"El saber de mis hijos hará mi grandeza"

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Algebraic and Geometric Aspects of the Averaging Method on $\mathbb{S}^1$-Spaces

T E S I S

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Resumen

Este trabajo aborda varios aspectos algebraicos y geométricos del método global de promedios para sistemas dinámicos en $\mathbb{S}^1$-variedades. Por $\mathbb{S}^1$-variedades, entendemos una variedad diferencial en la que actúa el grupo $\mathbb{S}^1$. Se estudian una amplia variedad de sistemas dinámicos perturbados, en el contexto del método de la transformada de Lie y en la teoría de formas normales, cuya caractística principal es que la parte no perturbada es invariante con respecto a una acción de $\mathbb{S}^1$ y siguiendo un enfoque libre de coordenadas.

Las investigaciones que se desarrollan en esta tesis se centran alrededor de las siguientes líneas: (i) enfoque algebraico de la ecuación homológica generalizada; (ii) formas normales globales y el teorema geométrico de promedios; (iii) el método de promedios en espacios fase con variables lentas y rápidas.

Una de la aportaciones de este trabajo está relacionada con la construcción de soluciones globales de la ecuación homológica tensorial asociada con flujos periódicos para el caso de campos tensoriales antisimétricos, covariantes y contravariantes, y que generalizan las fórmulas que Cushman derivó para el caso de funciones. Estos resultados son aplicados al problema de normalización de campos vectoriales con flujo periódico. Mas aún, se formula y se demuestra una versión Riemanniana del teorema de promedios la cual se usa para aproximar la dinámica real de un sistema dado, por medio de trayectorias de su sistema promediado en una escala de tiempo grande. Para este propósito, además de los argumentos de normalización, se utilizan estimaciones de tipo Gronwall para flujos en una variedad Riemanniana y las propiedades de la operación de levantamiento horizontal en un $\mathbb{S}^1$-haz principal.

Otra contribución de esta tesis consiste en presentar un enfoque de normalización geométrico de una dinámica Hamiltoniana perturbada en espacios con variables lentas y rápidas. Tales espacios aparecen en la teoría de aproximaciones adiabáticas y en sus generalizaciones. La principal característica de estos espacios es que se descomponen como el producto de un factor con variables lentas y otro factor con variables rápidas; lo cual se encuentra en correspondencia con la dependencia singular que tiene la forma simpléctica (ó corchete de Poisson) del parámetro de perturbación. Como consecuencia de esto, el sistema no-perturbado no hereda una estructura Hamiltoniana natural y por tanto no es posible aplicar directamente la teoría regular de perturbaciones para sistemas Hamiltonianos. Sin embargo, asumiendo ciertas hipótesis de simetría para la dinámica no perturbada, se derivan varios resultados de normalización basados en una versión paramétrica del método de homotopía de Moser y la técnica de promedios para formas (pre)simplécticas y conexiones no lineales en espacios fibrados, la cual se debe a Marsden, Montgomery y Ratiu.

Finalmente, en esta disertación se presentan algunos ejemplos, que tienen relación con problemas de Física-Matemática, en los cuales se ilustran las principales técnicas y resultados que son el fundamento de esta tesis.

Abstract

This dissertation is devoted to some algebraic and geometric aspects of the global averaging method for dynamical systems of $\mathbb{S}^1$-manifolds. In the context of Lie transform perturbation method and normal form theory, we present a free coordinate approach to the study of a wide class of perturbed dynamical systems whose unperturbed part is invariant with respect to a given $\mathbb{S}^1$-action. The thesis research centers around the following lines: (i) an algebraic approach to generalized homological equations; (ii) global normal forms and the geometric averaging theorem; (iii) the periodic averaging on slow-fast phase space.

The first contribution is related to the construction of global solutions to the tensor homological equations associated with periodic flows and generalizes the result by Cushman to the case of covariant and contravariant antisymmetric tensor fields of arbitrary order. These results are applied to the normalization problem to vector fields (not necessarily Hamiltonian) with periodic flow. Moreover, we formulate and prove a Riemannian version of the $\mathbb{S}^1$averaging theorem on the approximation of the true dynamics by the trajectories of an averaged system on a long time scale.

Another contribution of the thesis is a geometric approach to the normalization of perturbed Hamiltonian dynamics on the so-called slow-fast phase spaces which appear in the theory of adiabatic approximation and its generalizations. Such phase spaces separates into the product of the slow and fast factors according to the singular dependence of the symplectic form (the Poisson bracket) on a small perturbation parameter. As a consequence, the unperturbed system does not inherit any natural Hamiltonian structure and hence the regular perturbation theory for Hamiltonian systems can not be directly applies to this situation. Under an appropriate $\mathbb{S}^1$-symmetry hypothesis for the unperturbed dynamics, we derive various normalization results based on a parameter dependent version of the Moser homotopy method and the averaging technique for (pre)symplectic forms and nonlinear connections on fibered spaces due to Marsden, Montgomery and Ratiu. The dissertation concludes with illustrations of the main results on some examples.

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Introduction

This work is devoted to some algebraic and geometric aspects of the "global" averaging method for dynamical systems due to Moser [58] and Cushman [15] which refers to the flow on a phase space manifold rather than to a local coordinate description. A basic tool in the theory of normal forms for dynamical systems is the Lie transform method which was originally developed in the works by Deprit [21], Kamel [37], (see also [33, 35, 51]) . According to this method, formal normalization transformations of a perturbed system are constructed by means of formal Lie series. The existence of such transformations is provided by the solvability of linear nonhomogeneous equations (involving the Lie derivative along the unperturbed vector field) which are called the *homological equations* [5]. The "local" traditional approach (see for example [5, 7, 66, 67]) is based on the construction of solutions to homological equations and the corresponding normal forms within domains of local coordinate systems (such as action-angle variables in the Hamiltonian case). We are interested in a "global" normalization procedure which can be performed on an arbitrary relatively compact domain in a phase space. In this case, a normalization transformation can be defined as a flow of a time-dependent vector field where a small perturbation parameter plays the role of time. This setting leads to the study of the solvability of homological equations globally, on the whole phase space. We address this question to a class of perturbed dynamical systems on phase spaces with $\mathbb{S}^1$-symmetry, which includes, in particular, systems whose unperturbed parts have periodic flows. In the Hamiltonian case, a free coordinate formula for global solutions to a homological equation associated to a periodic flow was derived by Cushman [15]. Our point is to generalize this result to the case of homological equations for tensor fields of arbitrary type. The key observation here is that the averaging procedure with respect to an action of a compact Lie group (in particular, a circle action) on a manifold is well defined for a wide class of geometric objects including tensor fields.

A global averaging technique on phase spaces with symmetry was developed by S. Golin, A. Knauf, S. Marmi [28] and R. Montgomery [55] for the study of slowly varying integrable systems and extended to Poisson bundles by J. E. Marsden, R. Montgomery and T. Ratiu [47] in the context of the Hannay-Berry phases and the reduction method. One of the goals of the present work is to apply the averaging technique to a class of perturbed Hamiltonian systems [17, 19, 74, 76] which represents a generalization of the slow-fast Hamiltonian systems known in the theory of adiabatic approximation [7, 38, 43, 62]. The main feature of such perturbed Hamiltonian models is that the unperturbed system is no longer Hamiltonian and hence one can not apply the standard methods of the regular Hamiltonian perturbation theory. One of the main points here is to understand a geometric nature of normalization transformations of adiabatic type which is closely related to the averaging procedure of (pre)symplectic forms [28, 58], and Poisson brackets [75].

Remark also that a global perturbative setting is also important in the quantum averaging method under construction of the normalization transformation in the semiclassical approximation [31, 39, 71].

Therefore, the research lines of this thesis are:

1. *Algebraic approach to generalized homological equations.*

2. *Global normal forms and geometric averaging theorem.*

3. *The periodic averaging on slow-fast phase spaces.*

In more details the main results of the thesis are formulated as follows.

1. *Algebraic approach to generalized homological equations*. Let $\mathbf{A}_\varepsilon = A_0 + \varepsilon A_1 + \mathcal{O}(\varepsilon^2)$ be a perturbed vector on a manifold M. The existence of a (global) normalization of first order of $\mathbf{A}_\varepsilon$ relative to the unperturbed vector field A_0 is reduced to the existence of two vector fields Z and $\bar{A}_1$ on M satisfying the homological equation

$$[A_0, Z] = A_1 - \bar{A}_1 \tag{1}$$

and the condition

$$[A_0, \bar{A}_1] = 0. \tag{2}$$

If $(Z, \bar{A}_1)$ is a solution to this problem, then Z gives an infinitesimal generator of a normalization transformation and the vector field $\bar{A}_1$ represents the second term in the normal form. In general, the solvability of this problem is a nontrivial question. For different approaches see for example, [67]. We assume that the flow of the unperturbed vector field A_0 is periodic with frequency function $\omega : M \to \mathbb{R}$. In this case, we have an $\mathbb{S}^1$-action (not necessarily free) on M with an infinitesimal generator $\Upsilon = \frac{1}{\omega} A_0$. To get feeling for an algebraic nature of the homological equation, we study the problem (1), (2) in a more general setting on the exterior algebras of k-tensor fields $\chi^k(M) = \operatorname{Sec}\left(\bigwedge^k TM\right)$. For a given $F \in \chi^k(M)$, we are looking for k-vector fields Π and $\bar{F}$ on M satisfying the generalized homological equation

$$\mathcal{L}_{A_0} \Pi = F - \bar{F} \tag{3}$$

and the condition

$$\bar{F} \text{ is } \mathbb{S}^1\text{-}\textit{invariant}. \tag{4}$$

Here, $\mathcal{L}_{A_0} : \chi^k(M) \to \chi^k(M)$ is the Lie derivative along the vector field A_0 defined as the unique differential operator on the tensor algebra of the manifold M which coincides with the standard Lie derivative on the spaces of functions and vector fields on M. Moreover, the $\mathbb{S}^1$-action on M allows us to define the averaging operator $\mathcal{A} : \chi^k(M) \to \chi^k(M)$ and the resolvent operator $\mathcal{S} : \chi^k(M) \to \chi^k(M)$ by

$$\mathcal{A}(F) = \frac{1}{2\pi} \int_0^{2\pi} (\mathrm{Fl}^t_\Upsilon)^* F \, dt, \tag{5}$$

$$\mathcal{S}(F) := \frac{1}{2\pi}\int_0^{2\pi}(t-\pi)(\mathrm{Fl}_\Upsilon^t)^* F\,dt, \tag{6}$$

where Fl_Υ^t is the flow of Υ.

The operators $\mathcal{L}_\Upsilon, \mathcal{A}$ and $\mathcal{S}$ are related by

$$\mathcal{L}_\Upsilon \circ \mathcal{S} = \mathrm{id} - \mathcal{A} \tag{7}$$

Then, we observe that one can find explicit formulas for solutions to the generalized homological equation operating only with some algebraic properties of the triple $(\mathcal{L}_\Upsilon, \mathcal{A}, \mathcal{S})$.

Theorem 1 *Problem (3),(4) is globally solvable on M and every solution $(\Pi, \bar{F})$ is represented in the form*

$$\bar{F} = \mathcal{A}(F) + \frac{1}{\omega}A_0 \wedge \mathbf{i}_{d\omega}C, \tag{8}$$

$$\Pi = \frac{1}{\omega}\mathcal{S}(F) + \frac{1}{\omega^3}A_0 \wedge \mathcal{S}^2(\mathbf{i}_{d\omega}F) + C, \tag{9}$$

where C is an arbitrary $\mathbb{S}^1$-invariant k-vector field on M.

In the particular case when $k=0$ and $\chi^0(M) = C^\infty(M)$ is the space of smooth functions, formulas (8), (9) lead to the Cushman result [15]. Moreover, as a consequence of Theorem 1 for $k=1$ and $F = A_1$, we derive the following fact: if the frequency function ω is a first integral of the $\mathbb{S}^1$-average $\langle A_1\rangle = \mathcal{A}(A_1)$ of the perturbation vector field, that is,

$$\mathcal{L}_{\langle A_1\rangle}\omega = 0, \tag{10}$$

then the formulas (8), (9) give a global solution $\bar{A}_1 = \bar{F}$, $Z = \Pi$ of the problem (5), (6). Compatibility condition (10) always holds in the Hamiltonian case, when the perturbed vector field $\mathbf{A}_\varepsilon = X_{H_0+\varepsilon H_1+O(\varepsilon^2)}$ is Hamiltonian on a (pre) symplectic manifold. This is a consequence of the so-called period-energy relation due to Gordon [29] (see also [2, 10]) for the unperturbed Hamiltonian vector field $A_0 = X_{H_0}$ with periodic flow, which says that $d\omega \wedge dH_0 = 0$.

Moreover, we obtain similar formulas to (8), (9) for global solutions to the homological equation on the space $\Omega^k(M)$ of differential k -forms. As a consequence of the general results, we derive the following representation for the $\mathbb{S}^1$-average of an arbitrary closed k-form $\eta \in \Omega^k(M)$:

$$\langle\eta\rangle = \eta - d(\mathbf{i}_\Upsilon\mathcal{S}(\eta)), \tag{11}$$

which plays a key role in the averaging procedure for symplectic forms.

Finally, we observe that Theorem 1 remains true in an abstract setting, when we start with some linear operators $\mathcal{L}_\Upsilon$, $\mathcal{S}$ on the spaces of vector valued exterior forms on a Lie algebra which satisfy appropriate properties. In this case, formula (7) gives us the algebraic definition of the averaging operator $\mathcal{A}$.

2. ***Global normal forms and geometric averaging theorem.*** First of all, Theorem 1 allows us to show that a perturbed vector field $\mathbf{A}_\varepsilon$ whose unperturbed part A_0 has the period flow with frequency function ω can be reduced to an $\mathbb{S}^1$-invariant normal form of first order on any relatively compact domain for ε small enough. The corresponding normalization transformation is defined as the time-ε flow $\Phi_\varepsilon = \mathrm{Fl}_Z^t\,|_{t=\varepsilon}$ of the vector field

$$Z = \frac{1}{\omega}\mathcal{S}(A_1) + \frac{1}{\omega^3}\mathcal{S}^2(\mathcal{L}_{A_1}\omega)A_0 \tag{12}$$

If compatibility condition (10) is satisfied, then Φ_ε is a normalization transformation of first order for $\mathbf{A}_\varepsilon$ relative to A_0. If M is compact, then the normalization transformation Φ_ε is defined on the whole space M. In the Hamiltonian case, compatibility condition (10) holds and hence a global normalization always exists [15].

The $\mathbb{S}^1$-invariant normalization of $\mathbf{A}_\varepsilon$ is a first step in the coordinate proof of the classical averaging theorem (see, [7, 62]) which asserts that the true dynamics of $\mathbf{A}_\varepsilon$ is approximated by the trajectories of the averaged vector field $A_0+\varepsilon\langle A_1\rangle$ on the long time scale. We are interested in a geometric (free coordinate) version of this theorem. Assuming that the action of the circle with infinitesimal generator $\Upsilon = \frac{1}{\omega}A_0$ is free, we consider the principal $\mathbb{S}^1$-bundle $\rho : M \to \mathcal{O}$ over the orbit space $\mathcal{O} = M/\mathbb{S}^1$. Then, there is an $\mathbb{S}^1$-invariant splitting $TM = \mathbb{H}\oplus\mathbb{V}$, where $\mathbb{V} = \mathrm{Span}\{\Upsilon\}$ and $\mathbb{H}$ are the vertical and horizontal distributions, respectively. According to this splitting the $\mathbb{S}^1$-average $\langle A_1\rangle$ of the perturbation vector field admits the decomposition $\langle A_1\rangle = \mathrm{hor}(w)+c\Upsilon$, where $\mathrm{hor}(w)$ is a horizontal lift of a vector field $w \in \mathfrak{X}(\mathcal{O})$ determining the (reduced) *averaged system* on the orbit space. Then, fixing a metric $<,>^o$ on $\mathcal{O}$ and the corresponding distance function $\mathrm{dist}^o : \mathcal{O}\times\mathcal{O} \to \mathbb{R}$, we show that

$$\mathrm{dist}^o(\rho\circ\mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m^0),\ \ \mathrm{Fl}^{\varepsilon t}_w(\rho(m^0))) = O(\varepsilon)$$

for small enough ε and $t \sim \frac{1}{\varepsilon}$. One can try to prove this statement applying the local averaging theorem 1 to coordinate charts on M. But instead, we apply a global approach which allows us to get the estimations in the intrinsic terms of the Riemannian manifold $(M, <,>)$ (where $<,>$ is an induced metric on M such that the projection ρ is a Riemannian submersion). Geometrically, the proof of the classical averaging theorem for one-frequency systems on $M = \mathbb{S}^1\times\mathbb{R}^n$, besides of standard Gronwall estimates and the near identity transformation argument, is essentially based on the properties of minimal geodesic in the Euclidean space $\mathbb{R}^n$. This is just a main difficulty for the generalization to arbitrary Riemannian manifolds. Our idea is to associate to the normalized perturbed vector field $(\Phi_\varepsilon)^*\mathbf{A}_\varepsilon$ and the averaged vector field $A_0 + \varepsilon\langle A_1\rangle$ a smooth s-parameter family of vector fields whose flows induce a (s,t)-parametric surface for each $\varepsilon\in[0,\varepsilon_0]$,

$$\sigma_\varepsilon : \left[0, \frac{T_0}{\varepsilon}\right]\times[0,1] \ni (t,s) \mapsto \sigma_\varepsilon(t,s)$$

with properties

$$\frac{\partial}{\partial t}\sigma_\varepsilon(t,s) \in \mathbb{H}_{\sigma_\varepsilon(t,s)},$$
$$\sigma_\varepsilon(t,0) = \mathrm{Fl}^{\varepsilon t}_{\langle A_1\rangle^{\mathrm{hor}}}(m_0),$$
$$\rho\circ\sigma_\varepsilon(t,1) = \rho\circ\mathrm{Fl}^t_{(\Phi_\varepsilon)^*\mathbf{A}_\varepsilon}(m^0).$$

Then, using the properties of the horizontal lift on the principal $\mathbb{S}^1$-bundle $\rho : M \to \mathcal{O}$ and the properties of the covariant derivative on the Riemannian manifold $(M, <, >)$, we get the desired Gronwall type estimates for the arc length of the s-curves on this surface σ_ε. An important consequence of the averaging theorem is the following criterion: if $J_{\mathcal{O}} : \mathcal{O} \to \mathbb{R}$ is a first integral of the averaged vector field w, then the function $J = J_{\mathcal{O}} \circ \rho$ is an adiabatic invariant of $\mathbf{A}_\varepsilon$ that is, $| J \circ \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m^0) - J(m^0) | = O(\varepsilon)$ for small enough ε and $t \sim \frac{1}{\varepsilon}$.

