unimodularity property.

6.7 Coupling neighborhoods of a symplectic leaf

Let (M, Π) be a Poisson manifold and $\iota : S \hookrightarrow M$ an embedded symplectic leaf.

By a *coupling neighborhood* of S, we mean an open neighborhood N of S in M equipped with a surjective submersion $\pi : N \to S$ such that $\pi \circ \iota = \operatorname{Id}_S$,

$$\operatorname{rank} H = \dim S, \qquad \text{and} \qquad H \cap V = \{0\}, \tag{6.45}$$

where $V = \ker d\pi$ is the vertical subbundle of π and $H = \Pi^\sharp(V^\circ)$ is the horizontal subbundle associated to Π. It is clear that conditions in (6.45) are equivalent to the splitting $TN = H \oplus V$. Therefore the restriction $\Pi|_N$ is a $\mathcal{V}$-coupling Poisson structure on N, where the foliation $\mathcal{V} = \mathcal{V}^\pi := \{N_q = \pi^{-1}(q)\}_{q\in S}$ is given by the π-fibers. Taking into account that the symplectic leaf S is an orientable manifold, we conclude that the Reeb class of $\mathcal{V}^\pi$ is trivial (see Example 6.1). So, $\Pi|_N$ has a bigraded decomposition into a horizontal part of constant rank and a vertical Poisson tensor $P \in \Gamma(\wedge^2 V)$ vanishing at the points of S. The Poisson structure P is said to be a *transverse Poisson structure* of the leaf. The restriction $P_q := P|_{N_q}$ of P to the fiber N_q over every point $q \in S$ just gives the transverse Poisson structure of q due to Weinstein's splitting theorem [82]. Moreover, the Poisson connection γ on N is defined by $\ker\gamma = \Pi^\sharp(V^\circ)$ and the coupling form $\sigma \in \Gamma(\wedge^2 V^\circ)$ has the representation $\sigma = \pi^*\omega + \tilde{\sigma}$, where ω is the symplectic form on S and $\tilde{\sigma}$ is a horizontal 2-form vanishing at S.

For a given embedded symplectic leaf S, there exists always such a coupling neighborhood N [77]. In particular, one can choose N as a tubular neighborhood of S diffeomorphic to the normal bundle $E = T_SM \diagup TS$ of the symplectic leaf. If the normal bundle E is orientable, then N admits a volume form. Of course, this is true in the case when M is orientable. Hence, under the orientability hypothesis, the point is to study the germs at S of the modular vector fields of Π and the corresponding germified modular class.

First we formulate the following result.

Proposition 6.18. *If the Poisson structure* Π *is unimodular in a neighborhood of the embedded symplectic leaf* S*, then there exists a coupling neighborhood* N *of* S *such that the transverse Poisson structure* P *of* S *is also unimodular.*

Proof. The statement follows from Lemma 6.10, Proposition 6.3 and the fact that in a tubular neighborhood of S, the Reeb class of the fibration is trivial. ■

We say that the germ of the transverse Poisson structure at a point $q \in S$ is unimodular if there exists a submanifold N_q of M meeting the symplectic leaf S at q transversally, and such that

$$\operatorname{Mod}(N_q, P_q) = 0.$$

Theorem 6.19. *Let S be an embedded symplectic leaf of an orientable Poisson manifold (M, Π) and $q \in S$ a fixed point. Assume that the germ at $q \in S$ of the transverse Poisson structure P_q is unimodular. Then, one can choose a coupling neighborhood $(N \xrightarrow{\pi} S)$ of S with properties: there exists a leaf-wise volume form $\tau \in \Gamma(\wedge^{\mathrm{top}} V^*)$ of the vertical subbundle $\mathcal{V}^\pi$ such that the modular vector field of the Poisson foliation $(N, \mathcal{V}^\pi, P)$ vanishes. Furthermore, the modular vector field of $\Pi|_N$ with respect to the volume form $\Omega = \sigma^l \wedge \tau_\gamma$ is tangent to the symplectic foliation and the corresponding modular class is given by*

$$\operatorname{Mod}(N, \Pi|_N) = -(\Pi_H^\sharp)^*[\theta_\tau^\gamma].$$

Here, the divergence form θ_τ^γ induced by the pair (τ, γ) is a 1-cocycle of the cochain complex $(\mathcal{C}^\bullet, \overline{\mathrm{d}}^\gamma)$.

Proof. Choose a coupling neighborhood N such that the Poisson fiber bundle $(N \xrightarrow{\pi} S, P)$ is locally trivial with typical fiber (N_q, P_q). Then, by the proof of Corollary 6.4 we conclude that $\operatorname{Mod}(N, \mathcal{V}^\pi, P) = 0$. From Theorem 6.12, we derive the desired result. ■

Flat coupling neighborhoods. We say that a coupling neighborhood $N \xrightarrow{\pi} S$ over the leaf S is *flat* if there exists a regular foliation $\mathcal{F}$ on N such that (i) the tangent bundle $\mathbb{F} := T\mathcal{F}$ is complementary to the vertical subbundle V of π; (ii) each π-projectable section of $\mathbb{F}$ is an infinitesimal automorphism of the transverse Poisson structure $P \in \Gamma(\wedge^2 V)$; (iii) the foliation is compatible with the Poisson connection γ on N associated to the horizontal subbundle H in the following sense:

$$X - \gamma(X) \text{ is tangent to } P^\sharp(T^*M) \qquad \forall X \in \Gamma_{\mathrm{pr}}(\mathbb{F}).$$

Theorem 6.20. *Let S be an embedded symplectic leaf of an orientable Poisson manifold (M, Π) which admits a flat coupling neighborhood $(N \xrightarrow{\pi} S, \mathcal{F})$. Let P be the transverse Poisson structure on N of the leaf. If $H^1_{\overline{\partial}_\mathcal{F}} = \{0\}$, then the following assertions are equivalent:*

(a) the restriction of Π to N is unimodular;

(b) the Poisson manifold (N, P) is unimodular;

(c) the Poisson fibration $(N \xrightarrow{\pi} S, P)$ is unimodular,

$$\operatorname{Mod}(N, \mathcal{V}^\pi, P) = 0. \tag{6.46}$$

Proof. By Theorem 6.17, the assertions of items (a) and (c) are equivalent. The equivalence of (b) and (c) follows from Proposition 6.3 and the orientability of the symplectic leaf S. ■

Suppose we are given a flat Poisson fiber bundle $(\pi : M \to S, P, \mathcal{F})$ over a symplectic base (S, ω). Assume that S is an embedded submanifold of M, the inclusion map $\iota : S \hookrightarrow M$ is a section of π, $T_S\mathcal{F} = TS$ and the vertical Poisson structure $P \in \Gamma(\wedge^2 V)$ vanishes at the points of S. Let γ_0 be the flat Poisson connection on the Poisson fiber bundle $(\pi : M \to S, P)$ associated to the foliation $\mathcal{F}$. Denote by hor^{γ_0} the corresponding γ_0-horizontal lift and by $\psi \in \Gamma(\wedge^2 TS)$ the nondegenerated Poisson tensor of the symplectic manifold S. Then, putting $\sigma_0 = \pi^*\omega$, we get that formula (6.44) gives the following flat coupling Poisson tensor on M: $\Pi_{\mathrm{flat}} = \mathrm{hor}^{\gamma_0}(\psi) + P$. It is clear that (S, ω) is a symplectic leaf of Π_{flat}. Moreover, for a given horizontal 1-form $\mu \in \Gamma(V^\circ)$ on M vanishing along S, $\iota^*\mu = 0$, there exists a neighborhood N of S in M, such that the gauge transformation (6.32), (6.34) associated to μ is well-defined. Therefore, N is a flat coupling neighborhood of S for the deformed Poisson structure $\widetilde{\Pi}_{\mathrm{flat}}$. We get from Theorem 6.16 that the injectivity condition (6.41) together with $\mathrm{Mod}(\mathcal{F}) = 0$ and (6.46) provides the unimodularity of $\widetilde{\Pi}_{\mathrm{flat}}$.

6.8 The generalized Reeb class of a symplectic leaf

Let (M, Π) be a Poisson manifold and $S \hookrightarrow M$ an embedded symplectic leaf. Fix two exponential maps $\mathbf{f}, \widetilde{\mathbf{f}} : E \to M$ from the normal bundle E of S to a tubular neighborhood N of S. Consider the coupling Poisson structures $\mathbf{f}^*\Pi$ and $\widetilde{\mathbf{f}}^*\Pi$ and their associated geometric data (P, γ, σ) and $(\widetilde{P}, \widetilde{\gamma}, \widetilde{\sigma})$. Recall from Theorem 2.30 that the following relations hold for some $g \in \mathrm{Aut}(E)$ and $Q \in \Gamma(V^\circ)$:

$$g^*\widetilde{P} = P, \qquad \gamma - g^*\widetilde{\gamma} = P^\sharp(\mathrm{d}\, Q)^\flat, \qquad g^*\widetilde{\sigma} = \sigma - (\mathrm{d}^\gamma_{1,0}\, Q + \tfrac{1}{2}\{Q \wedge Q\}_P).$$

Proposition 6.21. *Let $(\mathcal{N}_0, \overline{\partial})$ and $(\widetilde{\mathcal{N}}_0, \widetilde{\overline{\partial}})$ be the respective de Rham - Casimir complexes. Then,*

1. *The pullback $g^* : (\widetilde{\mathcal{N}}_0, \widetilde{\overline{\partial}}) \to (\mathcal{N}_0, \overline{\partial})$ is a cochain complex isomorphism.*
2. *If the transverse Poisson structure of S is unimodular, then*

$$(g^*)_*\widetilde{\mathrm{Reeb}}(S) = \mathrm{Reeb}(S).$$

Proof. Since $g \in \mathrm{Aut}(E)$, g^* maps horizontal forms to horizontal forms. Moreover, $g^*\widetilde{P} = P$ implies $g^*\widetilde{\mathcal{N}}_0 = \mathcal{N}_0$. On the other hand, since the connections γ and $g^*\widetilde{\gamma}$ differ by Hamiltonian vector fields, it follows that their covariant derivatives coincide on $\mathcal{N}_0$. By $g \in \mathrm{Aut}(E)$, we have

$$g^* \circ \widetilde{\overline{\partial}} = g^* \circ \mathrm{d}^{\widetilde{\gamma}}_{1,0}|_{\widetilde{\mathcal{N}}_0} = \mathrm{d}^{g^*\widetilde{\gamma}}_{1,0}|_{g^*\widetilde{\mathcal{N}}_0} \circ g^* = \mathrm{d}^{\gamma}_{1,0}|_{\mathcal{N}_0} \circ g^* = \overline{\partial} \circ g^*.$$