3. The periodic averaging on slow-fast phase spaces. In the context of the normal form theory, we study a wide class of perturbed Hamiltonian systems on the so-called slow-fast phase spaces which appear in the the theory of adiabatic approximation [62] and its generalizations [16, 19, 74]. In applications, such perturbed models come from ε-dependent Hamiltonians which are slow or rapidly varying in some degrees of freedom. Geometrically, the perturbation theory for slow-fast systems deals with phase spaces equipped with symplectic forms (or Poisson brackets) depending on the perturbation parameter ε in a singular way at $\varepsilon = 0$. As a consequence, the main feature of our perturbed model is that, in the limit when $\varepsilon \to 0$, the unperturbed system does not inherits any natural Hamiltonian structure. Therefore, we deal with a slightly nonstandard setting in the Hamiltonian perturbation theory where the unperturbed dynamics is not Hamiltonian. This means that one can not apply to this situation some results of the regular perturbation theory for Hamiltonian systems.

By a slow-fast phase space we mean the product $M = S_1 \times S_2$ of two symplectic manifolds (S_1, σ_1) and (S_2, σ_2) equipped with rescaled product symplectic form $\sigma = \sigma_1 \oplus \varepsilon\sigma_2$. We think of M as the total space of the trivial fiber bundle $\pi_1 : M \to S_1$ over the "slow" base with "fast" fiber S_2. On such a phase space we consider a perturbed Hamiltonian system with Hamiltonian $H_\varepsilon = H_0 + \varepsilon H_1$, whose leading term H_0 depends on the slow variables $m_1 \in S_1$ and the fast variables $m_2 \in S_2$ appear only in the perturbation H_1. The corresponding Hamiltonian vector field V_{H_ε} is of the form $V_{H_\varepsilon} = \mathbb{V} + \varepsilon\mathbb{W}$, where the unperturbed vector field $\mathbb{V}$ is no longer Hamiltonian but projects to the Hamiltonian vector field v_f on (S_1, σ_1). In particular, when $H_0 \equiv 0$, we arrive at the adiabatic situation [7], [62].

We are interested in two types of normalization related to $\mathbb{S}^1$-actions. First, we show that in the resonant case, when the flow $\mathrm{Fl}^t_{\mathbb{V}}$ of the unperturbed system is periodic, the perturbed vector field $\mathbb{V}+\varepsilon\mathbb{W}$ admits a first order normalization relative to $\mathbb{V}$. Our main observation is that, although the unperturbed and perturbation vector fields $\mathbb{V}$ and $\mathbb{W}$ are not Hamiltonian, because of a special relationship between $\mathbb{V}$ and $\mathbb{W}$ and by the period-energy relation argument for the Hamiltonian vector field v_f, one can show that condition (10) holds . The term "resonance" is motivated by the following interpretation of the periodicity condition for the flow $\mathrm{Fl}^t_{\mathbb{V}}$. Since the flow $\mathrm{Fl}^t_{\mathbb{V}}$ is a fiber preserving mapping on the trivial symplectic bundle $S_1 \times S_2 \to S_1$, under the periodicity of the flow of v_f, one can introduce the monodromy map $g : S_1 \to \mathrm{Sym}(S_2, \sigma_2)$. Then, the flow $\mathrm{Fl}^t_{\mathbb{V}}$ is periodic if $g^k(m_1) = \mathrm{id}$ for all $m_1 \in S_1$and an integer $k \geq 1$. In the particular case, when $S_2 = \mathbb{R}^{2m}$ and H_1 is a quadratic function in the fast variables, this condition is precisely the resonance condition between the "tangential" and "normal" frequencies of the linearized Hamiltonian dynamics over S_1. Such perturbed models appear in the study of Hamiltonian dynamics near an invariant symplectic submanifold (S_1, σ_1) [39, 74].

The second normalization setting for $V_{H_\varepsilon} = \mathbb{V} + \varepsilon\mathbb{W}$ is motivated by the question on a geometric nature of normalization transformations in the proof of the classical adiabatic theorem [7, 38, 62]. In this case, the flow of $\mathbb{V}$ is not necessarily periodic and we only assume that $\mathbb{V}$ admits a circle first integral J. This means, the vertical Hamiltonian vector field V_J is an infinitesimal generator of an $\mathbb{S}^1$-action. Therefore we deal with the situation when the unperturbed vector field $\mathbb{V}$ is invariant with respect to the $\mathbb{S}^1$-action but not the symplectic form σ nor the Hamiltonian H_ε. To correct this "deffect", we are looking for a near identity transformation $\mathcal{T}_\varepsilon$ which brings the original perturbed model to a system which is ε^2 close to a $\mathbb{S}^1$-symmetric Hamiltonian system. We show that such a normalization transformation can be defined as a symplectomorphism between the symplectic structure σ and its $\mathbb{S}^1$-average $\langle\sigma\rangle$. The existence of such symplectomorphism follows from the following representation

$$\langle\sigma\rangle = \sigma - \varepsilon d\theta^o, \qquad \theta^o := \mathcal{S}(d_1 J),$$

which is a consequence of the general formula (11). Here, the 1-form θ^o induces an $\mathbb{S}^1$-invariant splitting of TM which is related with the notion of the Hannay-Berry connection on symplectic fiber bundles [55]. To construct a symplectomorphism $\mathcal{T}_\varepsilon$, for $\varepsilon \ll 1$, we use the Moser homotopy argument [30, 58], for a path of symplectic forms joining σ and $\langle\sigma\rangle$.

In the case of a Hamiltonian system with two degrees of freedom

$$H_\varepsilon = f(p_1, q_1) + \varepsilon F(p_1, q_1, p_2, q_2) \tag{13}$$

on the standard slow fast space

$$(M = \mathbb{R}^2 \times \mathbb{R}^2, \quad \sigma = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2), \tag{14}$$

our main result is formulated as follows.

Theorem 2 *If the unperturbed vector field $\mathbb{V}$ admits a circle first integral $J : M \to \mathbb{R}^1$, then for any open domain $N \subset \mathbb{R}^4$ and small enough $\varepsilon \neq 0$, there exists a symplectomorphism $\mathcal{T}_\varepsilon : N \to \mathbb{R}^4$ between σ^{inv} and $\langle\sigma\rangle$ such that the pull-back of the original Hamiltonian model (13), (14) is ε^2-close to the Hamiltonian system with $\mathbb{S}^1$-symmetry*

$$(\mathcal{N}, \langle\sigma\rangle, \langle H_\varepsilon\rangle = f \circ \pi_1 + \varepsilon\langle F\rangle) \tag{15}$$

in the sense that $H_\varepsilon \circ \mathcal{T}_\varepsilon = \langle H_\varepsilon\rangle + O(\varepsilon^2)$. Moreover, the $\mathbb{S}^1$-action with infinitesimal generator Υ is Hamiltonian relative to $\langle\sigma\rangle$ with momentum map εJ^0, where

$$J^0 := \mathbf{i}_\Upsilon \langle p_2 dq_2\rangle.$$

Therefore, J^0 is a first integral of the system (13), (14) related with J by $J - J^0 = g \circ \pi_1$, for a certain function $g \in C^\infty(\mathbb{R}^2)$. According to the reduction theory [2, 49, 70], restricting the $\mathbb{S}^1$-action and the system to a regular level set of J^0, we get a reduced Hamiltonian system with one degree of freedom.

A generalization of Theorem 2 to an arbitrary slow-fast space ($M = S_1 \times S_2$, $\sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)}$) with $\mathbb{S}^1$-symmetry associated to a circle first integral J of a Hamiltonian system $H_\varepsilon = f \circ \pi_1 + \varepsilon F$ which is related to so called adiabatic condition [47, 55]. The main observation here is that the $\mathbb{S}^1$-average $\langle d_1 J\rangle$ is a pull-back of a 1-form ς on S_1 whose cohomolgy class is independent of the choice of J. Thus, if $[\varsigma] = 0$, then the $\mathbb{S}^1$-action is Hamiltonian relative to $\langle\sigma\rangle$ and the approximate model $(N, \langle\sigma\rangle, \langle H_\varepsilon\rangle)$ is a Hamiltonian system with $\mathbb{S}^1$-symmetry. The corresponding moment map J^0 is just defined by the adiabatic condition $\langle d_1 J^0\rangle = 0$. Moreover, in the adiabatic case $f \equiv 0$, for the slow-fast Hamiltonian system $(M = S_1 \times S_2, \sigma = \sigma_1 \oplus \varepsilon\sigma_2, \varepsilon F)$, we get the following normalization result which give us a free action angle proof of the adibatic theorem. The second term of first order normal form for the corresponding Hamiltonian vector field is represented as follows $\mathbf{i}_{dF}\mathrm{hor}(\Pi_1) + \frac{1}{2}V^{(2)}_{\langle V_0\rangle}$, where $\mathrm{hor}(\Pi_1)$ is the horizontal lift of the Poisson tensor field Π_1 on S_1 with respect to the Hannay-Berry connection and $V^{(2)}_{\langle V_0\rangle}$ is the vertical Hamiltonian vector field on M of an $\mathbb{S}^1$-invariant function. This together with the averaging theorem proves the Montgomery conjecture [55]: a momentum map J_0 covariantly constant with respect to the Hannay-Berry connection is an adiabatic invariant of the slow-fast Hamiltonian system.

The thesis is organized as follows. In Chapter 1, we give an overview of the Lie transform method for perturbed dynamical system on manifolds including the Deprit and Hori schemes. Chapter 2 is devoted to generalized homological equations associated to periodic flows. The main results are presented in Theorem 2.4.1, Theorem 2.4.7 , and Proposition 2.4.13. At the end of the chapter we also discuss the energy-period relation for periodic Hamiltonian flows in the context of the solvability of the homological equation for vector fields. In Chapter 3, first, we formulate results (Theorem 3.1.1 and Theorem 3.1.3) on the global normalization of perturbed vector fields on a manifold which are based on the results of the previous chapter. Then, the rest of this chapter, section 3.2 deals with Gronwall types estimates for flows on a Riemannian manifold, the generalization of the $\mathbb{S}^1$-averaging theorem and its applications (Theorem 3.2.15 and Proposition 3.2.22). In Chapter 4, we present several normalization results for perturbed Hamiltonian systems on slow-fast phase spaces which also exploit the results of Chapter 2 and Chapter 3. Some motivations for possible perturbative settings are given in Section 4.1. In the resonance case, the results on the Deprit normalization and the structure of normal forms are presented in Section 4.2 (Theorem 4.2.1 and Theorem 4.2.2). In Section 4.3, we describe approximate Hamiltonian models with $\mathbb{S}^1$-symmetry for a class of Hamiltonian systems on $\mathbb{R}^4$ with slow or fast varying perturbations (Theorem 4.3.1). Subsection 4.3.3 details the averaging technique for (pre)symplectic forms and projectable dynamics. The main results of the chapter are collected in Subsection 4.3.4 and presented in Theorem 4.3.18 and Theorem 4.3.20. Finally, in Section 4.4 and Section 4.5, we illustrate the main results by some examples including the Hamiltonian systems of Yang-Mills type and the particle dynamics with spin in a magnetic field.

Chapter 1

Overview of the Lie Transform Method

A basic tool in the theory of normal form for dynamical systems is the Lie transform method which was originally developed in the works of Deprit [21], and extended by Kamel [37], (see also [33, 35, 51]). According to this method, formal normalization transformations of a perturbed system are constructed by means of formal Lie series. The existence of such transformations is provided by the solvability of linear nonhomogeneous equations (involving the Lie derivative along the unperturbed vector field) which are called the *homological equations* [5]. The "local" traditional approach (see for example [6, 7, 66, 67]) is based on the construction of solutions to homological equations and the corresponding normal forms on domains of local coordinate systems (such as action-angle variables in the Hamiltonian case).

1.1 Setting of the Normalization Problem.

Let M be a smooth manifold and $\mathfrak{X}(M)$ the space of vector fields on M. Let $\mathbf{A}(\varepsilon, x)$ be an ε-dependent vector field on M, that is, a smooth map $\mathbf{A} : \mathbb{R} \times M \to \mathrm{T}M$ such that $\mathbf{A}(\varepsilon, x) \in \mathrm{T}_x M$. In other words, the ε-dependent vector field $\mathbf{A}$ is a smooth family $\{\mathbf{A}_\varepsilon\}_{\varepsilon\in\mathbb{R}}$ of vector fields given by

$$\mathbf{A}_\varepsilon(x) := \mathbf{A}(\varepsilon, x).$$

We consider the Taylor expansion of $\mathbf{A}_\varepsilon(x)$ at $\varepsilon = 0$

$$\mathbf{A}_\varepsilon(x) = A_0(x) + \varepsilon A_1(x) + \cdots + \frac{\varepsilon^k}{k!} A_k(x) + \mathcal{O}(\varepsilon^{k+1}), \tag{1.1.1}$$

where $A_0, ..., A_k$ are vector fields on M which are defined by the relations

$$\mathcal{L}_{A_s} f = \left.\frac{d^s}{d\varepsilon^s}\right|_{\varepsilon=0} (\mathcal{L}_{\mathbf{A}_\varepsilon} f), \qquad (s = 1, ..., k), \tag{1.1.2}$$

for every $f \in C^\infty(M)$. Moreover, $\mathcal{O}(\varepsilon^{k+1})$ denotes an ε-dependent vector field which has zero at $\varepsilon = 0$ of order $k + 1$.

In the context of perturbation theory, for $\varepsilon \ll 1$, we consider the dynamical system of $\mathbf{A}_\varepsilon$

$$\frac{dx}{dt} = A_0(x) + \varepsilon A_1(x) + \cdots + \frac{\varepsilon^k}{k!} A_k(x) + \mathcal{O}(\varepsilon^{k+1}), \tag{1.1.3}$$

which is called the *perturbed system*. The limiting system as $\varepsilon \to 0$

$$\frac{dx}{dt} = A_0(x) \tag{1.1.4}$$

is called the *unperturbed system*. In practice, the unperturbed system usually has some"good" properties in the sense of the integrability theory and symmetries.

Definition 1.1.1 *An ε-dependent vector field $\mathbf{A}_\varepsilon$ on M is said to be in **normal form of order** k relative to the unperturbed vector field A_0 if the perturbation vector fields $A_1, ..., A_k$ commute with the unperturbed vector field A_0,*

$$\mathcal{L}_{A_0} A_s \equiv [A_0, A_s] = 0 \qquad (s = 1, ..., k), \tag{1.1.5}$$

or, equivalently,

$$A_s \in \ker(\mathcal{L}_{A_0}) \qquad (s = 1, ..., k). \tag{1.1.6}$$

This normalization approach provides a general setting due to Deprit [22].

Remark 1 *A more general definition of normal form of order k of an ε-dependent vector field $\mathbf{A}_\varepsilon$ is obtained under replacing condition (1.1.6) by the following*

$$A_s \in \ker(\mathcal{L}_{A_0}^l) \quad (s = 1, ..., k),$$

for a certain integer $l \geq 1$ [66].

Definition 1.1.2 *Let $N \subseteq M$ be an (nonempty) open domain and $\delta > 0$ a positive number. A smooth mapping $\Phi : (-\delta, \delta) \times N \to M$ is said to be a **near identity transformation** if for every $\varepsilon \in (-\delta, \delta)$ the map $\Phi_\varepsilon : N \to M$ given by*

$$\Phi_\varepsilon(x) = \Phi(\varepsilon, x) \tag{1.1.7}$$

is a diffeomorphism onto its image such and

$$\Phi_0 = \mathrm{id}\,. \tag{1.1.8}$$

The open subset N is called the domain of definition of the near identity transformation, usually denoted by Φ_ε. We have the following important property: the pull-back $(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon$ of the ε-dependent vector field $\mathbf{A}_\varepsilon$ by a near identity transformation Φ_ε is again an ε-dependent vector field on N such that

$$(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon\big|_{\varepsilon=0} = A_0. \tag{1.1.9}$$

This means that the near identity transformation Φ_ε preserves the unperturbed part of $\mathbf{A}_\varepsilon$.

In the case when N is a coordinate chart of M with coordinate functions x^i : $N \to \mathbb{R}$ $(i = 1, ..., \dim M)$, condition (1.1.8) can be expressed in the form

$$x^i \circ \Phi_\varepsilon^{-1} = x^i + O(\varepsilon).$$

Here, the functions $y^i = x^i \circ \Phi_\varepsilon^{-1}$ define a parameter dependent coordinate system on the image $\Phi_\varepsilon(N)$ for every $\varepsilon \in (-\delta, \delta)$.

Example 1.1.1 *Let M be a compact manifold and $\mathbf{Z}_\varepsilon$ a smooth ε-dependent (time-dependent) vector field on M. Then, the flow $\Phi_\varepsilon = \mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}$ of $\mathbf{Z}_\varepsilon$ is a near identity transformation on $N = M$ for all $\varepsilon \in \mathbb{R}$. Conversely, every near identity transformation $\Phi_\varepsilon : M \to M$ can be represented as the flow of the time-dependent vector field*

$$\mathbf{Z}_\varepsilon(x) = \frac{d\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}}{d\varepsilon}(\mathrm{Fl}^{-\varepsilon}_{\mathbf{Z}_\varepsilon}(x)) \quad x \in M.$$

Example 1.1.2 *Let $M = \mathbb{R}^n$ be the Euclidean space and $Z = \sum_{i=1}^n Z^i(x)\frac{\partial}{\partial x^i}$ be a vector field on $\mathbb{R}^n$. Then, for any open subset $N \subset \mathbb{R}^n$ with compact closure, there exists $\delta > 0$ such that the mapping*

$$x^i \mapsto x^i + \varepsilon Z^i(x)$$

is a near identity transformation with domain of definition N, for $\varepsilon \in (-\delta, \delta)$. The inverse of this mapping is of the form

$$x^i \mapsto x^i - \varepsilon Z^i(x) + O(\varepsilon^2).$$

A more general class of near-identity transformation is described in Proposition 1.1.1

Proposition 1.1.1 *Let $\Psi : \mathbb{R} \times M \to M$ be a smooth mapping, $(\varepsilon, x) \mapsto \Psi(\varepsilon, x)$ such that $\Psi_0 = \mathrm{id}$. Then, for any open domain $N \subset M$ with compact closure there exists $\delta > 0$, such that for each $\epsilon \in (\delta, \delta)$ the restriction*

$$\Phi_\varepsilon(\cdot) \stackrel{\mathrm{def}}{=} \Psi(\varepsilon, \cdot)|_N \tag{1.1.10}$$

is a diffeomorphism onto its image.