This proves the first part. For the second one, observe that if $\widetilde{\tau}$ is a fiber-wise volume form preserved under the Hamiltonian vector fields of $\widetilde{P}$, then $\tau := g^*\widetilde{\tau}$ is preserved under the Hamiltonian vector fields of P. Furthermore, the corresponding divergence 1-forms satisfy

$$g^*\theta^{\widetilde{\gamma}}_{\widetilde{\tau}} \wedge \tau = g^*\theta^{\widetilde{\gamma}}_{\widetilde{\tau}} \wedge g^*\widetilde{\tau} = g^*(\theta^{\widetilde{\gamma}}_{\widetilde{\tau}} \wedge \widetilde{\tau}) = g^*(\mathrm{d}^{\widetilde{\gamma}}_{1,0}\, \widetilde{\tau}) = \mathrm{d}^{g^*\widetilde{\gamma}}_{1,0}\, g^*\widetilde{\tau} = \mathrm{d}^{\gamma - P^\sharp(\mathrm{d}\, Q)^\flat}_{1,0}\, \tau = \mathrm{d}^{\gamma}_{1,0}\, \tau = \theta^{\gamma}_{\tau} \wedge \tau.$$

Here, in the penultimate step, we have applied the fact that τ is invariant under $\mathrm{Ham}(E, P)$. Therefore, $g^*\theta^{\widetilde{\gamma}}_{\widetilde{\tau}} = \theta^{\gamma}_{\tau}$. Taking into account the first part of this theorem, we get

$$(g^*)_*\widetilde{\mathrm{Reeb}}(S) = (g^*)_*[\theta^{\widetilde{\gamma}}_{\widetilde{\tau}}] = [g^*\theta^{\widetilde{\gamma}}_{\widetilde{\tau}}] = [\theta^{\gamma}_{\tau}] = \mathrm{Reeb}(S).$$

■

Due to this result, there is a well-defined notion of *the generalized Reeb class of a symplectic leaf.*

Part IV

Appendix

Appendix A

Some algebraic properties of coupling structures

Here, we give a proof of Lemma 2.3.

Lemma A.1. *For the Lagrangian subspace $D \subset \mathbb{W}$, the following assertions are equivalent:*

(a) D is V-coupling.

(b) $H \oplus V = W$.

(c) $V^\circ \oplus A = W^*$.

In this case, $\operatorname{Ann}(H) = A$.

Proof. Suppose that D is a V-coupling Lagrangian subspace of $\mathbb{W}$. If $w \in H \cap V$, then there exists $\alpha \in V^\circ$ such that $w \oplus \alpha \in D \cap \mathbb{V} = \{0\}$. Therefore, $w = 0$, so $H \cap V = \{0\}$. Moreover, if $w \in W$ is arbitrary, then there exists $\eta \in D$, $v \in V$, and $\alpha \in V^\circ$ such that $w \oplus 0 = \eta + v \oplus \alpha$. This implies that $(w - v) \oplus \alpha = \eta \in D$, with $\alpha \in V^\circ$, so $w - v \in H$. This shows that $w \in V + H$, which proves $(a \Rightarrow b)$. Conversely, suppose that $H \oplus V = W$. If $\eta \in D \cap \mathbb{V}$, then $\eta = v \oplus \alpha$ for some $v \in V$ and $\alpha \in V^\circ$. Then, $v \in H \cap V = \{0\}$, so $\eta = 0 \oplus \alpha$. Now, take $X \in H$ and $\beta \in V^\circ$ such that $X \oplus \beta \in D$. By the isotropy of D,

$$0 = \langle X \oplus \beta, 0 \oplus \alpha \rangle = \alpha(X).$$

Since $X \in H$ is arbitrary, we get $\alpha \in H^\circ \cap V^\circ = \{0\}$. This proves that $\eta = 0$. Therefore, $D \cap \mathbb{V} = \{0\}$, proving that D is V-coupling. This completes the proof of $(a \Leftrightarrow b)$. The equivalence $(a \Leftrightarrow c)$ is analogous: just change the role between vectors and covectors. Finally, to see that $\operatorname{Ann}(H) = A$ when D is V-coupling, just note that $A \subseteq \operatorname{Ann}(H)$ and, from (b) and (c), that

$$\dim A = \operatorname{codim} V^\circ = \dim V = \operatorname{codim} H = \dim \operatorname{Ann}(H).$$

■

Lemma A.2. *Let (P, γ, σ) be geometric data on (W, V). Then,*

$$D_{P,\gamma,\sigma} = \{(P^\sharp \mu + X) \oplus (\mu - \mathrm{i}_X \sigma) \mid X \in H^\gamma, \mu \in A^\gamma\} = \operatorname{Graph}(P^\sharp|_{A^\gamma}) \oplus \operatorname{Graph}(-\sigma^\flat|_{H^\gamma})$$

is a maximally isotropic subspace of $\mathbb{W}$ which is V-coupling.

Proof. Denote $D \equiv D_{P,\gamma,\sigma}$. First we observe that

$$\dim D = \dim \operatorname{Graph}(P^\sharp|_{A^\gamma}) + \dim \operatorname{Graph}(-\sigma^\flat|_{H^\gamma}) = \dim A^\gamma + \dim H^\gamma = W,$$

due to the fact that $A^\gamma = \mathrm{Ann}(H^\gamma)$. Now, pick $\eta_1, \eta_2 \in D$. Then, there exist $\mu_i \in A^\gamma$ and $X_i \in H^\gamma$ such that $\eta_i = (P^\sharp \mu_i + X_i) \oplus (\mu_i - \mathrm{i}_{X_i} \sigma)$, $i = 1, 2$. Hence,

$$\langle \eta_1, \eta_2 \rangle = -\mathrm{i}_{X_1} \sigma(X_2) + \mu_1(P^\sharp \mu_2) - \mathrm{i}_{X_2} \sigma(X_1) + \mu_2(P^\sharp \mu_1) = 0.$$

Here, we have taken into account that $P^\sharp \mu_i \in V$, $\mathrm{i}_{X_i} \sigma \in V^\circ$, and the skew-symmetry of P and σ. This shows that D is maximally isotropic. To see that D is V-coupling, take $\eta \in D \cap \mathbb{V}$. Then, $\eta = (P^\sharp \mu + X) \oplus (\mu - \mathrm{i}_X \sigma)$ for some $\mu \in A^\gamma$, $X \in H^\gamma$, and $\eta \in V \oplus V^\circ$. Since $P^\sharp \mu + X \in V$, it follows that $X \in H^\gamma \cap V = \ker(\gamma) \cap \mathrm{im}(\gamma)$; and since $\mu - \mathrm{i}_X \sigma \in V^\circ$, we get that $\mu \in V^\circ \cap A^\gamma = \ker(\gamma^*) \cap \mathrm{im}(\gamma^*)$. Since $\gamma^2 = \gamma$, we conclude that $\ker(\gamma) \cap \mathrm{im}(\gamma) = 0$, and $\ker(\gamma^*) \cap \mathrm{im}(\gamma^*)$. Hence, $\mu = 0$, $X = 0$, and $\eta = 0$. This proves that $D \cap \mathbb{V} = 0$, so D is V-coupling. ∎

Lemma A.3. *Let D be a V-coupling Lagrangian subspace of $\mathbb{W}$. Then, for each $X \in H(D, V)$ there exists a unique $\alpha \in V^\circ$ such that $X \oplus \alpha \in D$. Furthermore, there exists a unique $\sigma_D \in \wedge^2 V^\circ$ such that $X \oplus (-\mathrm{i}_X \sigma_D) \in D$ for all $X \in H(D, V)$. Similarly, there exists a unique $P_D \in \wedge^2 V$ such that $P_D^\sharp \mu \oplus \mu \in D$ for all $\mu \in A(D, V)$.*

Proof. Let $X \in H(D, V)$ and $\alpha, \alpha' \in V^\circ$ such that $X \oplus \alpha, X \oplus \alpha' \in D$. Then, $0 \oplus (\alpha - \alpha') \in D \cap \mathbb{V}$. Since D is V-coupling, it follows that $\alpha - \alpha' = 0$. Now, let $T : H(D, V) \to V^\circ$ be the map defined by the relation $X \oplus T(X) \in D$. Let us show that T is linear. If $X \oplus T(X), Y \oplus T(Y) \in D$, then $(X + rY) \oplus (T(X) + rT(Y)) \in D$ for each scalar r, so $T(X + rY) = T(X) + rT(Y)$. Moreover, observe that $T(X)(Y) = -T(Y)(X)$. Indeed,

$$T(X)(Y) + T(Y)(X) = \langle X \oplus T(X), Y \oplus T(Y) \rangle = 0,$$

due to the isotropy of D. Therefore, there exists a unique $\sigma_D \in \wedge^2 V^\circ$ such that $\sigma_D^\flat|_{H(D,V)} = T$, that is, $X \oplus (-\mathrm{i}_X \sigma_D) \in D$ for all $X \in H(D, V)$. The construction of $P_D \in \wedge^2 V$ is totally analogous. ∎

We have shown in Lemma A.2 that, given geometric data (P, γ, σ), there sum of the restricted graphs is a coupling Lagrangian subbundle $D_{P,\gamma,\sigma}$ on (W, V). Conversely, in Lemma A.3, it was shown that the triple $(P_D, \gamma_D, \sigma_D)$ is a geometric data on (W, V), where $\gamma_D : W \to W$ is the projection over V along the splitting $H(D, V) \oplus V = W$. It is left to be shown that both constructions define the one-to-one correspondence of Proposition 2.4.

Proposition A.4. *The correspondence between coupling Lagrangian subspaces and geometric data given by Lemmas A.2 and A.3 is one-to-one.*

Proof. Let D be a V-coupling Lagrangian subspace of $\mathbb{W}$, and consider the geometric data $(P_D, \gamma_D, \sigma_D)$ defined as in Lemma A.3. Now, consider the V-coupling Lagrangian subspace $D_{P_D,\gamma_D,\sigma_D}$ given as in Lemma A.2. We have to show that $D_{P_D,\gamma_D,\sigma_D} = D$. By definition, $P_D \in \wedge^2 V$ is such that $P_D^\sharp \mu \oplus \mu \in D$ for all $\mu \in A(D, V)$. Therefore, $\mathrm{Graph}(P_D^\sharp|_{A(D,V)}) \subset D$. Similarly, $\mathrm{Graph}(\sigma_D^\flat|_{H(D,V)}) \subset D$. Since $D_{P_D,\gamma_D,\sigma_D}$ is the sum of both graphs, we conclude that $D_{P_D,\gamma_D,\sigma_D} \subseteq D$. Finally, since $D_{P_D,\gamma_D,\sigma_D}$ and D are maximally isotropic subspaces, we get $D_{P_D,\gamma_D,\sigma_D} = D$. Conversely, let (P, γ, σ) be a triple of geometric data, and let $D_{P,\gamma,\sigma}$ be given as in Lemma A.2. Now, consider the triple of geometric data $(\tilde{P}, \tilde{\gamma}, \tilde{\sigma})$ defined by $D_{P,\gamma,\sigma}$ as in Lemma A.3. We must show that $(\tilde{P}, \tilde{\gamma}, \tilde{\sigma}) = (P, \gamma, \sigma)$. First note that, since $D_{P,\gamma,\sigma}$ is the sum of the graphs of $\sigma \in \wedge^2 V^\circ$ and $P \in \wedge^2 V$ restricted to H^γ and A^γ, respectively, we have that $H(D_{P,\gamma,\sigma}, V) = H^\gamma$, so $\tilde{\gamma} = \gamma$. Since $\tilde{\sigma} \in \wedge^2 V^\circ$ is defined in such a way that $X \oplus (-\mathrm{i}_X \tilde{\sigma}) \in D_{P,\gamma,\sigma}$ for all $X \in H(D_{P,\gamma,\sigma}, V)$, we conclude that $-\mathrm{i}_X \tilde{\sigma} = -\mathrm{i}_X \sigma$ for all $X \in H^\gamma$. By Lemma 2.3 (b), we get $\tilde{\sigma} = \sigma$. The proof of $\tilde{P} = P$ is analogous. ∎

Appendix B

Courant algebroids from 3-forms

B.1 Some useful formulas

Here we state and proof some useful computational facts which are needed in several parts of the text.

Lemma B.1. *Let $(E, q, [\cdot,\cdot]_E)$ be a Lie algebroid and $\Pi \in \Gamma(\wedge^2 E)$. Then, for all $\alpha, \beta, \theta \in \Gamma(E^*)$, we have*

$$[\Pi,\Pi]_E(\alpha_1,\alpha_2,\alpha_3) = -2\sum_{(1,2,3)} \Pi(\alpha_1, \partial\Pi(\alpha_2,\alpha_3)) - \alpha_3[\Pi^\sharp\alpha_1, \Pi^\sharp\alpha_2]_E.$$

Proof. By definition of the Schouten bracket,

$$[\Pi,\Pi]_E(\alpha_1,\alpha_2,\alpha_3) = -\,\mathrm{i}_{[\Pi,\Pi]_E}(\alpha_1\wedge\alpha_2\wedge\alpha_3) = -2\,\mathrm{i}_\Pi\,\partial\,\mathrm{i}_\Pi(\alpha_1\wedge\alpha_2\wedge\alpha_3) + \mathrm{i}_\Pi\,\mathrm{i}_\Pi(\alpha_1\wedge\alpha_2\wedge\alpha_3).$$

By straightforward computation, we get

$$\mathrm{i}_\Pi\,\partial\,\mathrm{i}_\Pi(\alpha_1\wedge\alpha_2\wedge\alpha_3) = \sum_{(1,2,3)} \Pi(\partial\Pi(\alpha_1,\alpha_2),\alpha_3) - \Pi(\alpha_1,\alpha_2)\,\mathrm{i}_\Pi\,\partial\alpha_3,$$

and

$$\mathrm{i}_\Pi\,\mathrm{i}_\Pi\,\partial(\alpha_1\wedge\alpha_2\wedge\alpha_3) = -2\sum_{(1,2,3)} \Pi(\alpha_1,\alpha_2)(\mathrm{i}_\Pi\,\partial\alpha_3) + 2\Pi(\alpha_1,\partial\Pi(\alpha_2,\alpha_3)) - \alpha_3[\Pi^\sharp\alpha_1,\Pi^\sharp\alpha_2]_E.$$

Therefore,

$$[\Pi,\Pi]_E(\alpha_1,\alpha_2,\alpha_3) = -2\sum_{(1,2,3)} \Pi(\alpha_1, \partial\Pi(\alpha_2,\alpha_3)) - \alpha_3[\Pi^\sharp\alpha_1, \Pi^\sharp\alpha_2]_E.$$

∎

Lemma B.2. *Let $(E, q, [\cdot,\cdot]_E)$ be a Lie algebroid, $P \in \Gamma(\wedge^2 E)$ a bivector, and $\psi \in \Gamma(\wedge^3 E^*)$ a 3-cocycle, $\partial\psi = 0$. Define*

$$\{\alpha,\beta\}_{P,\psi} := \mathrm{L}^{\partial}_{P^\sharp\alpha}\beta - \mathrm{i}_{P^\sharp\beta}\,\partial\alpha - \mathrm{i}_{P^\sharp\beta}\,\mathrm{i}_{P^\sharp\alpha}\,\psi, \qquad \forall \alpha,\beta \in \Gamma(E^*).$$

If $X \in \Gamma(E)$, $\mu,\nu \in \Gamma(E^*)$ are such that $\mu(X) = \nu(X) = 0$, then

$$\mathrm{i}_X\{\mu,\nu\}_{P,\psi} + P^\sharp\,\mathrm{i}_X\,\psi(\mu,\nu) = [X,P]_E(\mu,\nu).$$

Proof. By straightforward computations,

$$\begin{aligned}[X,P]_E(\mu,\nu) &= \mathrm{L}^{\partial}_X P(\mu,\nu) = \mathrm{L}_X(P(\mu,\nu)) - P(\mathrm{L}_X \mu,\nu) - P(\mu,\mathrm{L}_X \nu)\\ &= \mathrm{L}_X(P(\mu,\nu)) + \mathrm{L}_X \mu(P^\sharp\nu) - \mathrm{L}_X \nu(P^\sharp\mu)\\ &= \mathrm{L}_X(P(\mu,\nu)) + \mathrm{L}_X(\mu(P^\sharp\nu)) - \mu[X,P^\sharp\nu] - \mathrm{L}_X(\nu(P^\sharp\mu)) + \nu[X,P^\sharp\mu].\end{aligned}$$

Also, since $\mu(X)=\nu(X)=0$,

$$\begin{aligned}\mathrm{i}_X\{\mu,\nu\}_{P,\psi} + P^\sharp\,\mathrm{i}_X\,\psi(\mu,\nu) &= (\mathrm{L}^{\partial}_{P^\sharp\mu}\nu - \mathrm{i}_{P^\sharp\nu}\,\partial\mu - \mathrm{i}_{P^\sharp\nu}\,\mathrm{i}_{P^\sharp\mu}\,\psi)(X) + \mathrm{i}_X\,\psi(P^\sharp\mu,P^\sharp\nu)\\ &= \mathrm{L}^{\partial}_{P^\sharp\mu}\nu(X) - \mathrm{i}_{P^\sharp\nu}\,\partial\mu(X)\\ &= \mathrm{L}^{\partial}_{P^\sharp\mu}(\nu(X)) - \nu[P^\sharp\mu,X]_E - \mathrm{L}^{\partial}_{P^\sharp\nu}(\mu(X)) + \mathrm{L}^{\partial}_X(\mu(P^\sharp\nu)) + \mu[P^\sharp\nu,X]_E\\ &= -\,\mathrm{L}^{\partial}_X(P(\mu,\nu)) + \mu[P^\sharp\nu,X]_E - \nu[P^\sharp\mu,X]_E.\end{aligned}$$

■

In the case when $\psi=0$, we get the following immediate consequence of Lemma B.2.