Proof. We fix $x \in \bar{N}$. Since $\Psi(0, \cdot) = \mathrm{id}$, $\mathrm{D}_x\Psi(0, x)$ is an isomorphism. So, the Implicit Function Theorem implies that there exit a number $\delta_x > 0$, an open neighborhood W_x of $\Psi(0, x) = x$ in N, and a unique smooth mapping $g : (-\delta_x, \delta_x) \times W_x \to M$ such that for all $(\varepsilon, y) \in (-\delta_x, \delta_x) \times W_x$

$$\Psi(\epsilon, g(\epsilon, y)) = y.$$

In other words, for each $\epsilon \in (-\delta_x, \delta_x)$ the mapping Φ_ε is a diffeomorphism onto $\Phi_\varepsilon(W_x)$.

Since $\bar{N}$ is compact, it can be covered by a finite number k of neighborhoods $W_{x_1}, W_{x_2}, \dots W_{x_k}$. Each one of these neighborhoods has associated a number δ_{x_i}. Let δ be the minimum of $\delta_{x_1}, \delta_{x_2}, \dots, \delta_{x_k}$. Then, for each $\epsilon \in (-\delta, \delta)$, $\Phi_\varepsilon(x)$ is a diffeomorphism onto its image. ∎

Suppose that for a given ε-dependent vector field $\mathbf{A}_\varepsilon$, there exits a near identity transformation Φ_ε such that the pull-back $(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon$ is in normal form of order N,

$$(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon = A_0 + \varepsilon\tilde{A}_1 + \cdots + \frac{\varepsilon^N}{N!}\tilde{A}_N + \mathcal{O}(\varepsilon^{N+1}), \tag{1.1.11}$$

$$[A_0, \tilde{A}_s] = 0 \qquad (s = 1, ..., N). \tag{1.1.12}$$

In this case, $(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon$ is called a *normalization transformation* of order N. Consider the truncated vector field

$$A_0 + \varepsilon \tilde{\mathbf{A}}_\varepsilon^{(N)} \tag{1.1.13}$$

where

$$\tilde{\mathbf{A}}_\varepsilon^{(N)} \stackrel{\text{def}}{=} \tilde{A}_1 + \frac{1}{2}\varepsilon \tilde{A}_2 \cdots + \frac{\varepsilon^{N-1}}{N!}\tilde{A}_N. \tag{1.1.14}$$

Because of (1.1.12) the flow of the truncated vector field can be represented as the composition of the "slow" and "fast" components

$$\mathrm{Fl}^t_{A_0 + \tilde{\mathbf{A}}_\varepsilon^{(N)}} = \mathrm{Fl}^{\varepsilon t}_{\tilde{\mathbf{A}}_\varepsilon^{(N)}} \circ \mathrm{Fl}^t_{A_0} .$$

Long time scale. To complete this section we recall the following property of the flow of a perturbed vector field

Proposition 1.1.2 *Let $\mathbf{A}_\varepsilon = A_0 + \varepsilon \mathbf{R}_\varepsilon$ be an smooth ε-dependent vector field. Assume that the unperturbed vector field A_0 is complete on M. Then, for any open domain $N \subseteq M$ with compact closure and any constant $\delta > 0$ there is a constant $L > 0$ such that the flow $\mathrm{Fl}^t_{\mathbf{A}_\varepsilon}$ of $\mathbf{A}_\varepsilon$ is well-defined on N for all $t \in [0, \frac{L}{\varepsilon}]$ and each $\varepsilon \in (0, \delta]$.*

Proof. We will use the following fact which follows from standard properties of flows. The flows of two vector fields X and Y on M are related by

$$\mathrm{Fl}^t_X \circ \mathrm{Fl}^t_{P_t} = \mathrm{Fl}^t_Y . \tag{1.1.15}$$

where P_t is a time dependent vector field given by

$$P_t \stackrel{\text{def}}{=} -X + (\mathrm{Fl}^t_X)^* Y. \tag{1.1.16}$$

Let

$$(\mathrm{Fl}^t_{A_0})^* \mathbf{A}_\varepsilon - A_0 = \varepsilon \mathbf{R}_t(\varepsilon). \tag{1.1.17}$$

where $\mathbf{R}_t(\varepsilon) = (\mathrm{Fl}^t_{A_0})^* \mathbf{R}_\varepsilon$ depends on t and ε smoothly. Fix $\delta > 0$. The by the Flow Box Theorem and compactness of $\bar{N}$ there exists L such that the flow of $\mathbf{R}_{t,\varepsilon}$ is well-defined on N for $t \in [0, L]$. Applying formula (1.1.15) for $X = A_0$, $Y = \mathbf{A}_\varepsilon$ and $P_t = \mathbf{R}_{t,\varepsilon}$, we get

$$\mathrm{Fl}^t_{\mathbf{A}_\varepsilon} = \mathrm{Fl}^t_{A_0} \circ \mathrm{Fl}^{\varepsilon t}_{\mathbf{R}_{t,\varepsilon}},$$

and since $\mathrm{Fl}^t_{A_0}$ is well-defined for all $t \in \mathbb{R}$, we obtain the desired result. ∎

1.2 Lie Transforms on Manifolds

The idea of the Lie transform method is searching for a normalization transformation for a perturbed vector field as the flow of a time-dependent vector field where the small parameter ε plays the role of time. This method allows us to reduce the normalization problem to the study of the solvability of linear nonhomogeneous equations, involving the Lie derivative along the unperturbed vector field A_0, which are called the homological equations due to Arnold [5]. Usually, in the context of the formal normalization problem, the derivation of homological equations is given by using the formal Lie series and formal near identity transformations (see, for example [13, 15, 21, 33, 50, 59]). In this Section, we apply the Lie method to construct normal forms (in the sense of Definition 1.1.1) for perturbed systems of general type (which are not necessarily Hamiltonian), which consists of two steps: (1) Taylor expansions of flows and (2) the derivation of homological equations. Our considerations are based on the basic relationship in differential geometry between flows and Lie derivatives and is closed to the approach of Hernard and Roels [34].

We describe three ways for the construction of a near identity transformation Φ_ε:

(I) Deprit's version: Φ_ε is defined as a flow of a time-dependent vector field $\mathbf{Z}_\varepsilon$, where the perturbation parameter ε plays the role of time variable.

(II) Hori's version: Φ_ε is defined as the time-ε flow of an autonomous vector field $\mathbf{Z}(\varepsilon)$ smoothly depending on the parameter ε.

(III) Generalized version: Φ_ε is defined as the time-ε flow of a time-dependent vector field $\mathbf{Z}_\lambda(\varepsilon)$ smoothly depending on the parameter λ.

1.2.1 Deprit's method

Let $\mathbf{Z}_\varepsilon(x)$ be an smooth ε-dependent (time-dependent) vector field on M. For every integer $K \geq 0$, we have the Taylor expansion of $\mathbf{Z}_\varepsilon$ at $\varepsilon = 0$:

$$\mathbf{Z}_\varepsilon = \sum_{n=0}^{K} \frac{\varepsilon^k}{k!} Z_k + \mathcal{O}(\varepsilon^{K+1}). \tag{1.2.1}$$

Let $\Phi_\varepsilon = \mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}$ be the flow of $\mathbf{Z}_\varepsilon$,

$$\frac{d\,\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}}{d\varepsilon} = \mathbf{Z}_\varepsilon \circ \mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}, \tag{1.2.2}$$

$$\mathrm{Fl}^0_{\mathbf{Z}_\varepsilon} = \mathrm{id}\,. \tag{1.2.3}$$

Assume that there exist an open domain $N \subseteq M$ and $\delta > 0$ such that the flow $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}$ is well-defined on N for all $\varepsilon \in (-\delta, \delta)$. In other words, the map $\Phi_\varepsilon \overset{\text{def}}{=} \mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}$ is a near identity transformation with domain of definition N. In this case, the time dependent vector field $\mathbf{Z}_\varepsilon$ will be called a *generator* (or *generating vector field*) of Φ_ε.

Suppose we are given an ε-dependent vector field $\mathbf{A}_\varepsilon$ on M,

$$\mathbf{A}_\varepsilon = A_0 + \sum_{k=1}^{K} \frac{\varepsilon^k}{k!} A_k + O(\varepsilon^{K+1}). \tag{1.2.4}$$

Consider the pull-back of $\mathbf{A}_\varepsilon$ by the flow Φ_ε which is an ε-dependent vector field on N with Taylor expansion at $\varepsilon = 0$:

$$\tilde{\mathbf{A}}_\varepsilon \stackrel{\text{def}}{=} (\Phi^\varepsilon)^* \mathbf{A}_\varepsilon = A_0 + \sum_{n=1}^{K} \frac{\varepsilon^n}{n!} \tilde{A}_n + O(\varepsilon^{K+1}). \tag{1.2.5}$$

The point here is to compute the vector field coefficients $\tilde{A}_n \in \mathfrak{X}(N)$ of this decomposition in terms of the vector fields Z_n and A_n in (1.2.1) and (1.2.4).

By formulas (1.1.2), the coefficients $\tilde{A}_n$ in the Taylor expansion (1.2.5) are given by

$$\tilde{A}_k = \left.\frac{d^k}{d\varepsilon^k}\right|_{\varepsilon=0} ((\Phi^\varepsilon)^* \mathbf{A}_\varepsilon), \qquad \text{on} \quad N. \tag{1.2.6}$$

We have the following *basic formula* which describes the relationship between the Lie derivative and the flows of time-dependent vector fields

$$\frac{d}{d\varepsilon}((\Phi^\varepsilon)^* \mathbf{A}_\varepsilon) = (\Phi^\varepsilon)^* \left(\mathcal{L}_{Z_\varepsilon} \mathbf{A}_\varepsilon + \frac{\partial}{\partial \varepsilon} \mathbf{A}_\varepsilon \right), \tag{1.2.7}$$

where $\mathcal{L}_{Z_\varepsilon}$ is the Lie derivative along the vector field $\mathcal{L}_{Z_\varepsilon}$. Denote by $\mathfrak{X}(\mathbb{R} \times M)$ the space of all ε-dependent vector fields on M. Introduce the linear differential operator $\partial_{\mathbf{Z}_\varepsilon} : \mathfrak{X}(\mathbb{R} \times M) \to \mathfrak{X}(\mathbb{R} \times M)$ given by

$$\partial_{\mathbf{Z}_\varepsilon} \stackrel{\text{def}}{=} \mathcal{L}_{\mathbf{Z}_\varepsilon} + \frac{\partial}{\partial \varepsilon}.$$

Lemma 1.2.1 *For every integer $k \geq 1$, the following identity holds*

$$\tilde{A}_k = (\partial^k_{\mathbf{Z}_\varepsilon} \mathbf{A}_\varepsilon)\,|_{\varepsilon=0}, \tag{1.2.8}$$

where $\partial^k_{\mathbf{Z}_\varepsilon} = \partial_{\mathbf{Z}_\varepsilon} \circ \ldots \circ \partial_{\mathbf{Z}_\varepsilon}$ (k-times).

Proof. By formula (1.2.6), we just need to prove that

$$\frac{d^k}{d\varepsilon^k} [(\Phi^\varepsilon)^* \mathbf{A}_\varepsilon] = (\Phi^\varepsilon)^* \left(\partial^k_{\mathbf{Z}_\varepsilon} \mathbf{A}_\varepsilon \right), \tag{1.2.9}$$

for every $k \geq 1$. We proceed by induction. If $k = 1$, equation (1.2.8) coincides with basic formula (1.2.7). Then, we assume that (1.2.9) is true for $k = n - 1$. By direct computation, we get

$$\begin{aligned} \frac{d^n}{d\varepsilon^n} [(\Phi^\varepsilon)^* \mathbf{A}_\varepsilon] &= \frac{d}{d\varepsilon} \left(\frac{d^{n-1}}{d\varepsilon^{n-1}} [(\Phi^\varepsilon)^* \mathbf{A}_\varepsilon] \right), = \frac{d}{d\varepsilon} \left((\Phi^\varepsilon)^* \left(\partial^{n-1}_{\mathbf{Z}_\varepsilon} \mathbf{A}_\varepsilon \right) \right) \\ &= (\Phi^\varepsilon)^* \partial_{\mathbf{Z}_\varepsilon} \left(\partial^{n-1}_{\mathbf{Z}_\varepsilon} \mathbf{A}_\varepsilon \right) = (\Phi^\varepsilon)^* \left(\partial^n_{\mathbf{Z}_\varepsilon} \mathbf{A}_\varepsilon \right). \end{aligned}$$

■

On the other hand, the vector fields $\partial^k_{\mathbf{Z}_\varepsilon}\mathbf{A}_\varepsilon(x)$ also depend smoothly on ε. Let us suppose that $\partial^k_{\mathbf{Z}_\varepsilon}\mathbf{A}_\varepsilon(x)$ has the following Taylor expansion at $\varepsilon = 0$:

$$\partial^k_\varepsilon \mathbf{A}_\varepsilon = \sum_{m=0}^{K} \frac{\varepsilon^m}{m!} A^{(k)}_m + O(\varepsilon^{K+1}). \tag{1.2.10}$$

Now we prove a result which establishes a recursive relation between the coefficient of the Taylor expansion of vector fields (1.2.10).

Lemma 1.2.2 *The vector fields $A^{(k)}_m \in \mathfrak{X}(M)$ in (1.2.10) satisfy the recurrent relations*

$$A^{(k)}_n = A^{(k-1)}_{n+1} + \sum_{m=0}^{n} \mathrm{C}_n \mathcal{L}_{Z_m} A^{(k-1)}_{n-m}, \qquad \forall \quad k \geq 0. \tag{1.2.11}$$

Here $\mathrm{C}^n_m = \dfrac{n!}{m!(n-m)!}$.

Proof. In order to prove this lemma, we will use the following algebraic fact. For any two linear operators $\mathcal{T}$ and $\mathcal{D}$ on a vector space, we have the identity

$$[\mathcal{D}^n, \mathcal{T}] = \sum_{i=0}^{n-1} \mathrm{C}^n_i \,\mathrm{ad}^{n-i}_{\mathcal{D}}(\mathcal{T}) \cdot \mathcal{D}^i, \tag{1.2.12}$$

where $\mathrm{ad}_{\mathcal{D}}(\mathcal{T}) = [\mathcal{D}, \mathcal{T}]$. Equation (1.2.10) and formula (1.1.2) implies that

$$A^{(k)}_n = \left\{ \left(\frac{\partial^n}{\partial \varepsilon^n} \circ \partial^k_{\mathbf{Z}_\varepsilon} \right) \mathbf{A}_\varepsilon \right\}\Big|_{\varepsilon=0}. \tag{1.2.13}$$

By direct computation, we obtain

$$\begin{aligned}
\frac{\partial^n}{\partial \varepsilon^n} \circ \partial_{\mathbf{Z}_\varepsilon} &= \frac{\partial^n}{\partial \varepsilon^n} \circ \left(\mathcal{L}_{\mathbf{Z}_\varepsilon} + \frac{\partial}{\partial \varepsilon} \right) = \left(\mathcal{L}_{\mathbf{Z}_\varepsilon} + \frac{\partial}{\partial \varepsilon} \right) \circ \frac{\partial^n}{\partial \varepsilon^n} + \left[\frac{\partial^n}{\partial \varepsilon^n}, \partial_{\mathbf{Z}_\varepsilon} \right], \\
&= \mathcal{L}_{\mathbf{Z}_\varepsilon} \circ \frac{\partial^n}{\partial \varepsilon^n} + \frac{\partial^{n+1}}{\partial \varepsilon^{n+1}} + \left[\frac{\partial^n}{\partial \varepsilon^n}, \mathcal{L}_{\mathbf{Z}_\varepsilon} \right].
\end{aligned}$$

Taking into account that $\left[\frac{\partial^n}{\partial \varepsilon^n}, \mathcal{L}_{Z_\varepsilon}\right] = \mathcal{L}_{\frac{\partial^n}{\partial \varepsilon^n}\mathbf{Z}_\varepsilon}$, and applying formula (1.2.12) to operators $\mathcal{D} = \frac{\partial^n}{\partial \varepsilon^n}$ and $\mathcal{T} = \mathcal{L}_{\mathbf{Z}_\varepsilon}$, we obtain

$$\left[\frac{\partial^n}{\partial \varepsilon^n}, \mathcal{L}_{\mathbf{Z}_\varepsilon} \right] = \sum_{k=0}^{n-1} \mathrm{C}^n_k \mathcal{L}_{\frac{\partial^{n-k}}{\partial \varepsilon^{n-k}}\mathbf{Z}_\varepsilon} \frac{\partial^k}{\partial \varepsilon^k}.$$

Thus, we have $\frac{\partial^n}{\partial \varepsilon^n} \circ \partial_{\mathbf{Z}_\varepsilon} = \frac{\partial^{n+1}}{\partial \varepsilon^{n+1}} + \sum_{k=0}^{n} \mathrm{C}^n_k \mathcal{L}_{\frac{\partial^{n-k}}{\partial \varepsilon^{n-k}}\mathbf{Z}_\varepsilon} \frac{\partial^k}{\partial \varepsilon^k}$. Therefore, we get

$$\begin{aligned}
\left(\frac{\partial^n}{\partial \varepsilon^n} \circ \partial^k_{\mathbf{Z}_\varepsilon} \right) &= \left(\frac{\partial^n}{\partial \varepsilon^n} \circ \partial_{\mathbf{Z}_\varepsilon} \right) \circ \partial^{k-1}_{\mathbf{Z}_\varepsilon}, \\
&= \left(\frac{\partial^{n+1}}{\partial \varepsilon^{n+1}} \circ \partial^{k-1}_{\mathbf{Z}_\varepsilon} \right) + \sum_{k=0}^{n} \mathrm{C}^n_k \mathcal{L}_{\frac{\partial^{n-k}}{\partial \varepsilon^{n-k}}\mathbf{Z}_\varepsilon} \circ \left(\frac{\partial^k}{\partial \varepsilon^k} \circ \partial^{k-1}_{\mathbf{Z}_\varepsilon} \right)
\end{aligned} \tag{1.2.14}$$

$$\begin{array}{ccccccc}
A_0 & & & & & & \\
\downarrow & & & & & & \\
A_1 & \longrightarrow & A_0^{(1)} & & & & \\
\downarrow & & \downarrow & & & & \\
A_2 & \longrightarrow & A_1^{(1)} & \longrightarrow & A_0^{(2)} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A_3 & \longrightarrow & A_2^{(1)} & \longrightarrow & A_1^{(2)} & \longrightarrow & A_0^{(3)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Figure 1.1: Deprit's Triangle

Applying (1.2.14) to vector field $\mathbf{A}_\varepsilon$ and evaluating at $\varepsilon = 0$ we obtain (1.2.11). ■
The recursive formula (1.2.11) can be illustrated in Deprit's triangle, shown in figure 1.1, (see also [21, 50]).

Given an ε-dependent vector field $\mathbf{A}_\varepsilon$, the coefficients of its Taylor expansion (1.2.4) are located in the first column of the Deprit's triangle. We suppose that we have already calculated the terms of the first (k-1) rows and we want to compute the terms of the k-th row. We start with the computation of $A_{k-1}^{(1)}$. This computation involves only the terms on the first column which are above A_k (A_k itself, see formula (1.2.11)). Next, we compute $A_{k-2}^{(2)}$ using the term of the second column above $A_{k-1}^{(1)}$. We can continue with the computations of the terms of the k-th rows using formula (1.2.11).

We observe that the coefficients in the Taylor expansion of vector field $\widetilde{\mathbf{A}}_\varepsilon$ (1.2.5) are in the diagonal of Deprit's triangle. That is, $\widetilde{A}_k = A_0^{(k)}$. It follows by formula (1.2.11) that

$$\widetilde{A}_k = A_0^{(k)} = A_1^{(k-1)} + \mathcal{L}_{Z_0} A_0^{(k-1)}.$$

Finally, we can derive formulas for vector fields $\widetilde{A}_k$ in terms of the coefficients of Taylor expansion of vector fields $\mathbf{A}_\varepsilon$ and $\mathbf{Z}_\varepsilon$.

Proposition 1.2.3 ([21]) *The coefficients $\widetilde{A}_k$ are given by the formulas*

$$\widetilde{A}_k = A_k + \mathcal{L}_{Z_{k-1}} A_0 + R_{k-1}^D \tag{1.2.15}$$

*for $k = 1, 2, \ldots$, where the vector fields $R_{k-1}^D = R_{k-1}^D\{Z_0, \ldots, Z_{k-2}; A_0, \ldots, A_{k-1}\}$ are determined in terms of vector fields $Z_0, \ldots, Z_{k-2}$ and $A_0, \ldots, A_{k-1}$ by mean of recursive formulas (*1.2.11*).*

Proof. We just need to prove that

$$A_n^{(k)} = A_{n+k} + \mathcal{L}_{Z_{n+k-1}} A_0 + S_{n,k}^D, \tag{1.2.16}$$

where the vector fields $S^D_{n,k} = S^D_{n,k}\{Z_0, Z_1, \ldots, Z_{n+k-2}; A_0, A_1, \ldots, A_{n+k-1}\}$ are determined in terms of vector fields $Z_0, ..., Z_{n+k-2}$ and $A_0, ..., A_{n+k-1}$ by means of (1.2.11) for every non-negative integers $k \geq 1$, n. We proceed by induction over k. For $k = 1$, formula (1.2.11) reduces to

$$A^{(1)}_n = A_{n+1} + \sum_{m=0}^{n} C^n_m \mathcal{L}_{Z_m} A_{n-m} = A_{n+1} + \mathcal{L}_{Z_n} A_0 + \sum_{m=0}^{n-1} A_0 C^n_m \mathcal{L}_{Z_m} A_{n-m}.$$

Hence, we have $S^D_{n,1}\{Z_0, Z_1, \ldots, Z_n; A_0, A_1, \ldots, A_{n+1}\} = \sum_{m=0}^{n-1} C^n_m \mathcal{L}_{Z_m} A_{n-m}$. Now, we assume that (1.2.16) hold for $k = d$ and all integer n, that is

$$A^{(d)}_n = A_{n+d} + \mathcal{L}_{Z_{n+d-1}} A_0 + S^D_{n,d}. \tag{1.2.17}$$

Formula (1.2.11) gives $A^{(d+1)}_n = A^{(d)}_{n+1} + \sum_{m=0}^{n} C^n_m \mathcal{L}_{Z_m} A^{(d)}_{n-m}$. Since the vector fields $A^{(d)}_n$ are given by (1.2.17) for all n, we have

$$\begin{aligned} A^{(d+1)}_n &= A_{n+d+1} + \mathcal{L}_{Z_{n+d}} A_0 + S^D_{n,d} \\ &\quad + \sum_{m=0}^{n} C^n_m \mathcal{L}_{Z_m} \left(A_{n+d-m} + \mathcal{L}_{Z_{n+d-m-1}} A_0 + S^D_{n-m,d}\right). \end{aligned}$$

Taking

$$S^D_{n,d+1} \stackrel{\text{def}}{=} S^D_{n,d} + \sum_{m=0}^{n} C^n_m \mathcal{L}_{Z_m} \left(A_{n+d-m} + \mathcal{L}_{Z_{n+d-m-1}} A_0 + S^D_{n-m,d}\right), \tag{1.2.18}$$

we have that (1.2.16) also hold for $k = d + 1$ and for all n. Finally, $\widetilde{A}_k = A^{(k)}_0$ and equations (1.2.16) reduce to (1.2.15), where $R^D_{k-1} = S^D_{0,k}$. ■

As illustration of the recursive formulas (1.2.15), we compute some vector fields $\widetilde{A}_k$.

First order:

$$\widetilde{A}_1 = \mathcal{L}_{Z_0} A_0 + A_1, \quad \text{and} \quad R^D_0 = 0.$$

Second order:

$$\widetilde{A}_2 = A_2 + \mathcal{L}_{Z_1} A_0 + R^D_1, \quad \text{and} \quad R^D_1 = \mathcal{L}^2_{Z_0} A_0 + 2\mathcal{L}_{Z_0} A_1.$$

Third order:

$$\widetilde{A}_3 = A_3 + \mathcal{L}_{Z_2} A_0 + R^D_2,$$
$$R^D_2 = 3\mathcal{L}_{Z_0} A_2 + 3\mathcal{L}^2_{Z_0} A_1 + \mathcal{L}^3_{Z_0} A_0 + (2\mathcal{L}_{Z_0}\mathcal{L}_{Z_1} + \mathcal{L}_{Z_1}\mathcal{L}_{Z_0}) A_0 + 3\mathcal{L}_{Z_1} A_1.$$

In summary, if Φ_ε is the flow of the vector field $\mathbf{Z}_\varepsilon$ (1.2.1), then the coefficients of the Taylor expansion at $\varepsilon = 0$ of $\widetilde{\mathbf{A}}_\varepsilon = \Phi^*_\varepsilon \mathbf{A}_\varepsilon = A_0 + \varepsilon \widetilde{A}_1 + \frac{1}{2!}\varepsilon^2 \widetilde{A}_2 + \frac{1}{3!}\varepsilon^3 \widetilde{A}_3 + O(\varepsilon^4)$, are given by

$\widetilde{A}_1$	$\mathcal{L}_{Z_0} A_0 + A_1$
$\widetilde{A}_2$	$\mathcal{L}^2_{Z_0} A_0 + 2\mathcal{L}_{Z_0} A_1 + \mathcal{L}_{Z_1} A_0 + A_2$
$\widetilde{A}_3$	$3\mathcal{L}_{Z_0} A_2 + 3\mathcal{L}^2_{Z_0} A_1 + \mathcal{L}^3_{Z_0} A_0 + 2\mathcal{L}_{Z_0}\mathcal{L}_{Z_1} A_0 +$ $\mathcal{L}_{Z_1}\mathcal{L}_{Z_0} A_0 + 3\mathcal{L}_{Z_1} A_1 + \mathcal{L}_{Z_2} A_0 + A_3$

We remark that these formulas remain true if we replace the vector field $\mathbf{A}_\varepsilon$ by any ε-dependent tensor field on M. In particular, for an ε-dependent function $\mathbf{H}_\varepsilon = H_0 + \varepsilon H_1 + \frac{1}{2}\varepsilon^2 H_2$ we have

$$\mathbf{H}_\varepsilon \circ \Phi_\varepsilon = H_0 + \varepsilon\left(\mathcal{L}_{Z_0} H_0 + H_1\right) + \frac{\varepsilon^2}{2}\left(\mathcal{L}^2_{Z_0} H_0 + 2\mathcal{L}_{Z_0} H_1 + \mathcal{L}_{Z_1} H_0 + H_2\right) + O(\varepsilon^3).$$

1.2.2 Hori's method

Let $\mathbf{Z}(\varepsilon)$ be a ε-dependent vector field on a manifold M with Taylor expansion at $\varepsilon = 0$

$$\mathbf{Z}(\varepsilon) = \sum_{n=0}^{K} \frac{\varepsilon^k}{k!} Z_k + \mathcal{O}(\varepsilon^{K+1}). \tag{1.2.19}$$

We consider $\mathbf{Z}(\varepsilon)$ as an *autonomous* vector field on M smoothly depending on the parameter ε, let $\mathrm{Fl}^\lambda_{\mathbf{Z}(\varepsilon)}$ be the time-λ flow of $\mathbf{Z}(\varepsilon)$,

$$\frac{\mathrm{d}\,\mathrm{Fl}^\lambda_{\mathbf{Z}(\varepsilon)}}{\mathrm{d}\lambda} = \mathbf{Z}(\varepsilon) \circ \mathrm{Fl}^\lambda_{\mathbf{Z}(\varepsilon)}, \tag{1.2.20}$$

$$\mathrm{Fl}^0_{\mathbf{Z}(\varepsilon)} = \mathrm{id}\,. \tag{1.2.21}$$

We define the family of diffeomorphisms

$$\Phi_\varepsilon \stackrel{\text{def}}{=} \mathrm{Fl}^\lambda_{\mathbf{Z}(\varepsilon)}\Big|_{\lambda=\varepsilon}. \tag{1.2.22}$$

It is clear that $\Phi_0 = \mathrm{id}$. Therefore, the mapping Φ_ε (1.2.22) is a near identity transformation which is called *Hori's transformation.*

Assume that we are given an ε-dependent vector field $\mathbf{A}_\varepsilon$ on M

$$\mathbf{A}(\varepsilon) = \sum_{n=0}^{K} \frac{\varepsilon^k}{k!} A_k + \mathcal{O}(\varepsilon^{K+1}), \tag{1.2.23}$$

and Φ_ε is a Hori's transformation well-defined on an open subset $N \subset M$ and generated by the family of autonomous vector fields $\mathbf{Z}(\varepsilon)$ given by (1.2.19). We define the ε-dependent vector field $\widetilde{\mathbf{A}}_\varepsilon$ by

$$\widetilde{\mathbf{A}}_\varepsilon := \Phi^*_\varepsilon \mathbf{A}_\varepsilon = A_0 + \sum_{n=1}^{K} \widetilde{A}_n + O(\varepsilon^{K+1}). \tag{1.2.24}$$

Our goal is to get an expression for the vector fields $\widetilde{A}_n$ of the decomposition above in terms of the coefficients Z_n and A_n. For every fixed ε, formulas (1.1.2) and (1.2.7) imply the following decomposition of vector field $\left(\mathrm{Fl}^{\lambda}_{\mathbf{Z}(\varepsilon)}\right)^* \mathbf{A}_\varepsilon$ at $\lambda = 0$

$$(\mathrm{Fl}^{\lambda}_{\mathbf{Z}(\varepsilon)})^* \mathbf{A}(\varepsilon) = \sum_{m=0}^{K} \frac{\lambda^m}{m!} \mathcal{L}^m_{\mathbf{Z}(\varepsilon)} \mathbf{A}(\varepsilon). \tag{1.2.25}$$

Putting $\lambda = \varepsilon$ into (1.2.25) and using the Taylor expansion (1.2.23) of $\mathbf{A}_\varepsilon$, we get

$$\Phi^*_\varepsilon \mathbf{A}_\varepsilon = \sum_{n=0}^{K} \frac{\varepsilon^n}{n!} \sum_{m=0}^{n} \mathrm{C}^n_m \mathcal{L}^m_{\mathbf{Z}(\varepsilon)} A_{n-m} + O(\varepsilon^{K+1}).$$

In term of the coefficients of Taylor decomposition of $\mathbf{Z}(\varepsilon)$, the Lie derivative operator $\mathcal{L}^m_{\mathbf{Z}(\varepsilon)}$ takes the form

$$\mathcal{L}^m_{\mathbf{Z}(\varepsilon)} = (\mathcal{L}_{Z_0} + \varepsilon \mathcal{L}_{Z_1} + \frac{\varepsilon^2}{2!} \mathcal{L}_{Z_2} + \frac{\varepsilon^3}{3!} \mathcal{L}_{Z_3} + ...)^m.$$

The Lie operators $\mathcal{L}^m_{\mathbf{Z}(\varepsilon)}$ depend smoothly on ε. So, we have the following decomposition

$$\mathcal{L}^m_{\mathbf{Z}(\varepsilon)} = \sum_{j=0} \frac{\varepsilon^j}{j!} \hat{\mathbb{L}}^{(m)}_j, \tag{1.2.26}$$

where the differential operators $\hat{\mathbb{L}}^{(m)}_j$ are defined by the recurrent relations

$$\hat{\mathbb{L}}^{(m)}_j = \sum_{i=0}^{j} \mathrm{C}^k_i \mathcal{L}_{Z_i} \circ \hat{\mathbb{L}}^{(m-1)}_{j-i} \tag{1.2.27}$$

with

$$\begin{aligned} \hat{\mathbb{L}}^{(0)}_0 &= \mathrm{id}, \quad \hat{\mathbb{L}}^{(m)}_0 \equiv 0, \\ \hat{\mathbb{L}}^{(1)}_j &= \mathcal{L}_{Z_j}, \quad \text{and} \quad \hat{\mathbb{L}}^{(m)}_0 = \mathcal{L}^m_{Z_0}. \end{aligned}$$

Proposition 1.2.4 *Vector fields $\widetilde{A}_n$ in (1.2.24) are given by the recursive formulas*

$$\widetilde{A}_n = \sum_{m=0}^{n} \sum_{i=0}^{n-m} \mathrm{C}^n_m \mathrm{C}^{n-m}_i \hat{\mathbb{L}}^{(m)}_i A_{n-m-i}. \tag{1.2.28}$$

Proof. By (1.1.2), we have $\widetilde{A}_n = \frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n}\Big|_{\varepsilon=0} (\Phi^*_\varepsilon \mathbf{A}_\varepsilon)$. By direct computation, we obtain

$$\frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n} (\Phi^*_\varepsilon \mathbf{A}_\varepsilon) \quad = \quad \sum_{m=0}^{K} \frac{\mathrm{d}^n}{\mathrm{d}\varepsilon^n} \left(\frac{\varepsilon^m}{m!} \mathcal{L}^m_{\mathbf{Z}(\varepsilon)} \mathbf{A}_\varepsilon \right) = \sum_{m=0}^{K} \sum_{i=0}^{n} \mathrm{C}^n_i \frac{\varepsilon^{m+i-n}}{(m+i-n)!} \frac{\mathrm{d}^i}{\mathrm{d}\varepsilon^i} \left(\mathcal{L}^m_{\mathbf{Z}(\varepsilon)} \mathbf{A}_\varepsilon \right).$$

Thus

$$\widetilde{A}_k = \sum_{m=0}^{n} \mathrm{C}^n_{n-m} \left. \frac{\mathrm{d}^{n-m}}{\mathrm{d}\varepsilon^{n-m}} \right|_{\varepsilon=0} \left(\mathcal{L}^m_{\mathbf{Z}(\varepsilon)} \mathbf{A}_\varepsilon \right).$$

By (1.2.23) and (1.2.26), we have

$$\mathcal{L}^m_{\mathbf{Z}(\varepsilon)}\mathbf{A}_\varepsilon = \sum_{k=0}^{K}\frac{\varepsilon^k}{k!}\sum_{i=0}^{k}\mathrm{C}_i^k\hat{\mathbb{L}}_i^{(m)}A_{k-i}.$$

Hence,

$$\left.\frac{\mathrm{d}^{n-m}}{\mathrm{d}\varepsilon^{n-m}}\right|_{\varepsilon=0}\left(\mathcal{L}^m_{\mathbf{Z}(\varepsilon)}\mathbf{A}_\varepsilon\right) = \sum_{i=0}^{n-m}\mathrm{C}_i^{n-m}\hat{\mathbb{L}}_i^{(m)}A_{(n-m)-i}.$$

Therefore, we have

$$\widetilde{A}_n = \sum_{m=0}^{n}\sum_{i=0}^{n-m}\mathrm{C}_m^n\mathrm{C}_i^{n-m}\hat{\mathbb{L}}_i^{(m)}A_{n-m-i}.$$

■

Analogously to Deprit's method, formulas (1.2.28) of vector fields $\widetilde{A}_k$ (1.2.24) can be written as

$$\tilde{A}_k = A_k + k\mathcal{L}_{Z_{k-1}}A_0 + R^H_{k-1},$$

where the vector fields $R^H_{k-1} = R^H_{k-1}\{Z_0, ..., Z_{k-2}; A_0, ..., A_{k-1}\}$ are determined in terms of vector fields $Z_0, ..., Z_{k-2}$ and $A_0, ..., A_{k-1}$ by mean of recurrent formulas (1.2.27) and (1.2.28). Therefore, if Φ_ε is the near transformation (1.2.22), then the coefficients of the Taylor expansion at $\varepsilon = 0$ of $\widetilde{\mathbf{A}}_\varepsilon = \Phi^*_\varepsilon\mathbf{A}_\varepsilon = A_0 + \varepsilon\widetilde{A}_1 + \frac{1}{2!}\varepsilon^2\widetilde{A}_2 + \frac{1}{3!}\varepsilon^3\widetilde{A}_3 + O(\varepsilon^4)$, are given by

$\widetilde{A}_1$	$\mathcal{L}_{Z_0}A_0 + A_1$
$\widetilde{A}_2$	$\mathcal{L}^2_{Z_0}A_0 + 2\mathcal{L}_{Z_0}A_1 + 2\mathcal{L}_{Z_1}A_0 + A_2$
$\widetilde{A}_3$	$3\mathcal{L}_{Z_0}A_2 + 3\mathcal{L}^2_{Z_0}A_1 + \mathcal{L}^3_{Z_0}A_0 + 3\mathcal{L}_{Z_0}\mathcal{L}_{Z_1}A_0+$ $3\mathcal{L}_{Z_1}\mathcal{L}_{Z_0}A_0 + 6\mathcal{L}_{Z_1}A_1 + 3\mathcal{L}_{Z_2}A_0 + A_3$

1.2.3 Generalized scheme

Now, we suppose that a vector field $\mathbf{Z}_\lambda(\varepsilon)$ is given, and it is smoothly depending on the parameters λ and ε. Computing the Taylor expansion at $\lambda = 0$ and $\varepsilon = 0$

$$\mathbf{Z}_\lambda(\varepsilon) = \sum_{k=0}^{K}\sum_{m=0}^{K}\frac{\lambda^k\varepsilon^m}{k!m!}Z_{k,m} + \mathcal{O}(\lambda^{K+1}) + \mathcal{O}(\varepsilon^{K+1}), \tag{1.2.29}$$

where $Z_{k,m} = \left.\frac{\partial^{k+m}\mathbf{Z}_\lambda(\varepsilon)}{\partial\lambda^k\partial\varepsilon^m}\right|_{\lambda=0,\varepsilon=0}$. Let $\mathrm{Fl}^\lambda_{\mathbf{Z}_\lambda(\varepsilon)}$ be the flow of $\mathbf{Z}_\lambda(\varepsilon)$,

$$\frac{d\,\mathrm{Fl}^\lambda_{\mathbf{Z}_\lambda(\varepsilon)}}{d\lambda} = \mathbf{Z}_\lambda(\varepsilon)\circ\mathrm{Fl}^\lambda_{\mathbf{Z}(\varepsilon)},$$

$$\mathrm{Fl}^0_{\mathbf{Z}_\lambda(\varepsilon)} = \mathrm{id}\,.$$

Define the map

$$\Phi_\varepsilon \overset{\text{def}}{=} \mathrm{Fl}^{\lambda}_{\mathbf{Z}_\lambda(\varepsilon)}\Big|_{\lambda=\varepsilon} \tag{1.2.30}$$

with $\Phi_0 = \mathrm{id}$. Assuming that Φ_ε is well-defined on an open subset in M for all sufficiently small ε, we arrive at the following generalized version formulas of Deprit's method and Hori's method.