Lemma B.3. *Let $(E,q,[\cdot,\cdot]_E)$ be a Lie algebroid, $P\in\Gamma(\wedge^2E)$ a bivector, and*

$$\{\alpha,\beta\}_P := \mathrm{L}^{\partial}_{P^\sharp\alpha}\beta - \mathrm{i}_{P^\sharp\beta}\,\partial\alpha, \qquad \forall\alpha,\beta\in\Gamma(E^*).$$

If $X\in\Gamma(E)$, $\mu,\nu\in\Gamma(E^*)$ are such that $\mu(X)=\nu(X)=0$, then

$$\mathrm{i}_X\{\mu,\nu\}_P = [X,P]_E(\mu,\nu).$$

Lemma B.4. *Let $(E,q,[\cdot,\cdot]_E)$ be a Lie algebroid, $a\in\Gamma(E)$ and $\Pi\in\Gamma(\wedge^2E)$. Then, for all $\alpha,\beta\in\Gamma(E^*)$ and $a\in\Gamma(E)$, we have*

$$\mathrm{L}^{\partial}_a(\Pi(\alpha,\beta)) = [a,\Pi]_E(\alpha,\beta) + \Pi(\mathrm{L}^{\partial}_a\alpha,\beta) + \Pi(\alpha,\mathrm{L}^{\partial}_a\beta), \quad \textit{and} \quad [a,\Pi^\sharp\alpha]_E = \Pi^\sharp\,\mathrm{L}^{\partial}_a\alpha + \mathrm{i}_\alpha[a,\Pi]_E.$$

Proof. By definition of the Schouten bracket,

$$\begin{aligned}[a,\Pi]_E(\alpha,\beta) &= -\,\mathrm{i}_{[a,\Pi]_E}(\alpha\wedge\beta) = -[\mathrm{L}^{\partial}_a,\mathrm{i}_\Pi]_E(\alpha\wedge\beta)\\ &= \mathrm{L}^{\partial}_a(\Pi(\alpha,\beta)) + \mathrm{i}_\Pi(\mathrm{L}^{\partial}_a\alpha\wedge\beta + \alpha\wedge\mathrm{L}^{\partial}_a\beta)\\ &= \mathrm{L}^{\partial}_a(\Pi(\alpha,\beta)) - \Pi(L^{\partial}_a\alpha,\beta) - \Pi(\alpha,\mathrm{L}^{\partial}_a\beta),\end{aligned}$$

which proves the first formula. For the second one, just rewrite:

$$\begin{aligned}\mathrm{i}_\alpha[a,\Pi]_E(\beta) &= [a,\Pi]_E(\alpha,\beta) = \mathrm{L}^{\partial}_a(\Pi(\alpha,\beta)) - \Pi(\mathrm{L}^{\partial}_a\alpha,\beta) - \Pi(\alpha,\mathrm{L}^{\partial}_a\beta)\\ &= \mathrm{L}^{\partial}_a\,\mathrm{i}_{\Pi^\sharp\alpha}\beta - \Pi^\sharp\,\mathrm{L}^{\partial}_a\alpha(\beta) - \mathrm{i}_{\Pi^\sharp\alpha}\,\mathrm{L}^{\partial}_a\beta\\ &= \mathrm{i}_{[a,\Pi^\sharp\alpha]_E}\beta - \Pi^\sharp\,\mathrm{L}^{\partial}_a\alpha(\beta).\end{aligned}$$

■

B.2 The Courant algebroid of a Lie algebroid with background

Here we recall some basic facts of Courant algebroids which we use throughout the text.

Given a Lie algebroid $(E, \rho, [,])$ the vector bundle $\mathbb{E} := E \oplus E^*$ can be endowed with many structures of Courant algebroid. Consider the bilinear symmetric form on $\mathbb{E}$ given by

$$\langle a \oplus \alpha, b \oplus \beta \rangle := \beta(a) + \alpha(b) \tag{B.1}$$

and the projection $p : \mathbb{E} \to TM$ given by $p := \rho \circ p_E$, where $p_E : E \oplus E^* \to E$ is the projection on the first factor. Finally, for each 3-cocycle $\psi \in \Gamma(\wedge^3 E^*)$ of the exterior derivative ∂, define

$$[\![a \oplus \alpha, b \oplus \beta]\!]_\psi := [a, b] \oplus (\mathrm{L}^E_a \alpha - \mathrm{i}_b \partial\beta - \mathrm{i}_b \mathrm{i}_a \psi). \tag{B.2}$$

It is a straightforward computation to verify that $\mathbb{E}$ acquires a structure of Courant algebroid.

Proposition B.5. *Let $(E, \rho, [,])$ be a Lie algebroid and $\psi \in \Gamma(\wedge^3 E^*)$ be a 3-cocycle of exterior derivative ∂. Then, $\mathbb{E}_\psi := E \oplus E^*$ has a natural structure of Courant algebroid, where $\langle,\rangle$, $p : \mathbb{E} \to TM$, and $[\![,]\!] := [\![,]\!]_\psi$ are defined as in above.*

Proof. Suppose that $\eta, \eta', \eta_1, \eta_2 \in \Gamma(\mathbb{E}_\psi)$ are of form $\eta = a \oplus \alpha$, $\eta' = a' \oplus \alpha'$, $\eta_1 = a_1 \oplus \alpha_1$, $\eta_2 = a_2 \oplus \alpha_2$ and $f \in C^\infty(M)$. Let us show that each of the properties (CA1)-(CA3) hold.

(CA1) By the definition of the bracket $[\![,]\!]$, we get

$$\begin{aligned}
[\![\eta, [\![\eta_1, \eta_2]\!]]\!] &= [\![a \oplus \alpha, [a_1, a_2] \oplus (\mathrm{L}^E_{a_1} \alpha_2 - \mathrm{i}_{a_2} \partial\alpha_1 x - \mathrm{i}_{a_2} \mathrm{i}_{a_1} \psi)]\!] \\
&= [a, [a_1, a_2]] \oplus (\mathrm{L}^E_a (\mathrm{L}^E_{a_1} \alpha_2 - \mathrm{i}_{a_2} \partial\alpha_1 - \mathrm{i}_{a_2} \mathrm{i}_{a_1} \psi) - \mathrm{i}_{[a_1,a_2]} \partial\alpha - \mathrm{i}_{[a_1,a_2]} \mathrm{i}_a \psi).
\end{aligned}$$

Also, we have

$$\begin{aligned}
&[\![[\![\eta, \eta_1]\!], \eta_2]\!] = \\
&[\![[a, a_1] \oplus (\mathrm{L}^E_a \alpha_1 - \mathrm{i}_{a_1} \partial\alpha - \mathrm{i}_{a_1} \mathrm{i}_a \psi), a_2 \oplus \alpha_2]\!] = \\
&[[a, a_1], a_2] \oplus (\mathrm{L}^E_{[a,a_1]} \alpha_2 - \mathrm{i}_{a_2} \partial(\mathrm{L}^E_a \alpha_1 - \mathrm{i}_{a_1} \partial\alpha - \mathrm{i}_{a_1} \mathrm{i}_a \psi) - \mathrm{i}_{a_2} \mathrm{i}_{[a,a_1]} \psi) = \\
&[[a, a_1], a_2] \oplus (\mathrm{L}^E_{[a,a_1]} \alpha_2 - \mathrm{i}_{a_2} \partial \mathrm{L}^E_a \alpha_1 + \mathrm{i}_{a_2} \partial \mathrm{i}_{a_1} \partial\alpha + \mathrm{i}_{a_2} \partial \mathrm{i}_{a_1} \mathrm{i}_a \psi - \mathrm{i}_{a_2} \mathrm{i}_{[a,a_1]} \psi),
\end{aligned}$$

and

$$\begin{aligned}
[\![\eta_1, [\![\eta, \eta_2]\!]]\!] &= [\![a_1 \oplus \alpha_1, [a, a_2] \oplus (\mathrm{L}^E_a \alpha_2 - \mathrm{i}_{a_2} \partial\alpha - \mathrm{i}_{a_2} \mathrm{i}_a \psi)]\!] \\
&= [a_1, [a, a_2]] \oplus (\mathrm{L}^E_{a_1} (\mathrm{L}^E_a \alpha_2 - \mathrm{i}_{a_2} \partial\alpha - \mathrm{i}_{a_2} \mathrm{i}_a \psi) - \mathrm{i}_{[a,a_2]} \partial\alpha_1 - \mathrm{i}_{[a,a_2]} \mathrm{i}_{a_1} \psi) \\
&= [a_1, [a, a_2]] \oplus (\mathrm{L}^E_{a_1} \mathrm{L}^E_a \alpha_2 - \mathrm{L}^E_{a_1} \mathrm{i}_{a_2} \partial\alpha - \mathrm{i}_{[a,a_2]} \partial\alpha_1 - \mathrm{L}^E_{a_1} \mathrm{i}_{a_2} \mathrm{i}_a \psi - \mathrm{i}_{[a,a_2]} \mathrm{i}_{a_1} \psi).
\end{aligned}$$

Clearly, the Jacobi identity for the Lie bracket of $\Gamma(E)$ implies that the E-components of $[\![[\![\eta, \eta_1]\!], \eta_2]\!] + [\![\eta_1, [\![\eta, \eta_2]\!]]\!]$ and $[\![\eta, [\![\eta_1, \eta_2]\!]]\!]$ coincide. For E^*-component of $[\![[\![\eta, \eta_1]\!], \eta_2]\!] + [\![\eta_1, [\![\eta, \eta_2]\!]]\!]$, the part not involving ψ is

$$\begin{aligned}
&(\mathrm{L}^E_{[a,a_1]} \alpha_2 - \mathrm{i}_{a_2} \partial \mathrm{L}^E_a \alpha_1 + \mathrm{i}_{a_2} \partial \mathrm{i}_{a_1} \partial\alpha) + (\mathrm{L}^E_{a_1} \mathrm{L}^E_a \alpha_2 - \mathrm{L}^E_{a_1} \mathrm{i}_{a_2} \partial\alpha - \mathrm{i}_{[a,a_2]} \partial\alpha_1) = \\
&(\mathrm{L}^E_{[a,a_1]} + \mathrm{L}^E_{a_1} \mathrm{L}^E_a)\alpha_2 - (\mathrm{i}_{a_2} \partial \mathrm{L}^E_a + \mathrm{i}_{[a,a_2]} \partial)\alpha_1 + (\mathrm{i}_{a_2} \partial \mathrm{i}_{a_1} - \mathrm{L}^E_{a_1} \mathrm{i}_{a_2})\partial\alpha = \\
&(\mathrm{L}^E_a \mathrm{L}^E_{a_1})\alpha_2 - (\mathrm{L}^E_a \mathrm{i}_{a_2} \partial)\alpha_1 + (\mathrm{i}_{a_2} \partial \mathrm{i}_{a_1} + \mathrm{i}_{a_2} \mathrm{i}_{a_1} \partial - \mathrm{L}^E_{a_1} \mathrm{i}_{a_2})\partial\alpha = \\
&\mathrm{L}^E_a (\mathrm{L}^E_{a_1} \alpha_2 - \mathrm{i}_{a_2} \partial\alpha_1) + (\mathrm{i}_{a_2} \mathrm{L}^E_{a_1} - \mathrm{L}^E_{a_1} \mathrm{i}_{a_2})\partial\alpha = \\
&\mathrm{L}^E_a (\mathrm{L}^E_{a_1} \alpha_2 - \mathrm{i}_{a_2} \partial\alpha_1) - \mathrm{i}_{[a_1,a_2]} \partial\alpha,
\end{aligned}$$

which coincides with the E^*-component of $[\![\eta,[\![\eta_1,\eta_2]\!]]\!]$ not involving the 3-cocycle ψ. To complete the proof, we need to verify

$$\mathrm{i}_{a_2}\partial\,\mathrm{i}_{a_1}\mathrm{i}_a\psi-\mathrm{i}_{a_2}\mathrm{i}_{[a,a_1]}\psi-\mathrm{L}^E_{a_1}\mathrm{i}_{a_2}\mathrm{i}_a\psi-\mathrm{i}_{[a,a_2]}\mathrm{i}_{a_1}\psi=-\mathrm{L}^E_a\mathrm{i}_{a_2}\mathrm{i}_{a_1}\psi-\mathrm{i}_{[a_1,a_2]}\mathrm{i}_a\psi$$

which is equivalent to

$$\mathrm{i}_{a_2}\partial\,\mathrm{i}_{a_1}\mathrm{i}_a\psi-\mathrm{i}_{a_2}\mathrm{i}_{[a,a_1]}\psi-\mathrm{i}_{a_2}\mathrm{L}^E_{a_1}\mathrm{i}_a\psi+\mathrm{i}_{a_2}\mathrm{L}^E_a\mathrm{i}_{a_1}\psi=0.$$