Proposition 1.2.5 *For any open domain $N \subset M$ with compact closure there exists $\delta > 0$ such that for all $\varepsilon \in (-\delta, \delta)$, the mapping Φ_ε in (1.2.30) is well-defined on N and gives a diffeomorphism from N onto its image. Moreover, the coefficients of Taylor expansion of third order at $\varepsilon = 0$ of the pull-back $\Phi_\varepsilon^* \mathbf{A}_\varepsilon = A_0 + \varepsilon \widetilde{A}_1 + \frac{1}{2!}\varepsilon^2 \widetilde{A}_2 + \frac{1}{3!}\varepsilon^3 \widetilde{A}_3 + O(\varepsilon^4)$ are given by*

$\widetilde{A}_1$	$\mathcal{L}_{Z_{0,0}} A_0 + A_1$
$\widetilde{A}_2$	$\mathcal{L}^2_{Z_{0,0}} A_0 + 2\mathcal{L}_{Z_{0,0}} A_1 + 2\mathcal{L}_{Z_{0,1}} A_0 + \mathcal{L}_{Z_{1,0}} A_0 + A_2$
$\widetilde{A}_3$	$3\mathcal{L}_{Z_{0,0}} A_2 + 3\mathcal{L}^2_{Z_{0,0}} A_1 + 3\mathcal{L}_{Z_{1,0}} A_1 + \mathcal{L}^3_{Z_{0,0}} A_0 +$ $3\mathcal{L}_{Z_{0,0}}\mathcal{L}_{Z_{0,1}} A_0 + 3\mathcal{L}_{Z_{0,1}}\mathcal{L}_{Z_{0,0}} A_0 + 6\mathcal{L}_{Z_{0,1}} A_1 + 3\mathcal{L}_{Z_{0,2}} A_0 +$ $\mathcal{L}_{Z_{2,0}} A_0 + 2\mathcal{L}_{Z_{0,0}}\mathcal{L}_{Z_{1,0}} A_0 + \mathcal{L}_{Z_{1,0}}\mathcal{L}_{Z_{0,0}} A_0 + 3\mathcal{L}_{Z_{1,1}} A_0 + A_3$

Proof. By the flow box theorem for and compactness argument, there exists a $\delta > 0$ such that the flow $\mathrm{Fl}^{\lambda}_{\mathbf{Z}_\lambda(\varepsilon)}$ is well defined on $\bar{N}$ for all $\lambda \in (-\delta, \delta)$ and $\varepsilon \in [-1, 1]$. Now, we fix ε and consider the following decomposition $\mathbf{Z}_\lambda(\varepsilon) = \sum_{k=0}^{K} \frac{\lambda^k}{k!} Z_k(\varepsilon) + \mathcal{O}(\lambda^{K+1})$. Using formulas of Deprit, we obtain

$$\begin{aligned}\left(\mathrm{Fl}^{\lambda}_{\mathbf{Z}_\lambda(\varepsilon)}\right)^* \mathbf{A}(\varepsilon) &= \mathbf{A}(\varepsilon) + \lambda \mathcal{L}_{Z_0(\varepsilon)} \mathbf{A}(\varepsilon) + \frac{\lambda^2}{2!}(\mathcal{L}_{Z_1(\varepsilon)} + \mathcal{L}^2_{Z_0(\varepsilon)}) \mathbf{A}(\varepsilon) \\ &\quad + \frac{\lambda^3}{6!}(\mathcal{L}_{Z_2(\varepsilon)}) + \mathcal{L}^3_{Z_0(\varepsilon)} + 2\mathcal{L}_{Z_0(\varepsilon)}\mathcal{L}_{Z_1(\varepsilon)} + \mathcal{L}_{Z_1(\varepsilon)}\mathcal{L}_{Z_0(\varepsilon)}) + O(\lambda^4).\end{aligned}$$

Putting $\lambda = \varepsilon$ and using formulas (1.2.23), (1.2.29), we derive the desired formulas. ∎

Proposition (1.2.5) gives a general approach of Deprit's method and Hori's method, respectively. Indeed,

- in the Deprit case, vector field $\mathbf{Z}_\lambda(\varepsilon) = \mathbf{Z}_\lambda$ is independent of ε. Thus, $Z_{k,m} = 0$ if $m \geq 1$ and formulas of Proposition (1.2.5) coincides with formulas of Deprit's method;

- in the Hori case, $\mathbf{Z}_\lambda(\varepsilon) = \mathbf{Z}(\varepsilon)$ is independent of λ. It follows that $Z_{k,m} = 0$ if $k \geq 1$ and formulas of Proposition (1.2.5) reduce to formulas of Hori's method.

1.2.4 Homological equations

Now, we return to the normalization problem for a given vector field $\mathbf{A}_\varepsilon$. If we suppose that $\mathbf{A}_\varepsilon$ admits a normalization of order K and that normalization transformation $\Phi_\varepsilon : N \to M$ given as the flow of a vector field $\mathbf{Z}_\varepsilon = \sum_{i=0}^{K-1} \frac{\varepsilon^i}{i!} Z_i + O(\varepsilon^K)$, we obtain that vector fields A_k and Z_k satisfy certain equations called homological equations.

Proposition 1.2.6 *Assume that an ε-dependent vector field $\mathbf{A}_\varepsilon$ admits a near identity transformation $\Phi_\varepsilon : N \to M$ $(\varepsilon \in (-\delta, \delta))$ associated to an infinitesimal generator $\mathbf{Z}_\varepsilon$ which brings $\mathbf{A}_\varepsilon$ to normal form (*1.1.11*). (*1.1.12*) of order K for a certain vector fields $\tilde{A}_1, ..., \tilde{A}_K$. Then, the coefficients $Z_0, ..., Z_{K-1}$ of the Taylor expansion of $\mathbf{Z}_\varepsilon$ must satisfy the following equations on N:*

$$\mathcal{L}_{A_0} Z_{k-1} = A_k - \tilde{A}_k + R^D_{k-1}\{Z_0, ..., Z_{k-2}; A_0, ..., A_{k-1}\}, \tag{1.2.31}$$

$$\mathcal{L}_{A_0} \tilde{A}_k = 0, \tag{1.2.32}$$

for $k = 1, ..., K$. Here, the vector fields R^D_{k-1} are described in Proposition 1.2.3.

Proof. We assume that Φ_ε is well defined on N and is the normalization transformation of $\mathbf{A}_\varepsilon$ with generating vector field

$$\mathbf{Z}_\varepsilon = Z_0 + \varepsilon Z_1 + \ldots + \frac{\varepsilon^K - 1}{(K-1)!} Z_{K-1} + O(\varepsilon^K).$$

So, vector field

$$\Phi^*_\varepsilon \mathbf{A}_\varepsilon = A_0 + \varepsilon \widetilde{A_1} + \ldots + \frac{\varepsilon^K}{K!} A_K + O(\varepsilon^K),$$

is in normal form of order K relative to vector field A_0. That is,

$$\mathcal{L}_{A_0} \widetilde{A}_s = 0, \qquad s = 1, 2, \ldots, N.$$

Furthermore, Proposition 1.2.3 asserts that coefficients Z_k of Taylor expansion of $\mathbf{Z}_\varepsilon$ satisfy the following equations on N

$$\tilde{A}_k = A_k + \mathcal{L}_{Z_{k-1}} A_0 + R^D_{k-1}, \tag{1.2.33}$$

for $k = 1, 2, \ldots, K$, where vector fields $R^D_{k-1} = R^D_{k-1}\{Z_0, ..., Z_{k-2}; A_0, ..., A_{k-1}\}$ are determined in terms of vector fields $Z_0, ..., Z_{k-2}$ and $A_0, ..., A_{k-1}$ by mean of recurrent formulas (1.2.11). Taking into account that $\mathcal{L}_{Z_{K-1}} A_0 = -\mathcal{L}_{A_0} Z_{K-1}$, formulas (1.2.33) are equivalent to (1.2.31). ∎

The converse statement of Proposition above is true.

Proposition 1.2.7 *Assume that there exist an open domain $N \subseteq M$ and $\delta > 0$ such that the following conditions hold*

*(a) there are vector fields $Z_0, ..., Z_{K-1}$ and $\bar{A}_1, ..., \bar{A}_K$ satisfying on N equations (*1.2.31*),(*1.2.32*) on N for $k = 1, ..., K$.*

(b) the flow $\mathrm{Fl}^{\varepsilon}_{\mathbf{Z}_{\varepsilon}^{(K)}}$ *of the* ε*-dependent vector field*

$$\mathbf{Z}_{\varepsilon}^{(K)} = \sum_{n=0}^{K} \frac{\varepsilon^k}{k!} Z_k \tag{1.2.34}$$

is well defined on N *for all* $\varepsilon \in (-\delta, \delta)$.

Then, the near identity transformation

$$\Phi_\varepsilon = \mathrm{Fl}^{\varepsilon}_{\mathbf{Z}_{\varepsilon}^{(K)}} \tag{1.2.35}$$

brings the ε*-dependent vector field* $\mathbf{A}_\varepsilon$ *to normal form of order* $O(\varepsilon^K)$ *on* N *relative to* A_0*. In particular, if* M *is compact, then condition (b) holds on* $N = M$.

Proof. Let $\widetilde{\mathbf{A}}_\varepsilon := \Phi_\varepsilon^* \mathbf{A}_\varepsilon = A_0 + \varepsilon \widetilde{A}_1 + \ldots + \frac{\varepsilon^K}{K!} \widetilde{A}_K$. By Proposition 1.2.3, vector fields $\tilde{A}_k$ are given by

$$\widetilde{A}_k = A_k + \mathcal{L}_{Z_{k-1}} A_0 + R^D_{k-1}.$$

Hence, we have that $\widetilde{A}_k = \bar{A}_k$ for all $k = 1, 2, \ldots, K$. Since, vector fields $\bar{A}_k$ satisfy equations (1.2.32), the ε-dependent vector field $\widetilde{\mathbf{A}}_\varepsilon$ is in normal form and Φ_ε is a normalization transformation. ■

If M is compact then the unperturbed vector field A_0 is complete. Proposition 1.1.2 implies that the near identity transformations Φ_ε is well defined in any open domain $N \subset M$ with compact closure. Since M is compact, we have $\bar{M} = M$. So, Φ_ε in well defined in $N = M$.

Proposition 1.2.7 states that the normalization problem for a given ε-dependent vector $\mathbf{A}_\varepsilon$ depends on the solvability of equations (1.2.31), (1.2.32). It means that if we want to find an infinitesimal generator (1.2.34) of normalization transformation (1.2.35) then we have to solve , on several steps, the equations for vector fields Z and $\bar{W}$ on N of the form

$$\mathcal{L}_{A_0} Z = W - \bar{W}, \tag{1.2.36}$$

$$\mathcal{L}_{A_0} \bar{W} = 0, \tag{1.2.37}$$

where W is a given vector field.

Indeed, on the first step we have to find vector fields Z_0 and $\tilde{A}_1$ satisfying the equations

$$\mathcal{L}_{A_0} Z_0 = A_1 - \tilde{A}_1, \tag{1.2.38}$$

$$\mathcal{L}_{A_0} \tilde{A}_1 = 0.$$

On the second step, we need to find the vector fields Z_1 and $\tilde{A}_1$ satisfying the equations

$$\mathcal{L}_{A_0} Z_1 = A_2 - \tilde{A}_2 + \mathcal{L}^2_{Z_0} A_0 + 2\mathcal{L}_{Z_0} A_1,$$

$$\mathcal{L}_{A_0} \tilde{A}_2 = 0,$$

where the vector fields Z_0 and $\tilde{A}_1$ are given from the previous step. If after $(k-1)$ steps, we have the vector fields $Z_0, ..., Z_{k-2}$ and $\tilde{A}_1, ..., \tilde{A}_{k-1}$, then on the k-th step we have to find the solutions $Z = Z_{k-1}$ and $\bar{W} = \tilde{A}_k$ of equations (1.2.36),(1.2.37), where

$$W = A_k + R_{k-1}\{Z_0, ..., Z_{k-2}; A_0, ..., A_{k-1}\}.$$

In the context of averaging method, equation (1.2.36) is called a *homological equation*, [5]. The solvability conditions of equations (1.2.36),(1.2.37) clearly depends on the properties of the unperturbed vector field A_0.

By Proposition 1.2.7 and Proposition 1.1.2, we have the following facts.

Corollary 1.2.8 *Suppose that homological equations (*1.2.31*),(*1.2.32*) are solvable on an open domain $N_0 \subseteq M$ for $k = 1, ..., N$ and let $Z_0, ..., Z_{N-1}$ and $\tilde{A}_1, ..., \tilde{A}_N$ be solutions. Then, for every open domain $N \subseteq N_0$ with compact closure, formula (*1.2.35*) defines a near identity transformation which is well defined on N and takes the ε-dependent vector field $\mathbf{A}_\varepsilon$ into normal form of order $O(\varepsilon^N)$ on N.*

Corollary 1.2.9 (Normalization of first order) *Let $\mathbf{A}_\varepsilon(x)$ be an ε-dependent vector field with Taylor expansion at $\varepsilon = 0$ is $\mathbf{A}_\varepsilon = A_0 + \varepsilon A_1 + O(\varepsilon^2)$. If there exists a vector field Z_0 such that*

(a) the flow $\mathrm{Fl}^t_{Z_0}$ of Z_0 is well defined on an open domain $N \subset M$,

(b) satisfies on N the equation

$$\mathcal{L}_{A_0}\left(\mathcal{L}_{A_0} Z_0 - A_1\right) = 0. \tag{1.2.39}$$

Then, the near identity transformation $\Phi_\varepsilon = \mathrm{Fl}^t_{Z_0}$ sends the vector field $\mathbf{A}_\varepsilon$ to normal form of first order on N relative to A_0, that is, $\Phi^*_\varepsilon \mathbf{A}_\varepsilon = A_0 + \tilde{A}_1 + O(\varepsilon^2)$, where $\tilde{A}_1 = A_1 - \mathcal{L}_{A_0} Z_0$.

1.2.5 The Hamiltonian case

Here, we shall express the homological equation and recursive formulas (1.2.15) in terms of the Poisson bracket.

Recall that a *Poisson bracket* on a smooth manifold M is a $\mathbb{R}$-bilinear antisymmetric operation $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ compatible with the pointwise product of smooth functions by the Leibnitz rule and satisfying the Jacobi identity,

$$\{F, GH\} = \{F, G\}H + \{F, H\}G, \tag{1.2.40}$$

$$\underset{(F,G,H)}{\mathfrak{S}} \{F, \{G, H\}\} = 0,$$

where $\mathfrak{S}$ denotes the cyclic sum.

The pair $(M, \{,\})$ is called a *Poisson manifold* and $(C^\infty(\mathcal{M}), \{,\})$ is a Lie algebra. For every $H \in C^\infty(M)$, we define the adjoint operator $\mathrm{ad}_H : C^\infty(M) \to C^\infty(M)$ given by $\mathrm{ad}_H(\cdot) = \{H, \cdot\}$. A smooth vector field X on a Poisson manifold $(M, \{,\})$ is said to be *Hamiltonian* relative to the Poisson bracket $\{,\}$ if there exists a function $H \in C^\infty(M)$ such that the Lie derivative along X coincides with the adjoint operator of H,

$$\mathcal{L}_X = \mathrm{ad}_H\,. \tag{1.2.41}$$

By the Leibnitz identity (1.2.40), every function $H \in C^\infty(M)$ admits a unique Hamiltonian vector field in (1.2.41) which is denoted by $X = X_H$. In local coordinates, Hamiltonian dynamical system generated by X_H is written in the bracket form by

$$\dot{x}^i = \{H, x^i\}, \quad i = 1, 2, \ldots, 2n.$$

The set $\mathrm{Ham}(M)$ of all Hamiltonian vector fields is a Lie subalgebra in $\mathfrak{X}(M)$ and the correspondence $H \mapsto X_H$ is a Lie algebra homomorphism,

$$[X_{H_1}, X_{H_2}] = X_{\{H_1, H_2\}}. \tag{1.2.42}$$

whose kernel is just $\mathrm{Casim}(M)$. A vector field P on the Poisson manifold M is said to be an *infinitesimal Poisson automorphism* (or, a *Poisson vector field*) if its Lie derivative is a derivation of the Poisson algebra $(C^\infty(M), \{,\})$,

$$\mathcal{L}_P\{F_1, F_2\} = \{\mathcal{L}_P F_1, F_2\} + \{F_1, \mathcal{L}_P F_2\},$$

for any $F_1, F_2 \in C^\infty(M)$. It is clear that every Hamiltonian vector field is Poisson. The space of all Poisson vector fields form a Lie algebra, denoted by $\mathrm{Poiss}(M)$. It follows from (1.2.42) that

$$[P, X_H] = X_{\mathcal{L}_P H} \tag{1.2.43}$$

for any $P \in$ Poiss (M) and $H \in C^\infty(M)$. This property says that $\mathrm{Ham}(M)$ is an ideal of $\mathrm{Poiss}(M)$.

The Poisson bracket is called *nondegenerate* if every Casimir function $K \in \mathrm{Casim}(M)$ is a constant function. In this case there exists a unique nondegenerate closed 2-form σ on M, which is compatible with Poisson bracket by the condition

$$\sigma(X_{F_1}, X_{F_2}) = \{F_1, F_2\}.$$

The pair (M, σ), where σ is a nondegenerate closed 2- form, is called a symplectic manifold. In terms of the symplectic structure σ, condition (1.2.41) tells that a vector field X is Hamiltonian if there exists $H \in C^\infty(M)$ such that

$$\mathbf{i}_X \sigma = -dH.$$

Suppose that the ε-dependent vector field $\mathbf{A}_\varepsilon$ is Hamiltonian relative to $\mathbf{H}_\varepsilon = H_0 + \varepsilon H_1 + ...$, that is,

$$\mathbf{A}_\varepsilon = X_{\mathbf{H}_\varepsilon} = X_{H_0} + \varepsilon X_{H_1} + ...,$$

If $Z_0 = X_{G_0}, Z_1 = X_{G_1}, ..., Z_{k-2} = X_{G_{k-2}}$ are Hamiltonian vector fields of functions $G_0, G_1, ..., G_{k-1} \in C^\infty(M)$, Proposition 1.2.6 implies that the vector field

$$R^D_{k-1}\{X_{G_0}, ..., X_{G_{k-2}}; X_{H_0}, ..., X_{H_{k-1}}\} = X_{\mathcal{R}_{k-1}},$$

described in Proposition 1.2.3 are also Hamiltonian relative to the functions $\mathcal{R}_{k-1} = \mathcal{R}^D_{k-1}\{G_0, ..., G_{k-2}; H_0, ..., H_{k-1}\}$. In particular,

$$\mathcal{R}_0 = 0, \quad \mathcal{R}_1 = 2\{G_0, H_1\{G_0, H_0\}\},$$

and

$$\mathcal{R}_2 = \{G_0, 3H_2 + \{G_0, 3H_1 + \{G_0, H_0\}\}\} + \{G_1, 2H_0\} + \{G_1, 3H_1 + \{G_0, H_0\}\}.$$

An advantage of the Hamiltonian case is that the problem (1.2.36),(1.2.37) can be reduced to the study of homological equations for functions. Assume that vector fields $A_0 = X_{H_0}$, $W = X_F$ are Hamiltonian on a Poisson manifold $(M, \{,\})$. If there exist smooth functions G and $\bar{F}$ satisfying the equations

$$\{H_0, G\} = F - \bar{F}, \quad (1.2.44)$$
$$\{H_0, \bar{F}\} = 0, \quad (1.2.45)$$

then the Hamiltonian vector fields $Z = X_G$ and $\bar{W} = X_{\bar{F}}$ are solution to the problem (1.2.36),(1.2.37).

Consider the following generalization of the Hamiltonian case. Suppose we have a perturbed vector field of the form

$$\mathbf{A}_\varepsilon = P + \varepsilon X_{H_1} + \frac{\varepsilon^2}{2} X_{H_2} ...,$$

where P is a Poisson vector field on M, which plays the role of the unperturbed vector field. But the perturbation vector field remains Hamiltonian corresponding to an ε-dependent function $\varepsilon H_1 + \frac{\varepsilon^2}{2} H_2 + ..$. Then, $A_0 = P$, $W = X_{H_1}$. Putting again $Z = X_G$, $\bar{W} = X_{\bar{F}}$ into (1.2.36),(1.2.37) and using (1.2.43), we get the following equations for functions $G, \bar{F}$:

$$\mathcal{L}_P G = H_1 - \bar{F},$$

$$\mathcal{L}_P \bar{F} = 0.$$

1.3 Normalization Transformations Around Invariant Submanifolds

According to Proposition 1.2.7, the normalization of an ε-dependent vector field can be proceeded in two steps: (a) solving homological type problems (1.2.36), (1.2.37) and (b) studying the domain of definition of the flow of time-dependent vector field (1.2.34).

Here, we consider a class of perturbed systems on a manifold M (not necessarily compact) for which condition (b) of Corollary 1.2.7 holds.

Suppose we are given an ε-dependent vector field $\mathbf{A}_\varepsilon$ on M which has an *invariant submanifold* $S \subset M$ ($\dim S < \dim M$) and the restriction of $\mathbf{A}_\varepsilon$ to S does not depend on ε,

$$\mathbf{A}_\varepsilon(x) \in \mathrm{T}_x S \quad \forall x \in S, \varepsilon \in \mathbb{R}, \quad (1.3.1)$$

$$v \overset{def}{=} \mathbf{A}_\varepsilon|_S \ \text{ is independent of } \ \varepsilon. \quad (1.3.2)$$

In terms of the coefficients A_k of Taylor expansion (1.2.4) these conditions can be reformulated as follows: the submanifold S is invariant with respect to the flow of the unperturbed vector field A_0 and the perturbation vector fields $A_1, A_2, ...$ vanish at S, that is,

$$A_0(x) \in \mathrm{T}_x S \quad \forall x \in S, \tag{1.3.3}$$

$$A_k|_S = 0 \ (k = 1, 2, ...). \tag{1.3.4}$$

Definition 1.3.1 *We say that a near identity transformation $\Phi_\varepsilon : N \to M$ ($\varepsilon \in (-\delta, \delta)$) is compatible with submanifold $S \subset M$ (or, shortly S-compatible) if Φ_ε is a diffeomorphism from N onto another open neighborhood of S in M*

$$S \subset \Phi_\varepsilon(N) \tag{1.3.5}$$

and

$$\Phi_\varepsilon \mid_S = \mathrm{id} \tag{1.3.6}$$

for all $\varepsilon \in (-\delta, \delta)$.

An important class of S-compatible near identity transformations can be obtained in the following way.

Lemma 1.3.1 *Let $\mathbf{Z}_\varepsilon$ be an ε-dependent vector field on M vanishing at the submanifold $S \subset M$, $\mathbf{Z}_\varepsilon|_S = 0$, $\forall \varepsilon \in \mathbb{R}$. Then, for every $\delta > 0$ there exists an open neighborhood $N = N_\delta$ of S in M such that the flow $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}$ of $\mathbf{Z}_\varepsilon$ is well defined on N for all $\varepsilon \in (-\delta, \delta)$. Moreover, $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon} : N \to M$ is a near identity transformation compatible with S.*

Proof. We fix $\delta > 0$. By the flow box theorem, for every $\xi \in S$ there exists an open neighborhood U_ξ of ξ on M such that the flow $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}$ is well defined on U_ξ for all $\varepsilon \in (-\delta, \delta)$. Let $N_\delta = \bigcup_{\xi \in S} U_\xi$ be an open neighborhood of S. Thus, $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon} : N_\delta \to M$ is well defined for all $\varepsilon \in (-\delta, \delta)$ and a diffeomorphism onto its image . Since $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}$ vanishes at S, we have that $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}(\xi) = \xi$ for all ε. It follows that $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}\big|_S = \mathrm{id}$ and $S \subset \mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}(N_\delta)$, for $\varepsilon \in (-\delta, \delta)$. Therefore, $\mathrm{Fl}^\varepsilon_{\mathbf{Z}_\varepsilon}$ is a S-compatible near identity transformation. ■

Definition 1.3.2 *We say that the perturbed vector field $\mathbf{A}_\varepsilon$ satisfying (1.3.1), (1.3.2) admits a normalization of order K around the invariant submanifold S if there exists a S-compatible near identity transformation Φ_ε such that the pull-back $(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon$ is in normal form (1.1.11), (1.1.12).*

We denote by $C^\infty_S(M)$ and $\mathfrak{X}_S(M)$ the Lie subalgebra of smooth functions vector fields on M vanishing at S. It is clear that the Lie derivative $\mathcal{L}_{A_0}$ along A_0 leaves invariant these subalgebras. Moreover, $\mathfrak{X}_S(M)$ is a $C^\infty_S(M)$-module.

If $Z_0, ..., Z_{k-2}$ are vector fields vanishing at S. Then, the vector field

$$R^D_{k-1}\{Z_0, ..., Z_{k-2}; A_0, ..., A_{k-1}\},$$

where the vector fields R^D_{k-1} are described in Proposition 1.2.3, also vanishing at S.

Therefore, if there exists S-compatible normalization then the vector field $(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon$ automatically vanish at S, that is,

$$\tilde{A}_1|_S = ... = \tilde{A}_K|_S = 0.$$

Proposition 1.3.2 *Let $\mathbf{A}_\varepsilon = A_0 + \ldots + \frac{\varepsilon^K}{K!}$ be an $\varepsilon-$dependent vector field vanishing at S. Assume that there exist vector fields $Z_0, ..., Z_{N-1} \in \mathfrak{X}_S(M)$ and $\tilde{A}_1, ..., \tilde{A}_K \in \mathfrak{X}_S(M)$ satisfying the recurrent equations*

$$\mathcal{L}_{A_0} Z_{k-1} = A_k - \tilde{A}_k + R_{k-1}\{Z_0, ..., Z_{k-2}; A_0, ..., A_{k-1}\} \tag{1.3.7}$$

$$\mathcal{L}_{A_0} \tilde{A}_k = 0 \tag{1.3.8}$$

for $k = 1, ..., N$. Then, $\mathbf{A}_\varepsilon$ admits normalization of order K, where the S-compatible near identity transformation Φ_ε is defined as the flow of the time-dependent vector field $\mathbf{Z}_\varepsilon = Z_0 + \varepsilon Z_1 \cdots + \frac{\varepsilon^{K-1}}{K!} Z_{K-1} \in \mathfrak{X}_S(M)$,

$$\Phi_\varepsilon = \mathrm{Fl}^{\varepsilon}_{\mathbf{Z}_\varepsilon^{(K)}} \, . \tag{1.3.9}$$

Chapter 2

Homological Equations for Tensor Fields associated to Periodic Flows

The so-called homological equations usually appear in the context of normal forms and the method of averaging for perturbed dynamical systems (see, for example, [67]). According to the Lie transform method [21, 33], the infinitesimal generators of normalization transformations for perturbed dynamics systems are defined as the solutions to homological equations for vector fields. In the Hamiltonian case, the normalization problem for vector fields with periodic flow is reduced to the solvability of homological equations for functions, [15].

Our goal is to study homological equations of tensor type associated to periodic flows on a manifold. We generalize the Cushman intrinsic formula [15] to the case of multivector fields and differential forms. Applications of this formula to normal forms and the averaging method for perturbed Hamiltonian systems on slow-fast phase spaces can be found in Chapter 4.

2.1 Lie Group Actions. Basic Notions

Here, we recall some necessary definitions and facts about the actions of Lie groups on manifolds (for more details, see, for example, [2, 49, 52]).

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A left action of G on a manifold M is a smooth mapping $\Psi : G \times M \to M$ such that $\Psi(g, \Psi(h, m)) = \Psi(gh, m)$ for all $g, h \in G$ and $m \in M$. This implies that the mapping $g \mapsto \Psi_g$ (where $\Psi_g(m) = \Psi(g, m)$ is a homomorphism between the groups G and $\mathfrak{Diff}(M)$. In this case, the triple (M, G, Ψ) is called a G-space. An infinitesimal generator $\Upsilon_a \in \mathfrak{X}(M)$ of the G-action associated to $a \in \mathfrak{g}$ is a complete vector field given by $\Upsilon_a(m) = \frac{d}{dt}\big|_{t=0} \Psi_{\exp(ta)}(m)$. The map $a \mapsto \Upsilon_a$ is linear but not necessarily a Lie algebra homomorphism. A tensor field Ξ on M is said to be G-invariant if $\Psi_g^* \Xi = \Xi \ \ \forall g \in G$. In infinitesimal terms, this condition reads $\mathcal{L}_{\Upsilon_a} \Xi = 0 \ \forall a \in \mathfrak{g}$. Let $\mathcal{D} \subset TM$ be a distribution and $\mathfrak{X}_{\mathcal{D}}(M)$ the subspace of vector field which is tangent to $\mathcal{D}$. Then, $\mathcal{D}$ is G-inviariant if $(d_m \Psi_g)\mathcal{D}_m = \mathcal{D}_{\Psi_g(m)}$ or equivalently $[\Upsilon_a, \mathfrak{X}_{\mathcal{D}}(M)] \subset \mathfrak{X}_{\mathcal{D}}(M)$.

Consider the orbit $G \cdot m = \{\Psi_g(m) | g \in G\}$ through $m \in M$. Then, the quotient M/G is called the orbit space. The isotropy of $m \in M$ is the closed subgroup $G_x = \{g \in G | \Psi_g(m) = m\}$ of G. The action Ψ is said to be (i) *transitive* if there is only one orbit; (ii) effective (or faithful) if $\Phi_g = \mathrm{id}_M$ implies $g = e$; and (iii) *free* if there are no fixed points, that is, $\Phi_g(m) = m$ implies $g = e$, or equivalently, if for each $m \in M$, the mapping $g \mapsto \Phi_g(x)$ is one to one. For example, if $G = \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$

is the circle , then the $\mathbb{S}^1$-action is free if and only if the flow of the infinitesimal generator Υ is minimally 2π-periodic.

Now suppose that G is connected compact Lie group. Then, for every tensor field Ξ on M one can define its G-average as [47]

$$\langle \Xi \rangle := \int_G \Psi_g^* \Xi dg$$

which is again well-defined tensor field on M of the same type as Ξ. Here, dg denotes the normalized Haar measure on G, $\int_G dg = 1$. A tensor field Ξ is G-invariant if and only if $\langle \Xi \rangle = \Xi$. If the action Ψ is free, then the orbit space M/G inherits a C^∞ manifold structure such that the canonical projection $\rho : M \to M/G$ is a smooth surjective submersion. In this case, the G-invariance of a tensor field Ξ means that Ξ is a pull back by ρ of a certain tensor field on the orbit space M/G.

Let (M, σ) be a symplectic manifold. A G-action Ψ is called a *symplectic or canonical* on (M, σ) if it acts by symplectomorphism, $\mathcal{L}_{\Upsilon_a} \sigma = 0 \ \forall a \in \mathfrak{g}$. A symplectic action Ψ admits a *momentum map* if there exists a smooth vector valued function $\mathbb{J} : M \to \mathfrak{g}^*$ such that $\mathbf{i}_{\Upsilon_a} \sigma = -dJ_a, \forall a \in \mathfrak{g}$, where $J_a(m) =< \mathbb{J}(m), a >$. We say that a Hamiltonian system (M, σ, H) is G-symmetric if the Lie group G acts canonically on (M, σ) and the Hamiltonian H is G-invariant. A symplectic action Ψ with momentum map $\mathbb{J}$ is said to be Hamiltonian if the mapping $a \mapsto J_a$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Poisson algebra $C^\infty(M)$ associated to the symplectic form σ. Let (M, σ, H) be a Hamiltonian system which is G-symmetric relative to a Hamiltonian action of G with momentum map $\mathbb{J}$. Then, we say that we have a Hamiltonian system with G-symmetry. In this case, we can apply to the Hamiltonian system the reduction procedure due to [2] for example in the situation when G is compact and the action is free.

2.2 Generalized Homological Equations

Let M be a smooth manifold and X a vector field on M. Recall that we denote by $\mathcal{T}_s^k(M)$ to the space of tensor fields of type (k, m) on M. In particular, $\mathcal{T}_0^0(M) = C^\infty(M)$ and $\mathcal{T}_0^1(M) = \mathfrak{X}(M)$. Moreover, we denote by $\mathcal{L}_X : \mathcal{T}_s^k(M) \to \mathcal{T}_s^k(M)$ the Lie derivative along X, that is, the unique differential operator on the tensor algebra of the manifold M which coincides with the standard Lie derivative $\mathcal{L}_X$ on $C^\infty(M)$ and $\mathfrak{X}(M)$ (see, for example [2]). We assume that X is a complete vector field on a manifold M with periodic flow. This means that there exists a smooth positive function $T : M \to \mathbb{R}$, called period function, such that

$$\mathrm{Fl}_X^{t+T(x)}(x) = \mathrm{Fl}_X^t(x),$$

for all $t \in \mathbb{R}$ and $x \in M$.

We are interested in the following problem: given a tensor field $\Xi \in \mathcal{T}_s^k(M)$, determine under which conditions there exist tensor fields $\eta, \bar{\Xi} \in \mathcal{T}_s^k$ satisfying the homological equation

$$\mathcal{L}_X \eta = \Xi - \bar{\Xi}, \tag{2.2.1}$$

and the condition

$$\mathcal{L}_X \bar{\Xi} = 0. \tag{2.2.2}$$

This problem is arisen in the context of normalization problem for vector fields (tensor field of type $(0,1)$) treated in Chapter 1. We have generalized equations (2.2.1), (2.2.2) to tensor fields of arbitrary type because the Lie derivative of X is a differential operator which can be extended to the full tensor algebra $\mathcal{T}(M)$ and left invariant each tensor space $\mathcal{T}_s^k(M)$. In order to study this type of problems, we review the algebraic properties of the $\mathbb{S}^1$-averaging.

2.3 Algebraic Properties of the $\mathbb{S}^1$-Averaging

Given $\Xi \in \mathcal{T}_s^k(M)$, we get a curve through $\Xi(m)$ in the fiber on m by using the flow of X. The derivative of this curve is the Lie derivative,

$$\frac{d}{dt}\left((\mathrm{Fl}_X^t)^*\Xi\right) = (\mathrm{Fl}_X^t)^*(\mathcal{L}_Y \Xi). \tag{2.3.1}$$

Now, suppose that we are given an action of the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ on M with infinitesimal generator Υ. Therefore, Υ is a complete vector field on M whose flow Fl_Υ^t is 2π-periodic. We admit that the $\mathbb{S}^1$-action is not necessarily free.

Definition 2.3.1 *For every tensor field $\Xi \in \mathcal{T}_s^k(M)$, its average with respect to the $\mathbb{S}^1$-action is a tensor field $\langle\Xi\rangle \in \mathcal{T}_s^k(M)$ of the same type which is defined as*

$$\langle\Xi\rangle := \frac{1}{2\pi}\int_0^{2\pi} (\mathrm{Fl}_\Upsilon^t)^*\Xi dt. \tag{2.3.2}$$

A tensor field $\Xi \in \mathcal{T}_s^k(M)$ is said to be invariant with respect to the $\mathbb{S}^1$-action (or $\mathbb{S}^1$-invariant) if $(\mathrm{Fl}_\Upsilon^t)^*\Xi = \Xi$ $(\forall t \in \mathbb{R})$ or, equivalently, $\mathcal{L}_\Upsilon \Xi = 0$.

Proposition 2.3.1 *For every $\Xi \in \mathcal{T}_s^k(M)$, the following properties holds:*

(i) Ξ *is invariant under the flow of* Υ *if and only if* $\langle\Xi\rangle = \Xi$,

(ii) $\mathcal{L}_\Upsilon\langle\Xi\rangle = 0$.

(iii) $\langle\langle\Xi\rangle\rangle = \langle\Xi\rangle$,

Proof. We assume that $\left(\mathrm{Fl}_\Upsilon^t\right)^*\Xi = \Xi$, then it is clear from the definition of averaging that $\langle\Xi\rangle = \Xi$. Conversely, if $\langle\Xi\rangle = \Xi$, it follows from basic properties of flows and vector fields (see [1]) that $\mathcal{L}_\Upsilon\Xi(x) = \Xi(\mathrm{Fl}_\Upsilon^{2\pi}(x)) - \Xi(x)$, for all x on M. Since the flow of Υ is 2π-periodic, we have that $\mathcal{L}_\Upsilon\Xi = 0$. Hence, item (i) holds. By the basic relation of pull-back and flows (2.3.1), it is possible deduce that $\mathcal{L}_\Upsilon\langle\Xi\rangle = 0$. Item (ii) implies that $\langle\Xi\rangle$ is invariant under the flow of Υ. By item (i), we get $\langle\langle\Xi\rangle\rangle = \langle\Xi\rangle$.
■

From Proposition 2.3.1, $\mathbb{S}^1$-invariance condition reads $\Xi = \langle\Xi\rangle$.