Adding the first to the third term and the second term to the fourth, we get

$$-\mathrm{i}_{a_2}\mathrm{i}_{a_1}\partial\,\mathrm{i}_a\psi+\mathrm{i}_{a_2}\mathrm{i}_{a_1}\mathrm{L}^E_a\psi=0.$$

Finally, the above sum is just $\mathrm{i}_{a_2}\mathrm{i}_{a_1}\mathrm{i}_a\partial\psi$, which is zero because of $\partial\psi=0$.

(CA2) By definition of the bracket $[\![,]\!]$,

$$\begin{aligned}\langle[\![\eta,\eta_1]\!],\eta_2\rangle&=\langle[a,a_1]\oplus(\mathrm{L}^E_a\alpha_1-\mathrm{i}_{a_1}\mathrm{d}_E\alpha-\mathrm{i}_{a_1}\mathrm{i}_a\psi),a_2\oplus\alpha_2\rangle\\&=\alpha_2([a,a_1])+(\mathrm{L}^E_a\alpha_1-\mathrm{i}_{a_1}\mathrm{d}_E\alpha-\mathrm{i}_{a_1}\mathrm{i}_a\psi)(a_2)\\&=\alpha_2([a,a_1])+\mathrm{L}^E_a\alpha_1(a_2)-\mathrm{i}_{a_2}\mathrm{i}_{a_1}\mathrm{d}_E\alpha-\mathrm{i}_{a_2}\mathrm{i}_{a_1}\mathrm{i}_a\psi\\&=\alpha_2([a,a_1])-\alpha_1([a,a_2])+\mathrm{L}^E_a(\alpha_1(a_2))-\mathrm{i}_{a_2}\mathrm{i}_{a_1}\mathrm{d}_E\alpha-\mathrm{i}_{a_2}\mathrm{i}_{a_1}\mathrm{i}_a\psi.\end{aligned}$$

Because of the symmetry of $\langle,\rangle$, an analogous computation leads to

$$\langle\eta_1,[\![\eta,\eta_2]\!]\rangle=\langle[\![\eta,\eta_2]\!],\eta_1\rangle=\alpha_1([a,a_2])-\alpha_2([a,a_1])+\mathrm{L}^E_a(\alpha_2(a_1))-\mathrm{i}_{a_1}\mathrm{i}_{a_2}\mathrm{d}_E\alpha-\mathrm{i}_{a_1}\mathrm{i}_{a_2}\mathrm{i}_a\psi.$$

Adding up both equations and cancelling out by the skew-symmetry of $\mathrm{d}_E\alpha$ and $\mathrm{i}_a\psi$, we get

$$\langle[\![\eta,\eta_1]\!],\eta_2\rangle+\langle\eta_1,[\![\eta,\eta_2]\!]\rangle=\mathrm{L}^E_a(\alpha_1(a_2))+\mathrm{L}^E_a(\alpha_2(a_1))=\mathrm{L}^E_a\langle\eta_1,\eta_2\rangle=\mathrm{L}_{p(\eta)}\langle\eta_1,\eta_2\rangle.$$

(CA3) Note that

$$[\![\eta_2,\eta_2]\!]=[a_2,a_2]\oplus(\mathrm{L}^E_{a_2}\alpha_2-\mathrm{i}_{a_2}\partial\alpha_2-\mathrm{i}_{a_2}\mathrm{i}_{a_2}\psi)=0\oplus\partial\,\mathrm{i}_{a_2}\alpha_2=0\oplus\tfrac{1}{2}\partial\langle\eta_2,\eta_2\rangle.$$

Then,

$$\langle\eta_1,[\![\eta_2,\eta_2]\!]\rangle=\langle a_1\oplus\alpha_1,0\oplus\tfrac{1}{2}\partial\langle\eta_2,\eta_2\rangle=\tfrac{1}{2}\partial\langle\eta_2,\eta_2\rangle(a_1)=\tfrac{1}{2}\mathrm{L}_{q(a_1)}\langle\eta_2,\eta_2\rangle=\tfrac{1}{2}\mathrm{L}_{p(\eta_1)}\langle\eta_2,\eta_2\rangle.$$

$\blacksquare$

Appendix C

Bigraded models on algebraic Lie algebroids

Following [40], let $\mathscr{R}$ be a ring, $\mathscr{C}$ a commutative $\mathscr{R}$-algebra, and $\operatorname{End}_{\mathscr{R}}\mathscr{C}$ the $\mathscr{C}$-module of $\mathscr{R}$-linear endomorphisms of $\mathscr{C}$. We say that $X \in \operatorname{End}_{\mathscr{R}}\mathscr{C}$ is a *derivation* if the Leibniz rule

$$X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$$

is satisfied for all $f, g \in \mathscr{C}$. It is clear that the set $\operatorname{Der}_{\mathscr{R}}\mathscr{C}$ of $\mathscr{R}$-linear derivations is a Lie subalgebra of $(\operatorname{End}_{\mathscr{R}}\mathscr{C}, [\cdot,\cdot])$, where the bracket denotes the commutator $[X,Y] := X \circ Y - Y \circ X$.

Recall that a Lie algebroid over $\mathscr{C}$ is a triple $(\mathscr{A}, \mathrm{q}, [\cdot,\cdot]_{\mathscr{A}})$ consisting of a faithful $\mathscr{C}$-module $\mathscr{A}$, together with a Lie bracket $[\cdot,\cdot]_{\mathscr{A}}$ on $\mathscr{A}$, and a $\mathscr{C}$-module morphism $\mathrm{q} : \mathscr{A} \to \operatorname{End}_{\mathscr{R}}\mathscr{C}$, compatible with the Lie bracket by

$$[a, fb]_{\mathscr{A}} = f[a,b]_{\mathscr{A}} + \mathrm{q}(a)(f)b,$$

for all $f \in \mathscr{C}$, $a, b \in \mathscr{A}$. The $\mathscr{C}$-module $\mathscr{A}$ is called the *total space*, and the $\mathscr{C}$-module morphism $\mathrm{q} : \mathscr{A} \to \operatorname{End}_{\mathscr{R}}\mathscr{C}$ is called the *anchor*. The previous formula is called the *Leibniz rule*. By the faithfulness of $\mathscr{A}$ and the Leibniz rule, straightforward computations show that q is a representation of $\mathscr{A}$ on $\mathscr{C}$ by derivations, that is, $\mathrm{q}(a) \in \operatorname{Der}_{\mathscr{R}}\mathscr{C}$, and $\mathrm{q}[a,b]_{\mathscr{A}} = [\mathrm{q}(a), \mathrm{q}(b)]_{\mathscr{A}}$ for all $a, b \in \mathscr{A}$.

Example C.1. A well-known example of a Lie algebroid over a commutative $\mathscr{R}$-algebra $\mathscr{C}$ is the *Lie algebroid of derivations*, $(\mathscr{A} := \operatorname{Der}_{\mathscr{R}}\mathscr{C}, \mathrm{Id}_{\mathscr{A}}, [\cdot,\cdot])$, where the anchor is precisely the identity map, and $[\cdot,\cdot]$ denotes the commutator of derivations. ▼

Let us consider the dual of $\mathscr{A}$ as a $\mathscr{C}$-module, $\mathscr{A}^* := \operatorname{Hom}_{\mathscr{C}}(\mathscr{A}, \mathscr{C})$. We now assume that the following condition holds,

$$\bigcap_{\alpha \in \mathscr{A}^*} \ker(\alpha) = \{0\}, \tag{C.1}$$

which is equivalent to say that every element $a \in \mathscr{A}$ is uniquely determined by the values $\alpha(a) \in \mathscr{C}$, where α runs over all $\mathscr{A}^*$. In other words, the canonical map $\mathscr{A} \ni a \mapsto \mathrm{i}_a \in \operatorname{Hom}_{\mathscr{C}}(\mathscr{A}^*, \mathscr{C})$, given on each $\alpha \in \mathscr{A}^*$ by $\mathrm{i}_a\,\alpha := \alpha(a)$, is injective.

For each $k \in \mathbb{Z}$, let $\mathcal{C}^k$ be the $\mathscr{C}$-module of $\mathscr{C}$-multilinear skew-symmetric maps

$$\alpha : \underbrace{\mathscr{A} \times \cdots \times \mathscr{A}}_{k \text{ times}} \to \mathscr{C}.$$

In particular, $\mathcal{C}^1 = \mathscr{A}^*$ is the dual $\mathscr{C}$-module of $\mathscr{A}$, $\mathcal{C}^0 = \mathscr{C}$, and $\mathcal{C}^k = \{0\}$ for $k < 0$. Then, $\mathcal{C}^\bullet := \bigoplus_{k \in \mathbb{Z}} \mathcal{C}^k$ is a graded commutative and associative $\mathscr{C}$-algebra with the exterior product $\wedge$ defined as usual.