Corollary 2.3.2 *The kernel of the linear operator $\mathcal{L}_\Upsilon : \mathcal{T}_s^k(M) \to \mathcal{T}_s^k(M)$ is given by*

$$\ker \mathcal{L}_\Upsilon = \{\Xi \in \mathcal{T}_s^k(M) | \langle\Xi\rangle = \Xi\}.$$

We shall denote by $\mathcal{A} : \mathcal{T}^k_s(M) \to \mathcal{T}^k_s(M)$ the averaging operator, $\mathcal{A}(\Xi) = \langle \Xi \rangle$ which is an $\mathbb{R}$-linear operator. By Proposition 2.3.1, the operator $\mathcal{A}$ has the following properties:

- $\mathcal{A}^2 = \mathcal{A}$ (projection map),
- the image of $\mathcal{A}$ consists of all $\mathbb{S}^1$-invariant tensor fields,
- a tensor field belongs to $\operatorname{Ker}\mathcal{A}$ if its $\mathbb{S}^1$-average is zero.

Therefore, we have the $\mathbb{S}^1$-invariant splitting

$$\mathcal{T}^k_s(M) = \operatorname{Im}\mathcal{A} \oplus \operatorname{Ker}\mathcal{A}. \tag{2.3.3}$$

We introduce also the $\mathbb{R}$-linear operator $\mathcal{S} : \mathcal{T}^k_s(M) \to \mathcal{T}^k_s(M)$ given by

$$\mathcal{S}(\Xi) := \frac{1}{2\pi}\int_0^{2\pi} (t-\pi)(\mathrm{Fl}^t_\Upsilon)^*\Xi dt. \tag{2.3.4}$$

It is easy to see that if $\eta \in \mathcal{T}^p_q(M)$ is $\mathbb{S}^1$-invariant, $(\mathrm{Fl}^t_\Upsilon)^*\eta = \eta$, then

$$\mathcal{A}(\eta \otimes \Xi) = \eta \otimes \mathcal{A}(\Xi) \tag{2.3.5}$$

$$\mathcal{S}(\eta \otimes \Xi) = \eta \otimes \mathcal{S}(\Xi), \quad \forall \Xi \in \mathcal{T}^k_s. \tag{2.3.6}$$

It follows directly from definitions that the operators $\mathcal{L}_\Upsilon, \mathcal{A}$ and $\mathcal{S}$ pairwise commute and satisfy the relations

$$\mathcal{A} \circ \mathcal{L}_\Upsilon = \mathcal{L}_\Upsilon \circ \mathcal{A} = 0, \tag{2.3.7}$$

$$\mathcal{A} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{A} = 0. \tag{2.3.8}$$

Moreover, we have the following important property.

Proposition 2.3.3 *The following identity holds*

$$\mathcal{L}_\Upsilon \circ \mathcal{S} = \mathrm{id} - \mathcal{A}, \tag{2.3.9}$$

Proof. For every tensor field $\Xi \in \mathcal{T}^k_s(M)$, by definition (2.3.4), we have

$$\begin{aligned}(\mathrm{Fl}^\tau_\Upsilon)^*\mathcal{S}(\Xi) &= \frac{1}{2\pi}\int_0^{2\pi}(t-\pi)(\mathrm{Fl}^{t+\tau}_\Upsilon)^*\Xi dt \\ &= \frac{1}{2\pi}\int_\tau^{2\pi+\tau}(t-\tau-\pi)(\mathrm{Fl}^t_\Upsilon)^*\Xi dt\end{aligned}$$

Differentiating the both sides of this equality in τ and using the 2π-periodicity of the flow $\mathrm{Fl}^\tau_\Upsilon$, we get

$$\begin{aligned}\frac{d}{d\tau}(\mathrm{Fl}^\tau_\Upsilon)^*\mathcal{S}(\Xi) &= \frac{1}{2\pi}[(t-\tau-\pi)(\mathrm{Fl}^t_\Upsilon)^*\Xi]\Big|_\tau^{2\pi+\tau} - \frac{1}{2\pi}\int_\tau^{2\pi+\tau}(\mathrm{Fl}^t_\Upsilon)^*\Xi dt \\ &= (\mathrm{Fl}^\tau_\Upsilon)^*(\Xi - \langle\Xi\rangle)\end{aligned}$$

Comparing this equality with the identity

$$\frac{d}{d\tau}(\mathrm{Fl}^\tau_\Upsilon)^*\mathcal{S}(\Xi) = (\mathrm{Fl}^\tau_\Upsilon)^*(\mathcal{L}_\Upsilon\mathcal{S}(\Xi)) \tag{2.3.10}$$

gives $\mathcal{L}_\Upsilon(\mathcal{S}(\Xi)) = \Xi - \langle\Xi\rangle$. ∎

Corollary 2.3.4 *For every tensor field $\Xi \in \mathcal{T}_s^k(M)$, the following assertions are equivalent*

- $\mathcal{S}(\Xi) = 0$;
- $\mathcal{S}(\Xi)$ *is* $\mathbb{S}^1$*-invariant;*
- Ξ *is* $\mathbb{S}^1$*-invariant.*

Proof. The equivalence of the first two conditions follows from property (2.3.8) which says that $\langle \mathcal{S}(\Xi) \rangle = 0$. Property (2.3.9) implies the equivalence of the last two assertions. ∎

Proposition 2.3.5 *The following relations hold*

$$\operatorname{Ker}\mathcal{S} = \operatorname{Ker}\mathcal{L}_\Upsilon = \operatorname{Im}\mathcal{A}, \tag{2.3.11}$$

$$\operatorname{Im}\mathcal{S} = \operatorname{Im}\mathcal{L}_\Upsilon = \operatorname{Ker}\mathcal{A}. \tag{2.3.12}$$

Proof. Taking into account that the kernel of the Lie derivative $\mathcal{L}_\Upsilon : \mathcal{T}_s^k(M) \to \mathcal{T}_s^k(M)$ consists of all $\mathbb{S}^1$-invariant tensor fields and by the Corollary 2.3.4, we derive (2.3.11). By (2.3.7), we have $\operatorname{Im}\mathcal{L}_\Upsilon \subseteq \operatorname{Ker}\mathcal{A}$. On the other hand, it follows from (2.3.9) that

$$\mathcal{L}_\Upsilon \mathcal{S}(\Xi) = \Xi \ \forall \Xi \in \operatorname{Ker}\mathcal{A} \tag{2.3.13}$$

and hence $\operatorname{Ker}\mathcal{A} \subseteq \operatorname{Im}\mathcal{L}_\Upsilon$. Therefore, $\operatorname{Im}\mathcal{L}_\Upsilon = \operatorname{Ker}\mathcal{A}$. By (2.3.7)-(2.3.9) we have the identities $\mathcal{S} = \mathcal{L}_\Upsilon \circ \mathcal{S}^2$ and $\mathcal{L}_\Upsilon = \mathcal{S} \circ \mathcal{L}_\Upsilon^2$ which say that $\operatorname{Im}\mathcal{S} = \operatorname{Im}\mathcal{L}_\Upsilon$. ∎

As a consequence of (2.3.3) and (2.3.11), (2.3.12), we get also the decomposition

$$\mathcal{T}_s^k(M) = \operatorname{Ker}\mathcal{L}_\Upsilon \oplus \operatorname{Im}\mathcal{L}_\Upsilon \tag{2.3.14}$$

which together with (2.3.13) implies that the restriction of $\mathcal{L}_\Upsilon$ to $\operatorname{Im}\mathcal{L}_\Upsilon$ is an isomorphism whose inverse is just $\mathcal{S}$.

Finally, we notice that operators $\mathcal{A}$, $\mathcal{S}$ and are natural with respects to operation of pull-back. This means that for any diffeomorphism $\Phi : M \to M$, we have $\Phi^* \circ \mathcal{A} = \mathcal{A} \circ \Phi^*$, $\Phi^* \circ \mathcal{S} = \mathcal{S} \circ \Phi^*$. This follows directly from definition and properties of pull-back.

Remark 2 *Actually, the definition of operators $\mathcal{A}$, $\mathcal{S}$ (2.3.2), (2.3.4) are not so important for the results present in the next section. These results can be obtained in a general setting where the operators the operators $\mathcal{A}$, $\mathcal{S}$ and $\mathcal{L}_\Upsilon$ satisfies the conditions (2.3.5), (2.3.6), (2.3.9), (2.3.11) and (2.3.12).*

2.4 The Global Solvability of Generalized Homological Equations

Let X be a vector field with periodic flow. We shall introduce the frequency function $\omega : M \to \mathbb{R}$ given by $\omega = \frac{2\pi}{T}$. It is clear that ω is a first integral of X.

Here, we present the solvability condition of equations (2.2.1), (2.2.2) and formulas for the solutions for k-vector fields and k-forms on M

2.4.1 Homological equations for k-vector fields.

Let $\chi^k(M) = \mathrm{Sec}(\bigwedge^k \mathrm{T}M)$ be the space of all k-multivector fields on M. In particular, $\chi^0(M) = C^\infty(M)$ and $\chi^1(M) = \mathfrak{X}(M)$. It is clear that the operators $\mathcal{L}_\Upsilon, \mathcal{S}$ and $\mathcal{A}$ leave invariant the subspaces $\chi^k(M) \subset \mathcal{T}_0^k(M)$. For every k-vector field $A \in \chi^k(M)$ and an arbitrary 1-form α on M, denote by $\mathbf{i}_\alpha A \in \mathcal{T}_0^{k-1}(M)$ a $(k-1)$-vector field defined by

$$(\mathbf{i}_\alpha A)(\alpha_1, ..., \alpha_{k-1}) = A(\alpha, \alpha_1, ..., \alpha_{k-1})$$

for all 1-forms $\alpha_1, ..., \alpha_{k-1}$ on M. By definition, $\mathbf{i}_\alpha A = 0$, for every 0-vector field A, (smooth function on M).

Consider the $\mathbb{S}^1$-action on M associated to the periodic flow Fl_X^t. Denote by

$$\chi^k_{\mathrm{inv}}(M) = \mathrm{Ker}\,\mathcal{L}_\Upsilon$$

the subspace of all $\mathbb{S}^1$-invariant k-vector fields on M. Then, according to (2.3.14), we have the splitting

$$\chi^k(M) = \chi^k_{\mathrm{inv}}(M) \oplus \chi^k_0(M), \tag{2.4.1}$$

where $\chi^k_0(M) = \mathrm{Im}\,\mathcal{L}_\Upsilon$ denotes the subspace of all k-vector fields on M with zero average.

Theorem 2.4.1 *Let X be a vector field on M with periodic flow and frequency function ω. Then, for a given $B \in \chi^k(M)$, all k-vector fields A and $\bar{B}$ on M satisfying the homological equation*

$$\mathcal{L}_X A = B - \bar{B} \tag{2.4.2}$$

and the condition

$$\bar{B} \text{ is } \mathbb{S}^1\text{-invariant} \tag{2.4.3}$$

are of the form

$$\bar{B} = \langle B \rangle + \frac{1}{\omega} X \wedge \mathbf{i}_{\mathrm{d}\omega} C, \tag{2.4.4}$$

$$A = \frac{1}{\omega}\mathcal{S}(B) + \frac{1}{\omega^3} X \wedge \mathcal{S}^2(\mathbf{i}_{\mathrm{d}\omega} B) + C, \tag{2.4.5}$$

where $C \in \chi^k_{\mathrm{inv}}(M)$ is an arbitrary $\mathbb{S}^1$-invariant k-vector field. Here, the average $\langle\cdot\rangle$ is taken with respect to the $\mathbb{S}^1$-action on M associated to the flow of X.

We shall use the following identity. Let $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, and $A \in \chi^k(M)$; then

$$\mathcal{L}_{fX} A = f\mathcal{L}_X A - X \wedge \mathrm{i}_{\mathrm{d}f} A, \tag{2.4.6}$$

see [52].

Proof of Theorem 2.4.1 By (2.4.6), we get

$$\mathcal{L}_X A = \mathcal{L}_{\omega\Upsilon} A = \omega \mathcal{L}_\Upsilon A - \Upsilon \wedge \mathbf{i}_{\mathrm{d}\omega} A. \tag{2.4.7}$$

We rewrite equation (2.4.2) in the form

$$\mathcal{L}_\Upsilon A - \frac{1}{\omega}\Upsilon \wedge \mathbf{i}_{\mathrm{d}\omega} A = \frac{1}{\omega}(B - \bar{B}). \tag{2.4.8}$$

Applying the averaging operator to the both sides of this equation and taking into account conditions (2.3.7) and (2.4.3), we get

$$\bar{B} = \langle \bar{B} \rangle = \langle B \rangle + \Upsilon \wedge \mathbf{i}_{\mathrm{d}\omega} \langle A \rangle. \tag{2.4.9}$$

According to decomposition (2.4.1), we have

$$A = \langle A \rangle + A_0, \qquad \langle A_0 \rangle = 0, \tag{2.4.10}$$

$$B = \langle B \rangle + B_0, \qquad \langle B_0 \rangle = 0.$$

Putting these representations together with (2.4.9) into (2.4.8), we see that the original problem (2.4.2), (2.4.3) is reduced to the following equation for $A_0 \in \chi_0^k(M)$:

$$\mathcal{L}_\Upsilon A_0 - \frac{1}{\omega}\Upsilon \wedge \mathbf{i}_{d\omega} A_0 = \frac{1}{\omega} B_0.$$

Looking for A_0 in the form $A_0 = \frac{1}{\omega}\mathcal{S}(B_0) + \tilde{A}_0$ and using property (2.3.9), we conclude that $\tilde{A}_0 \in \chi_0^k(M)$ must satisfy the equation

$$\mathcal{L}_\Upsilon \tilde{A}_0 - \frac{1}{\omega}\Upsilon \wedge \mathbf{i}_{\mathrm{d}\omega} \tilde{A}_0 = \frac{1}{\omega^2}\Upsilon \wedge \mathbf{i}_{\mathrm{d}\omega} \mathcal{S}(B_0). \tag{2.4.11}$$

Next, taking into account that $\mathbf{i}_{\mathrm{d}\omega}\Upsilon = 0$ and putting $\tilde{A}_0 = \Upsilon \wedge D$, we reduce (2.4.11) to the following equation for $D \in \chi_0^{k-1}(M)$:

$$\Upsilon \wedge \mathcal{L}_\Upsilon D = \frac{1}{\omega^2}\Upsilon \wedge \mathbf{i}_{\mathrm{d}\omega} \mathcal{S}(B_0).$$

By property (2.3.9), the $(k-1)$-vector field $D = \frac{1}{\omega^2}\mathcal{S}^2(\mathbf{i}_{\mathrm{d}\omega} B_0)$ satisfies the relation $\mathcal{L}_\Upsilon \mathcal{S}(D) = \frac{1}{\omega^2}\mathbf{i}_{\mathrm{d}\omega}\mathcal{S}(B_0)$. Therefore, the solutions to problem (2.4.2), (2.4.3) are given by (2.4.9) and (2.4.10), where

$$A_0 = \frac{1}{\omega}\mathcal{S}(B_0) + \frac{1}{\omega^3} X \wedge \mathcal{S}^2(\mathbf{i}_{\mathrm{d}\omega} B_0) \tag{2.4.12}$$

and $\langle A \rangle$ is an arbitrary $\mathbb{S}^1$-invariant k-vector field on M. Finally, property (2.3.8) says that $\mathcal{S}(B_0) = \mathcal{S}(B)$ and hence formulas (2.4.4) and (2.4.5) follow from (2.4.9) and (2.4.12) with $C = \langle A \rangle$. ■

As a straightforward consequence of Theorem 2.4.1, we get the following results.

Corollary 2.4.2 *The kernel of the Lie derivative* $\mathcal{L}_X : \chi^k(M) \to \chi^k(M)$ *is*

$$\mathrm{Ker}(\mathcal{L}_X) = \chi^k_{\mathrm{inv}}(M) \cap \{C \in \chi^k(M) \mid X \wedge \mathbf{i}_{\mathrm{d}\omega} C = 0\}. \tag{2.4.13}$$

Corollary 2.4.3 *For a given $B \in \chi^k(M)$ $(k \geq 1)$, the homological equation*

$$\mathcal{L}_X A = B \tag{2.4.14}$$

is solvable relative to a k-vector field A on M if and only if

$$\langle B \rangle = X \wedge \mathbf{i}_{\mathrm{d}\omega} P \tag{2.4.15}$$

for a certain $\mathbb{S}^1$-invariant k-vector field $P \in \chi^k_{\mathrm{inv}}(M)$. Under this condition, every solution of (2.4.14) is given by (2.4.5), where $C = P + C'$, $C' \in \ker(\mathcal{L}_X)$.

It follows from (2.4.15) that the necessary conditions for the solvability of (2.4.14) are the following

$$X(m) = 0 \quad \to \quad \langle B \rangle(m) = 0, \tag{2.4.16}$$

$$X \wedge \langle B \rangle = 0, \tag{2.4.17}$$

$$\mathbf{i}_{\mathrm{d}\omega} \langle B \rangle = 0. \tag{2.4.18}$$

Therefore, if one of these conditions does not hold, then equation (2.4.14) is unsolvable.

Corollary 2.4.4 *There exist k-vector fields A and $\bar{B}$ on M satisfying the equations*

$$\mathcal{L}_X A = B - \bar{B}, \tag{2.4.19}$$

$$\mathcal{L}_X \bar{B} = 0 \tag{2.4.20}$$

if and only if

$$X \wedge \mathbf{i}_{\mathrm{d}\omega} \langle B \rangle = 0. \tag{2.4.21}$$

Under this condition, all solutions $(A, \bar{B})$ to (2.4.19),(2.4.20) are given by formulas (2.4.4), (2.4.5). Moreover, the k-vector field A in (2.4.5) can be represented in the form $A = \dfrac{1}{\omega}\mathcal{S}(B) + (\mathbb{S}^1$-invariant k-vector field $)$ if and only if $\Upsilon \wedge \mathbf{i}_{\mathrm{d}\omega} B = 0$.