The Lie algebroid structure of $\mathscr{A}$ naturally induces a derivation on $\mathcal{C}$, which is a coboundary operator on $\mathcal{C}$ of degree 1, $\partial \in \mathrm{Der}^1_{\mathscr{C}}\mathcal{C}$. Indeed, for each $\alpha \in \mathcal{C}^k$, $\partial\alpha \in \mathcal{C}^{k+1}$ is given by the Koszul's formula

$$\begin{aligned}\partial\alpha(a_1, \dots, a_{k+1}) := &\sum_{\sigma\in S_{(1,k+1)}} \mathrm{sgn}(\sigma)\, \mathrm{q}(a_{\sigma_1})(\alpha(a_{\sigma_2}, \dots, a_{\sigma_k})) \\ &- \sum_{\sigma\in S_{(2,k)}} \mathrm{sgn}(\sigma)\alpha([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}, \dots, a_{\sigma_{k+1}}),\end{aligned} \tag{C.2}$$

where $a_1, \dots, a_{k+1} \in \mathscr{A}$. Here, $S_{(i,j)}$ denotes the set of all shuffle permutations $\sigma \in S_{i+j}$, with $\sigma_1 < \cdots < \sigma_i$, and $\sigma_{i+1} < \cdots < \sigma_{i+j}$. A similar notation applies for $S_{(i,j,k)}$. Then, the Lie algebroid cohomology of $\mathscr{A}$ is defined by $H^\bullet(\mathscr{A}) := H^\bullet(\mathcal{C}, \partial)$.

To see that $\partial^2 = 0$, fix $\alpha \in \mathcal{C}^k$ and $a_1, \dots, a_{k+2} \in \mathscr{A}$. Then, a straightforward computation leads

$$\begin{aligned}\partial^2\alpha(a_1, \dots, a_{k+2}) = &\sum_{\sigma\in S_{(1,1,k)}} \mathrm{sgn}(\sigma)\, \mathrm{q}(a_{\sigma_1})(\mathrm{q}(a_{\sigma_2})(\alpha(a_{\sigma_3}, \dots, a_{\sigma_{k+2}}))) \\ &- \sum_{\sigma\in S_{(1,2,k-1)}} \mathrm{sgn}(\sigma)\, \mathrm{q}(a_{\sigma_1})(\alpha([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3} \dots, a_{\sigma_{k+2}})) \\ &- \sum_{\sigma\in S_{(2,k)}} \mathrm{sgn}(\sigma)\, \mathrm{q}[a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}(\alpha(a_{\sigma_3}, \dots, a_{\sigma_{k+2}})) \\ &+ \sum_{\sigma\in S_{(1,2,k-1)}} \mathrm{sgn}(\sigma)\, \mathrm{q}(a_{\sigma_1})(\alpha([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3} \dots, a_{\sigma_{k+2}})) \\ &+ \sum_{\sigma\in S_{(2,1,k-1)}} \mathrm{sgn}(\sigma)\alpha([[a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}]_{\mathscr{A}}, a_{\sigma_4}, \dots, a_{\sigma_{k+2}}) \\ &+ \sum_{\sigma\in S_{(2,2,k-2)}} \mathrm{sgn}(\sigma)\alpha([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, [a_{\sigma_3}, a_{\sigma_4}]_{\mathscr{A}}, a_{\sigma_5} \dots, a_{\sigma_{k+2}}).\end{aligned}$$

Clearly, the second and fourth sums cancel out with each other. Moreover, the last sum is identically zero, due to the skew-symmetry of α. Since q is a representation, the first and third sums also cancel out with each other. Finally, the fifth sum is zero due to the Jacobi identity of $[\cdot,\cdot]_{\mathscr{A}}$. A similar computation shows that $\partial \in \mathrm{Der}^1_{\mathscr{C}}\mathcal{C}$. Furthermore, condition (C.1) implies that the coboundary operator ∂ is determined by a unique Lie algebroid structure on $\mathscr{A}$. Indeed, the anchor map q and the Lie bracket can be expressed in terms of ∂ in (C.2) for $k = 0$ and $k = 1$, respectively.

Now, recall that for each $a \in \mathscr{A}$, the insertion $\mathrm{i}_a : \mathcal{C} \to \mathcal{C}$, given by

$$\mathrm{i}_a\, \alpha(a_1, \dots, a_{l-1}) := \alpha(a, a_1, \dots, a_{l-1}), \qquad \alpha \in \mathcal{C}^l, \quad a_i \in \mathscr{A},$$

is a graded derivation of degree -1, $\mathrm{i}_a \in \mathrm{Der}^{-1}_{\mathscr{C}}\mathcal{C}$. More generally, if $K : \mathscr{A} \times \cdots \times \mathscr{A} \to \mathscr{A}$ is a k-linear (over $\mathscr{C}$) skew-symmetric map, then $\mathrm{i}_K \in \mathrm{Der}^{k-1}_{\mathscr{C}}\mathcal{C}$, where

$$\mathrm{i}_K\, \alpha(a_1, \dots, a_{k+l-1}) := \sum_{\sigma\in S_{(k,l-1)}} \mathrm{sgn}(\sigma)\alpha(K(a_{\sigma_1}, \dots, a_{\sigma_k}), a_{\sigma_{k+1}}, \dots, a_{\sigma_{k+l-1}}), \quad \alpha \in \mathcal{C}^l, \quad a_i \in \mathscr{A}.$$

Hence, $\mathrm{L}^{\partial}_K := [\mathrm{i}_K, \partial]$ is a graded derivation of degree k, $\mathrm{L}^{\partial}_K \in \mathrm{Der}^k_{\mathscr{C}}\mathcal{C}$. In particular, if $K = \mathrm{Id}_{\mathscr{A}}$ is the identity map on $\mathscr{A}$, then $\mathrm{i}_{\mathrm{Id}_{\mathscr{A}}}\, \alpha = k\alpha$ for all $\alpha \in \mathcal{C}^k$, and hence, $\mathrm{L}^{\partial}_{\mathrm{Id}_{\mathscr{A}}} = \partial$.

Bigrading on $\mathscr{C}$-modules. Let $\mathscr{A}$ be a module over the commutative $\mathscr{R}$-algebra $\mathscr{C}$. An endomorphism $\mathrm{p} \in \operatorname{End}_{\mathscr{C}} \mathscr{A}$ is said to be a *connection* on $\mathscr{A}$ if it is a projection map, $\mathrm{p}^2 = \mathrm{p}$. Let us define the p-horizontal and p-vertical submodules of $\mathscr{A}$ by $\mathcal{H}^{\mathrm{p}} := \ker(\mathrm{p})$ and $\mathcal{V}^{\mathrm{p}} := \operatorname{im}(\mathrm{p})$. We will drop the superscript p when the choice of the connection is clear. Observe that $\mathscr{A} = \mathcal{H} \oplus \mathcal{V}$. Furthermore, the transpose (dual) map $\mathrm{p}^* \in \operatorname{End}_{\mathscr{C}} \mathscr{A}^*$ is a projection on $\mathscr{A}^*$ such that $\ker(\mathrm{p}^*) = \operatorname{Ann}(\mathcal{V})$ and $\operatorname{im}(\mathrm{p}^*) = \operatorname{Ann}(\mathcal{H})$. In particular, $\mathscr{A}^* = \operatorname{Ann}(\mathcal{V}) \oplus \operatorname{Ann}(\mathcal{H})$.

Remark C.2. *Note that the role of the horizontal and vertical submodules $\mathcal{H}^{\mathrm{p}}$ and $\mathcal{V}^{\mathrm{p}}$ is symmetric. In fact, if* p *is a connection, then* $\mathrm{Id}_{\mathscr{A}} - \mathrm{p}$ *is another connection such that* $\mathcal{H}^{\mathrm{Id}_{\mathscr{A}} - \mathrm{p}} = \mathcal{V}^{\mathrm{p}}$ *and* $\mathcal{V}^{\mathrm{Id}_{\mathscr{A}} - \mathrm{p}} = \mathcal{H}^{\mathrm{p}}$.

Let us extend the bigrading of $\mathcal{C}^1 = \mathscr{A}^*$ to the whole algebra $\mathcal{C}$. We define

$$\mathcal{C}^{p,q} := \{\alpha \in \mathcal{C}^{p+q} \mid \alpha(a_1, \ldots, a_{k-s}, b_1, \ldots, b_s) = 0 \quad \forall s \neq q,\ a_i \in \mathcal{H}, b_j \in \mathcal{V}\}$$

as the submodule of $\mathscr{C}$-linear skew-symmetric maps of bidegree (p, q). It is straightforward to check that $\mathcal{C}^{p,q} \wedge \mathcal{C}^{p',q'} \subseteq \mathcal{C}^{p+p',q+q'}$ for all $p, p', q, q' \in \mathbb{Z}$. On the other hand, since $\mathrm{p} : \mathcal{C} \to \mathcal{C}$ is $\mathscr{C}$-linear, it induces the derivation $\mathrm{i_p} \in \operatorname{Der}^0_{\mathscr{C}} \mathcal{C}$ of degree 0 in $\mathcal{C}$. The elements of $\mathcal{C}^{p,q}$ turn out to be the q-eigenvectors of $\mathrm{i_p}$ on $\mathcal{C}^{p+q}$, $\mathrm{i_p}\, \alpha = q\alpha$ if $\alpha \in \mathcal{C}^{p,q}$ (equivalently, $\mathrm{i}_{\mathrm{Id}_{\mathscr{A}} - \mathrm{p}}\, \alpha = p\alpha$). The converse is also true if $\mathscr{C}$ is free as $\mathbb{Z}$-module.

Clearly, $\mathcal{C}^{i,j} = \{0\}$ whenever $i < 0$ or $j < 0$, and $\mathcal{C}^{p,q} \cap \mathcal{C}^{p',q'} = \{0\}$ whenever $p \neq p'$ or $q \neq q'$. Furthermore,

Lemma C.3. *For each $k \in \mathbb{Z}$, one has*

$$\mathcal{C}^k = \bigoplus_{p+q=k} \mathcal{C}^{p,q}.$$

Proof. For the purposes of this proof, let us denote $\mathrm{p}_0 := \mathrm{Id}_{\mathscr{A}} - \mathrm{p}$, and $\mathrm{p}_1 := \mathrm{p}$. Now, for each $k, p, q \in \mathbb{Z}$, denote by $A_k := \{0, 1\}^k$ the set sequences of length k consisting of 0's and 1's, and by $A_{p,q}$ the subset of A_{p+q} which take the value 0 exactly p times, and hence, the value 1 exactly q times. Now, fix $\alpha \in \mathcal{C}^k$. Since $\mathrm{p}_0 + \mathrm{p}_1 = \mathrm{Id}_{\mathscr{A}}$, and $\bigcup_{p+q=k} A_{p,q} = A_k$, it is easy to see that, for any $a_1, \ldots, a_k \in \mathscr{A}$,

$$\alpha(a_1, \ldots, a_k) = \sum_{p+q=k} \sum_{s \in A_{p,q}} \alpha(\mathrm{p}_{s_1} a_1, \ldots, \mathrm{p}_{s_k} a_k). \tag{C.3}$$

Moreover, it is straightforward to verify that, for each $p, q \in \mathbb{Z}$ with $p + q = k$, the map $\alpha_{p,q} : \mathscr{A} \times \cdots \times \mathscr{A} \to \mathscr{C}$, given on $a_1, \ldots, a_k \in \mathscr{A}$ by

$$\alpha_{p,q}(a_1, \ldots, a_k) := \sum_{s \in A_{p,q}} \alpha(\mathrm{p}_{s_1} a_1, \ldots, \mathrm{p}_{s_k} a_k),$$

is $\mathscr{C}$-multilinear, skew-symmetric, and has bidegree (p, q), $\alpha_{p,q} \in \mathcal{C}^{p,q}$. By (C.3), one has $\alpha = \sum_{p+q=k} \alpha_{p,q}$, as desired. ∎

Finally, consider the $\mathscr{C}$-submodule $\operatorname{Der}^{r,s}_{\mathscr{R}} \mathcal{C} \subseteq \operatorname{Der}^{r+s}_{\mathscr{R}} \mathcal{C}$ consisting of the $\mathscr{R}$-linear derivations D of bidegree (r, s), that is, such that $D(\mathcal{C}^{p,q}) \subseteq \mathcal{C}^{p+r,q+s}$ for all $p, q \in \mathbb{Z}$. It is straightforward to check that $[\mathrm{i}_{\mathrm{p}}, D] = sD$ and, equivalently, $[\mathrm{i}_{\mathrm{Id}_{\mathscr{A}}-\mathrm{p}}, D] = rD$, for all $D \in \mathrm{Der}^{r,s}_{\mathscr{R}}\,\mathcal{C}$. Here, $[D, D'] := D \circ D' - (-1)^{|D||D'|} D' \circ D$ denotes the graded commutator of endomorphisms. Finally, we denote by

$$\widehat{\mathrm{Der}}^{k}_{\mathscr{R}}\mathcal{C} := \bigoplus_{r+s=k} \mathrm{Der}^{r,s}_{\mathscr{R}}\,\mathcal{C}$$

the set of graded derivations which admit a bigraded decomposition. For each $D \in \widehat{\mathrm{Der}}^{k}_{\mathscr{R}}\mathcal{C}$, the corresponding element in $\mathrm{Der}^{r,s}_{\mathscr{R}}\,\mathcal{C}$ will be denoted by $D^{\mathrm{p}}_{r,s}$.

Bigrading on Lie algebroids. Now, assume that $\mathscr{A}$ is the total space of a Lie algebroid $(\mathscr{A}, \mathrm{q}, [\cdot,\cdot]_{\mathscr{A}})$ over the commutative $\mathscr{R}$-algebra and free $\mathbb{Z}$-module $\mathscr{C}$. Let us fix a connection p on $\mathscr{A}$, $\mathrm{p}^2 = \mathrm{p}$. Because of the Leibniz rule on the Lie algebroid, and the skew-symmetry of $[\cdot,\cdot]_{\mathscr{A}}$, the maps $R^{\mathrm{p}}, S^{\mathrm{p}} : \mathcal{C} \times \mathcal{C} \to \mathscr{A}$ given by

$$R^{\mathrm{p}}(a,b) = \mathrm{p}[(\mathrm{Id}_{\mathscr{A}} - \mathrm{p})a, (\mathrm{Id}_{\mathscr{A}} - \mathrm{p})b]_{\mathscr{A}}, \qquad S^{\mathrm{p}}(a,b) = (\mathrm{Id}_{\mathscr{A}} - \mathrm{p})[\mathrm{p}\,a, \mathrm{p}\,b]_{\mathscr{A}}$$

are $\mathscr{C}$-bilinear and skew-symmetric. It is clear that $\mathcal{H} := \ker(\mathrm{p})$ and $\mathcal{V} := \mathrm{im}(\mathrm{p})$ are Lie $\mathscr{R}$-subalgebras of $\mathscr{A}$ if and only if $R^{\mathrm{p}} = 0$ and $S^{\mathrm{p}} = 0$, respectively. [35]. Thus, R^{p} and S^{p} are said to be the *curvature* and *co-curvature* of p, respectively.

Proposition C.4. *Consider the cochain complex $(\mathcal{C}, \partial)$ associated with $(\mathscr{A}, \mathrm{q}, [\cdot,\cdot]_{\mathscr{A}})$. For any connection* p *on $\mathscr{A}$, one has*

$$\partial = \partial^{\mathrm{p}}_{-1,2} + \partial^{\mathrm{p}}_{0,1} + \partial^{\mathrm{p}}_{1,0} + \partial^{\mathrm{p}}_{2,-1}.$$

Furthermore,

$$\partial^{\mathrm{p}}_{-1,2} = -\,\mathrm{i}_{S^{\mathrm{p}}}, \quad \partial^{\mathrm{p}}_{0,1} = \mathrm{L}^{\partial}_{\mathrm{p}} + 2\,\mathrm{i}_{S^{\mathrm{p}}} - \mathrm{i}_{R^{\mathrm{p}}}, \quad \partial^{\mathrm{p}}_{1,0} = \mathrm{L}^{\partial}_{\mathrm{Id}_{\mathscr{A}}-\mathrm{p}} + 2\,\mathrm{i}_{R^{\mathrm{p}}} - \mathrm{i}_{S^{\mathrm{p}}}, \quad \textit{and} \quad \partial^{\mathrm{p}}_{2,-1} = -\,\mathrm{i}_{R^{\mathrm{p}}}.$$

Proof. To prove the first part, we have to check that $\partial_{i,j} = 0$ whenever $i < -1$ or $j < -1$. Fix $\alpha \in \mathcal{C}^{p,q}$, $a_1, \ldots, a_{p+i} \in \mathcal{H}$ and $a_{p+i+1}, \ldots, a_{p+q+1} \in \mathcal{V}$. By definition,

$$\begin{aligned}\partial_{i,j}\alpha(a_1, \ldots, a_{p+q+1}) = &\sum_{\sigma \in S_{(1,k+1)}} \mathrm{sgn}(\sigma)\,\mathrm{q}(a_{\sigma_1})(\alpha(a_{\sigma_2}, \ldots, a_{\sigma_k})) \\ &- \sum_{\sigma \in S_{(2,k)}} \mathrm{sgn}(\sigma)\alpha([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}, \ldots, a_{\sigma_{k+1}}).\end{aligned}$$

If $i < 0$ (resp. $j < 0$), then on every term in the first sum α has less (resp. more) than p arguments belonging to $\mathcal{H}$. So, it follows that the first sum is zero whenever i or j is negative. A similar argument shows that the second sum is zero whenever $i < -1$ or $j < -1$. This proves the first part. Now, let us show, for $i = 2, j = -1$, that $\partial_{2,-1} = -\,\mathrm{i}_{R^{\mathrm{p}}}$. To do so, it suffices to prove that

$$\alpha([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}, \ldots, a_{\sigma_{k+1}}) = \alpha(\mathrm{p}[(\mathrm{Id}_{\mathscr{A}} - \mathrm{p})a_{\sigma_1}, (\mathrm{Id}_{\mathscr{A}} - \mathrm{p})a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}, \ldots, a_{\sigma_{k+1}})$$

holds for each shuffle permutation σ. In the case when $\sigma_2 \leq p + 2$, such identity follows from $a_{\sigma_1}, a_{\sigma_2} \in \mathcal{H}$ and $\alpha \in \mathcal{C}^{p,q}$. If $\sigma_2 > p + 2$, then both sides are identically zero. The proof of $\partial^{\mathrm{p}}_{-1,2} = -\,\mathrm{i}_{S^{\mathrm{p}}}$ is analogous. Finally, taking into account the properties of $[\mathrm{i}_{\mathrm{p}}, \cdot]$, we get

$$\begin{aligned}\mathrm{L}^{\partial}_{\mathrm{p}} &= [\mathrm{i}_{\mathrm{p}}, \partial] = [\mathrm{i}_{\mathrm{p}}, \partial^{\mathrm{p}}_{-1,2}] + [\mathrm{i}_{\mathrm{p}}, \partial^{\mathrm{p}}_{0,1}] + [\mathrm{i}_{\mathrm{p}}, \partial^{\mathrm{p}}_{1,0}] + [\mathrm{i}_{\mathrm{p}}, \partial^{\mathrm{p}}_{2,-1}] \\ &= 2\partial^{\mathrm{p}}_{-1,2} + \partial^{\mathrm{p}}_{0,1} - \partial^{\mathrm{p}}_{2,-1} = -2\,\mathrm{i}_{S^{\mathrm{p}}} + \partial^{\mathrm{p}}_{0,1} + \mathrm{i}_{R^{\mathrm{p}}}.\end{aligned}$$

This proves $\partial^{\mathrm{p}}_{0,1} = \mathrm{L}^{\partial}_{\mathrm{p}} + 2\,\mathrm{i}_{S^{\mathrm{p}}} - \mathrm{i}_{R^{\mathrm{p}}}$. The proof of $\partial^{\mathrm{p}}_{1,0} = \mathrm{L}^{\partial}_{\mathrm{Id}_{\mathscr{A}}-\mathrm{p}} + 2\,\mathrm{i}_{R^{\mathrm{p}}} - \mathrm{i}_{S^{\mathrm{p}}}$ is analogous. ∎

Remark C.5. *If $\mathscr{A}$ is a finitely generated and projective $\mathscr{C}$-module, then $\mathcal{C}^k = \wedge^k \mathscr{A}^*$. In this case, $\widehat{\mathrm{Der}}_{\mathscr{R}}\mathcal{C} = \mathrm{Der}_{\mathscr{R}}\mathcal{C}$. Furthermore, $D = D_{-1,2} + D_{0,1} + D_{1,0} + D_{2,-1}$ for any $D \in \mathrm{Der}^1_{\mathscr{R}}\mathcal{C}$. Indeed, the fact that $D_{i,j} = 0$ whenever $i < -2$ or $j < -2$, follows from the well-known property that the only derivation of $\wedge^k \mathscr{A}^*$ vanishing on $\mathcal{C}^0 = \mathscr{C}$ and $\mathcal{C}^1 = \mathscr{A}^*$ is the zero derivation. Moreover, $D_{2,-1}$ and $D_{-1,2}$ are also of the form i_K and i_L for some $K \in \mathcal{C}^{2,0} \otimes \mathscr{A}$ and $L \in \mathcal{C}^{0,2} \otimes \mathscr{A}$. Similar formulas can be derived for $D_{1,0}$ and $D_{0,1}$.*

The following fact can be found in [26, Section 5] for the case of connections of zero co-curvature on the tangent Lie algebroid of a manifold.

Corollary C.6. *Let p be a connection on the Lie algebroid $(\mathscr{A}, \mathrm{q}, [\cdot,\cdot]_{\mathscr{A}})$. If the vertical submodule $\mathcal{V}$ is involutive, then the bigraded components of the exterior differential satisfy*

$$\partial^{\mathrm{p}}_{-1,2} = 0, \qquad \partial^{\mathrm{p}}_{1,0} = \mathrm{L}^{\partial}_{\mathrm{Id}_{\mathscr{A}} - \mathrm{p}} + 2\,\mathrm{i}_{R^{\mathrm{p}}}, \qquad \partial^{\mathrm{p}}_{0,1} = \mathrm{L}^{\partial}_{\mathrm{p}} - \mathrm{i}_{R^{\mathrm{p}}}, \qquad \partial^{\mathrm{p}}_{2,-1} = -\,\mathrm{i}_{R^{\mathrm{p}}}, \qquad (\partial^{\mathrm{p}}_{1,0})^2 = \mathrm{L}^{\partial}_{R^{\mathrm{p}}}.$$

In particular, if $\mathcal{H}$ is also an involutive submodule, then $\partial^{\mathrm{p}}_{2,-1} = 0$, and $\partial^{\mathrm{p}}_{1,0}$ is a coboundary.

Proof. Recall that $\mathcal{V}$ is involutive if and only if $S^{\mathrm{p}} = 0$. By Proposition C.4, we get $\partial^{\mathrm{p}}_{-1,2} = 0$. So, it remains to prove that $(\partial^{\mathrm{p}}_{1,0})^2 = \mathrm{L}^{\partial}_{R^{\mathrm{p}}}$. Since $\partial = \partial^{\mathrm{p}}_{1,0} + \partial^{\mathrm{p}}_{0,1} + \partial^{\mathrm{p}}_{2,-1}$ and $(\partial^{\mathrm{p}})^2 = 0$, it follows that

$$(\partial^{\mathrm{p}}_{1,0})^2 = -(\partial^{\mathrm{p}}_{0,1}\partial^{\mathrm{p}}_{2,-1} + \partial^{\mathrm{p}}_{0,1}\partial^{\mathrm{p}}_{2,-1}) = -[\partial^{\mathrm{p}}_{0,1}, \partial^{\mathrm{p}}_{2,-1}] = -[\partial, \partial^{\mathrm{p}}_{2,-1}] = [\partial, \mathrm{i}_{R^{\mathrm{p}}}] = \mathrm{L}^{\partial}_{R^{\mathrm{p}}}.$$

Thus, if $\mathcal{H}$ is involutive, then $R^{\mathrm{p}} = 0$, which implies that $\partial^{\mathrm{p}}_{2,-1} = 0$, and $(\partial^{\mathrm{p}}_{1,0})^2 = 0$. ■

Corollary C.7. *Let p be a connection on the Lie algebroid $(\mathscr{A}, \mathrm{q}, [\cdot,\cdot]_{\mathscr{A}})$ such that the vertical submodule satisfy the following properties:*

1. *The isotropy algebra contains $\mathcal{V}$: $\mathcal{V} \subseteq \ker(\mathrm{q})$.*
2. *The vertical submodule is an ideal: $[\mathcal{V}, \mathscr{A}]_{\mathscr{A}} \subseteq \mathcal{V}$.*
3. *The vertical submodule is an Abelian Lie subalgebra of $\mathscr{A}$: $[\mathcal{V}, \mathcal{V}]_{\mathscr{A}} = \{0\}$.*

Then, $\partial^{\mathrm{p}}_{-1,2} = 0$, and $\partial^{\mathrm{p}}_{0,1} = 0$.

Proof. Since $[\mathcal{V}, \mathcal{V}]_{\mathscr{A}} = \{0\}$, it is clear that $\mathcal{V}$ is involutive, so $\partial^{\mathrm{p}}_{-1,2} = 0$. It is left to show that $\partial^{\mathrm{p}}_{0,1} = 0$. Fix $\alpha \in \mathcal{C}^{p,q}$, $a_1, \dots, a_p \in \mathcal{H}$, and $a_{p+1}, \dots, a_{p+q+1} \in \mathcal{V}$. By definition,

$$\begin{aligned}\partial^{\mathrm{p}}_{0,1}\alpha(a_1, \dots, a_{p+q+1}) = &\sum_{\sigma \in S_{(1,k+1)}} \mathrm{sgn}(\sigma)\, \mathrm{q}(a_{\sigma_1})(\alpha(a_{\sigma_2}, \dots, a_{\sigma_{p+q+1}})) \\ &- \sum_{\sigma \in S_{(2,k)}} \mathrm{sgn}(\sigma) \alpha([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}, \dots, a_{\sigma_{p+q+1}}).\end{aligned}$$

If $\sigma \in S_{(1,k+1)}$ is such that $\sigma_1 \geq p+1$, then $a_{\sigma_1} \in \mathcal{V} \subseteq \ker(\mathrm{q})$, so $\mathrm{q}(a_{\sigma_1}) = 0$. If, on the contrary, $\sigma \in S_{(1,k+1)}$ satisfies $\sigma_1 \leq p$, then $a_{\sigma_1} \in \mathcal{H}$, so the sequence $(a_{\sigma_2}, \dots, a_{\sigma_{p+q+1}})$ consists of $p-1$ elements of $\mathcal{H}$ and $q+1$ elements of $\mathcal{V}$. Since $\alpha \in \mathcal{C}^{p,q}$, we get $\alpha(a_{\sigma_2}, \dots, a_{\sigma_{p+q+1}}) = 0$. In any case, every term of the first sum is zero. For the second sum, we consider the following cases:

- If $\sigma_1 \geq p+1$, then $a_{\sigma_1}, a_{\sigma_2} \in \mathcal{V}$, and hence $[a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}} = 0$.

- If $\sigma_1 \leq p$ and $\sigma_2 \geq p+1$, then $a_{\sigma_1} \in \mathcal{H}$, and $a_{\sigma_2} \in \mathcal{V}$. Since $\mathcal{V}$ is an ideal, one has $[a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}} \in \mathcal{V}$. Thus, $([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}, \ldots, a_{\sigma_{p+q+1}})$ consists of $p-1$ elements in $\mathcal{H}$ and $q+1$ elements in $\mathcal{V}$.
- If $\sigma_2 \leq p$, then $a_{\sigma_1}, a_{\sigma_2} \in \mathcal{H}$, and hence the sequence $([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}, \ldots, a_{\sigma_{p+q+1}})$ has as least $q+1$ elements in $\mathcal{V}$.

Since $\alpha \in \mathcal{C}^{p,q}$, in any case we get $\alpha([a_{\sigma_1}, a_{\sigma_2}]_{\mathscr{A}}, a_{\sigma_3}, \ldots, a_{\sigma_{p+q+1}}) = 0$. ■

Corollary C.8. *Let* p *be a connection on the Lie algebroid* $(\mathscr{A}, \mathrm{q}, [\cdot,\cdot]_{\mathscr{A}})$ *such that the vertical submodule* $\mathcal{V}$ *is involutive, and the horizontal submodule satisfy the following properties:*

1. *The isotropy algebra contains* $\mathcal{H}$*:* $\mathcal{H} \subseteq \ker(\mathrm{q})$.
2. *The horizontal submodule is an ideal:* $[\mathcal{H}, \mathscr{A}]_{\mathscr{A}} \subseteq \mathcal{H}$.
3. *The horizontal submodule is an Abelian Lie subalgebra of* $\mathscr{A}$*:* $[\mathcal{H}, \mathcal{H}]_{\mathscr{A}} = \{0\}$.

Then, $\eth = \eth^{\mathrm{p}}_{0,1}$.

Proof. By interchanging the role of $\mathcal{H}$ and $\mathcal{V}$, we conclude from Corollary C.7 that $\eth^{\mathrm{p}}_{2,-1} = 0$ and $\eth^{\mathrm{p}}_{1,0} = 0$. Furthermore, the involutivity of $\mathcal{V}$ implies that $\eth^{\mathrm{p}}_{-1,2} = 0$. ■

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