The Case of $C^\infty(M)$. We apply the results on solvability of homological equation to the tensor space $\mathcal{T}^0_0(M) = C^\infty(M)$. For a given function $G \in C^\infty(M)$, we are looking for smooth functions F and $\bar{G}$ satisfying the homological equation

$$\mathcal{L}_X F = G - \bar{G}, \tag{2.4.22}$$

and the condition

$$\mathcal{L}_X \bar{G} = 0. \tag{2.4.23}$$

By identity (2.4.42), the solvability condition (2.4.21) for $C^\infty(M)$ is equivalent to $\mathcal{L}_X \langle G \rangle = 0$ which always holds for every smooth function G. From this observation and Corollary 2.4.12 the global solutions of the homological equation (2.4.22) and the condition (2.4.23) always exist and are given by

$$\bar{G} = \langle G \rangle, \tag{2.4.24}$$

and

$$F = \mathcal{S}(G). \tag{2.4.25}$$

In terms of time averaging, this formula can written as follows

$$F(m) = \frac{1}{T(m)} \int_0^{T(m)} (t - \frac{T(m)}{2}) G(\mathrm{Fl}_X^t(m)) dt.$$

We remark that formulas (2.4.24) and (2.4.25) were obtained in [15] by Cushman.
The Case of Vector Fields. Now, we study the particular case of vector fields. By Theorem 2.4.1 , Corollary 2.4.2 and Corollary 2.4.4 (where $A = Z$ and $B = W$ are vector fields on M) we deduce the following facts.

Proposition 2.4.5 *Let X be a vector field with periodic flow and frequency function ω. Then, for a given $W \in \mathfrak{X}(M)$, all vector fields $Z \in \mathfrak{X}(M)$ and $\bar{W} \in \mathfrak{X}_{\mathrm{inv}}(M)$ on M satisfying the equation*

$$\mathcal{L}_X Z = W - \bar{W}, \tag{2.4.26}$$

and the condition

$$\bar{W} \textit{ is } \mathbb{S}^1\textit{-invariant} \tag{2.4.27}$$

are given by the formulas

$$\bar{W} = \langle W \rangle + \frac{1}{\omega} \mathcal{L}_Y(\omega) X, \tag{2.4.28}$$

$$Z = \frac{1}{\omega} \mathcal{S}(W) + \frac{1}{\omega^3} \mathcal{S}^2(\mathcal{L}_W \omega) X + Y, \tag{2.4.29}$$

where $Y \in \mathfrak{X}_{\mathrm{inv}}(M)$.

Proof. By Theorem 2.4.1, the solutions of equations (2.4.26), (2.4.27) are given by

$$\bar{W} = \langle W \rangle + \frac{1}{\omega} X \wedge \mathbf{i}_{\mathrm{d}\omega} Y, \tag{2.4.30}$$

$$Z = \frac{1}{\omega} \mathcal{S}(W) + \frac{1}{\omega^3} X \wedge \mathcal{S}^2(\mathbf{i}_{\mathrm{d}\omega} W) + Y, \tag{2.4.31}$$

where $Y \in \chi^1(M) = \mathfrak{X}(M)$ is an arbitrary $\mathbb{S}^1$-invariant vector field. Now, for every vector field Y, we have

$$\mathbf{i}_{\mathrm{d}\omega} Y = \mathrm{d}\omega(Y) = \mathcal{L}_Y \omega.$$

Thus,

$$X \wedge \mathbf{i}_{\mathrm{d}\omega} Y = (\mathcal{L}_Y \omega) X,$$

and

$$X \wedge \mathcal{S}^2(\mathbf{i}_{\mathrm{d}\omega} W) = \mathcal{S}^2(\mathcal{L}_W \omega) X.$$

Therefore, formulas (2.4.30) and (2.4.31) reduce to (2.4.28), (2.4.29). ∎

Corollary 2.4.6 *Given a vector field W, there exist vector fields Z and $\bar{W}$ on M satisfying the equations*

$$\mathcal{L}_X Z = W - \bar{W}, \tag{2.4.32}$$

$$\mathcal{L}_X \bar{W} = 0 \tag{2.4.33}$$

if and only if

$$\mathcal{L}_{\langle W\rangle}\omega = 0. \tag{2.4.34}$$

Under this condition, all solutions $(\bar{W}, Z)$ to (2.4.32),(2.4.33) are given by formulas (2.4.28), (2.4.29). Moreover, the vector field Z in (2.4.29) can be represented in the form $Z = \frac{1}{\omega}\mathcal{S}(W) + (\mathbb{S}^1$-invariant vector field) if and only if $\mathcal{L}_W\omega = 0$.

Proof. Corollary 2.4.4 asserts that equations (2.4.32), (2.4.33) are solvable and their solutions are given by (2.4.28), (2.4.29) if and only if condition condition (2.4.21) holds. For vector fields, we have

$$X \wedge \mathbf{i}_{\mathrm{d}\omega}\langle W\rangle = (\mathcal{L}_{\langle W\rangle}\omega)X$$

Since $X \neq 0$, we get that condition (2.4.21) is equivalent to (2.4.34). ∎

One direct consequence of Corollary 2.4.6 is the solvability of the homological equation

$$\mathcal{L}_X Z = W. \tag{2.4.35}$$

for given W. Equation (2.4.35) is solvable for Z if and only if $\langle W\rangle = \mathcal{L}_{\tilde{Y}}(\omega)X$ for a certain $\mathbb{S}^1$-invariant vector field $\tilde{Y}$.

Let $\mathrm{Reg}(X) = \{m \in M \mid X(m) \neq 0\}$ be the set of points regular of X. If $\mathrm{Reg}(X)$ is everywhere dense in M, Corollary 2.4.2 implies that the kernel of the Lie derivative $\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is

$$\mathrm{Ker}(\mathcal{L}_X) = \mathfrak{X}_{\mathrm{inv}}(M) \cap \{Y \in \mathfrak{X}(M) \mid \mathcal{L}_Y\omega = 0\}. \tag{2.4.36}$$

2.4.2 Homological equations for k-forms.

Consider the space $\Omega^k(M) = \mathrm{Sec}(\bigwedge^k T^*M)$ of k-forms on M. Then, subspace $\Omega^k(M) \subset \mathcal{T}^0_k(M)$ is invariant with respect to the action of the operators $\mathcal{L}_\Upsilon, \mathcal{S}$ and $\mathcal{A}$. By $\mathbf{i}_Y\alpha \in \Omega^{k-1}(M)$ we denote the interior product of a vector field Y and a k-form α on M which is defined by the usual formula:

$$(\mathbf{i}_Y\alpha)(Y_1, ..., Y_{k-1}) = \alpha(Y, Y_1, ..., Y_{k-1}).$$

Let $\Omega^k_{\mathrm{inv}}(M) = \mathrm{Ker}\,\mathcal{L}_\Upsilon$ and $\Omega^k_0(M) = \mathrm{Im}\,\mathcal{L}_\Upsilon$. Then, we have the $\mathbb{S}^1$-invariant splitting

$$\Omega^k(M) = \Omega^k_{\mathrm{inv}}(M) \oplus \Omega_0.^k(M), \tag{2.4.37}$$

There is the following covariant analog of Theorem 2.4.1.

Theorem 2.4.7 *For a given $\eta \in \Omega^k(M)$, all k-forms θ and $\bar{\eta}$ on M satisfying the homological equation*

$$\mathcal{L}_X\theta = \eta - \bar{\eta} \tag{2.4.38}$$

and the condition

$$\bar{\eta} \text{ is } \mathbb{S}^1\text{-invariant}, \tag{2.4.39}$$

are represented as

$$\bar{\eta} = \langle \eta \rangle - \frac{1}{\omega} \mathrm{d}\omega \wedge \mathbf{i}_X \mu, \tag{2.4.40}$$

$$\theta = \frac{1}{\omega} \mathcal{S}(\eta) - \frac{1}{\omega^3} \mathrm{d}\omega \wedge \mathcal{S}^2(\mathbf{i}_X \eta) + \mu, \tag{2.4.41}$$

where $\mu \in \Omega^k_{\mathrm{inv}}(M)$ is an arbitrary $\mathbb{S}^1$-invariant k-form.

The proof of this theorem goes in the same line as the proof Theorem 2.4.1, where instead of identity 2.4.6 we have to use its covariant analog result: Let $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\alpha \in \Omega^k(M)$; then,

$$\mathcal{L}_{fX} \alpha = f \mathcal{L}_X \alpha + \mathrm{d}f \wedge \mathbf{i}_X \alpha. \tag{2.4.42}$$

Proof of Theorem 2.4.7 Applying formula (2.4.42) to left hand side of equation (2.4.38), we get

$$\mathcal{L}_{\omega\Upsilon} \theta = \omega \mathcal{L}_\Upsilon \theta + \mathrm{d}\omega \wedge \mathbf{i}_\Upsilon \theta.$$

We rewrite equation (2.4.38) in the form

$$\mathcal{L}_\Upsilon \theta + \frac{1}{\omega} \mathrm{d}\omega \wedge \mathbf{i}_\Upsilon \theta = \frac{1}{\omega}(\eta - \bar{\eta}). \tag{2.4.43}$$

Applying the averaging operator to the both sides of this equation and taking into account conditions (2.3.7) and (2.4.39), we get

$$\bar{\eta} = \langle \bar{\eta} \rangle = \langle \eta \rangle - \mathrm{d}\omega \wedge \mathbf{i}_\Upsilon \langle \theta \rangle. \tag{2.4.44}$$

According to decomposition (2.4.37), we have

$$\theta = \langle \theta \rangle + \theta_0, \qquad \langle \theta_0 \rangle = 0, \tag{2.4.45}$$

$$\eta = \langle \eta \rangle + \eta_0, \qquad \langle \eta_0 \rangle = 0.$$

Putting these representations together with (2.4.44) into (2.4.43), we see that the original problem (2.4.38), (2.4.39) is reduced to the following equation for $\theta_0 \in \Omega^k_0(M)$:

$$\mathcal{L}_\Upsilon \theta_0 + \frac{1}{\omega} \mathrm{d}\omega \wedge \mathbf{i}_\Upsilon \theta_0 = \frac{1}{\omega} \eta_0.$$

We look for θ_0 in the form $\theta_0 = \frac{1}{\omega}\mathcal{S}(\eta_0) + \tilde{\theta}_0$ and using property (2.3.9), we conclude that $\tilde{\theta}_0 \in \Omega^k_0(M)$ must satisfy the equation

$$\mathcal{L}_\Upsilon \tilde{\theta}_0 + \frac{1}{\omega} \mathrm{d}\omega \wedge \mathbf{i}_\Upsilon \tilde{\theta}_0 = -\frac{1}{\omega^2} \mathrm{d}\omega \wedge \mathbf{i}_\Upsilon \mathcal{S}(\eta_0). \tag{2.4.46}$$

Taking into account that $\mathbf{i}_\Upsilon \mathrm{d}\omega = \mathcal{L}_\Upsilon \omega = 0$ and putting $\tilde{\theta}_0 = \mathrm{d}\omega \wedge \varrho$, we reduce (2.4.46) to the following equation for $\varrho \in \Omega^{k-1}_0(M)$:

$$\mathrm{d}\omega \wedge \mathcal{L}_\Upsilon \varrho = \mathrm{d}\omega \wedge (-\frac{1}{\omega^2} \mathrm{i}_\Upsilon S(\eta_0))$$

By property (2.3.9), the (k-1)-form $\varrho = -\frac{1}{\omega^2}\mathcal{S}^2(\mathbf{i}_\Upsilon \eta_0)$ satisfies the relation

$$\mathcal{L}_\Upsilon \mathcal{S}(\varrho) = -\frac{1}{\omega^2}\mathbf{i}_\Upsilon \mathcal{S}(\eta_0).$$

Therefore, the solutions to problem (2.4.38), (2.4.39) are given by (2.4.44) and (2.4.45), where

$$\theta_0 = \frac{1}{\omega}\mathcal{S}(\eta_0) - \frac{1}{\omega^3}\mathrm{d}\omega \wedge \mathcal{S}^2(\mathbf{i}_X \eta_0) \tag{2.4.47}$$

and $\langle\theta\rangle$ is an arbitrary $\mathbb{S}^1$-invariant k-form on M. Finally, property (2.3.8) says that $\mathcal{S}(\eta_0) = \mathcal{S}(\eta)$ and hence formulas (2.4.40) and (2.4.41) follow from (2.4.44) and (2.4.47) with $\mu = \langle\theta\rangle$. ■

From Theorem 2.4.7, we deduce the following consequences

Corollary 2.4.8 *The kernel of the Lie derivative $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$ is*

$$\mathrm{Ker}(\mathcal{L}_X) = \Omega^k_{\mathrm{inv}}(M) \cap \{\mu \in \Omega^k(M) \mid d\omega \wedge \mathbf{i}_X \mu = 0\}. \tag{2.4.48}$$

Corollary 2.4.9 *For a given $\eta \in \Omega^k(M)$ $(k \geq 1)$, the homological equation*

$$\mathcal{L}_X \theta = \eta \tag{2.4.49}$$

is solvable relative to a k-form θ on M if and only if

$$\langle\eta\rangle = \mathrm{d}\omega \wedge \mathbf{i}_X \alpha \tag{2.4.50}$$

for a certain $\mathbb{S}^1$-invariant k-form $\alpha \in \Omega^k_{\mathrm{inv}}(M)$.

It follows from (2.4.50) that the necessary conditions for the solvability of equation (2.4.49) are

$$X(m) = 0 \Longrightarrow \langle\eta\rangle(m) = 0, \tag{2.4.51}$$

$$\mathrm{d}\omega \wedge \langle\eta\rangle = 0, \tag{2.4.52}$$

$$\mathbf{i}_X \langle\eta\rangle = 0. \tag{2.4.53}$$

Corollary 2.4.10 *There exist k-forms θ and η on M satisfying the equations*

$$\mathcal{L}_X \theta = \eta - \bar{\eta}, \tag{2.4.54}$$

$$\mathcal{L}_X \bar{\eta} = 0, \tag{2.4.55}$$

if and only if the following condition holds

$$\mathrm{d}\omega \wedge \mathbf{i}_X \langle\eta\rangle = 0. \tag{2.4.56}$$

*Under this conditions, solutions (θ, η) of (*2.4.54*) (*2.4.55*) are given by (*2.4.40*) and (*2.4.41*).*

The Case of 1-forms. Now, we derive the formulas of solution for homological equation and the compatibility condition for 1-forms.

Proposition 2.4.11 *For a given 1-form $\beta \in \Omega^1(M)$, all 1-forms α and $\bar{\beta}$ on M satisfying the homological equation*

$$\mathcal{L}_X \alpha = \beta - \bar{\beta}, \tag{2.4.57}$$

and the condition

$$\bar{\beta} \text{ is } \mathbb{S}^1\text{-invariant} \tag{2.4.58}$$

are written as follows

$$\bar{\beta} = \langle \beta \rangle - \frac{1}{\omega}(\mathbf{i}_X \mu) d\omega, \tag{2.4.59}$$

$$\alpha = \frac{1}{\omega}\mathcal{S}(\eta) - \frac{1}{\omega^3}\mathcal{S}^2(\mathbf{i}_X \beta) d\omega + \mu, \tag{2.4.60}$$

with $\mu \in \Omega^1_{\mathrm{inv}}(M)$.

Proof. It follows from Theorem 2.4.7, solution of equation (2.4.57), (2.4.58) are given by (2.4.40), (2.4.41). These formulas reduce to (2.4.59), (2.4.60) because of for any 1-forms β and any vector field X, $\mathbf{i}_X \beta$ is a smooth function. ■
From Corollary 2.4.8, we have the following consequences:

- the solvability condition for homological equation

$$\mathcal{L}_X \alpha = \beta \tag{2.4.61}$$

  reads

$$\langle \beta \rangle = \frac{1}{\omega}(\mathbf{i}_X \mu) d\omega \tag{2.4.62}$$

  for a certain $\mathbb{S}^1$-invariant 1-form $\mu \in \Omega^1_{\mathrm{inv}}(M)$.

- Let $\mathrm{Reg}(\omega) = \{m \in M \mid \mathrm{d}_m \omega \neq 0\}$ be the set of regular points of the frequency function ω. If $\mathrm{Reg}(\omega)$ is everywhere dense in M, then

$$\mathrm{Ker}(\mathcal{L}_X) = \Omega^1_{\mathrm{inv}}(M) \cap \{\mu \in \Omega^k(M) \mid \mathbf{i}_X \mu = 0\}. \tag{2.4.63}$$

Corollary 2.4.12 *For any $\beta \in \Omega^1(M)$, there exist 1-forms α and $\bar{\beta}$ satisfying the equations*

$$\mathcal{L}_X \alpha = \beta - \bar{\beta}, \tag{2.4.64}$$

$$\mathcal{L}_X \bar{\beta} = 0, \tag{2.4.65}$$

if and only if

$$\mathbf{i}_X \langle \beta \rangle = 0. \tag{2.4.66}$$

Under this conditions, the formulas (2.4.59) and (2.4.60) give the solution of (2.4.64), (2.4.65).

2.4.3 The $\mathbb{S}^1$-average of closed forms

Let us consider the case of closed forms. Let $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior derivative. By standard properties of the exterior derivative, we conclude that d commutes with operators $\mathcal{L}_\Upsilon, \mathcal{S}$ and $\mathcal{A}$, in particular, $\langle \mathrm{d}\eta \rangle = \mathrm{d}\langle \eta \rangle$ for any $\eta \in \Omega^k(M)$. Moreover, splitting (2.4.37) is invariant with respect to d, in the sense that if $\eta \in \Omega^k(M)$ has the decomposition

$$\eta = \langle \eta \rangle + \eta_0,$$

where $\langle \eta \rangle \in \Omega^k_{\mathrm{inv}}(M), \eta_0 \in \Omega^k_0(M)$, then

$$\mathrm{d}\eta = \langle \mathrm{d}\eta \rangle + (\mathrm{d}\eta)_0 = \mathrm{d}\langle \eta \rangle + \mathrm{d}\eta_0.$$

It follows that if η is closed then, the components $\langle \eta \rangle$ and η_0 are also closed k-forms. In this case, according to (2.4.37), a solution to the equation $\mathcal{L}_\Upsilon \theta_0 = \eta_0$ is given by $\theta_0 = \mathcal{S}(\eta_0)$. Then, $\mathrm{d}\theta_0 = \mathcal{S}(\mathrm{d}\eta_0) = 0$ and hence $\eta_0 = \mathcal{L}_\Upsilon \theta_0 = \mathrm{d} \circ \mathbf{i}_\Upsilon \theta_0$. This proves the following assertion.

Proposition 2.4.13 *For every closed k-form η on M, we have the decomposition*

$$\eta = \langle \eta \rangle + \mathrm{d}(\mathbf{i}_\Upsilon \theta_0), \tag{2.4.67}$$

where $\theta_0 = \mathcal{S}(\eta) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} (t - \pi)(\mathrm{Fl}^t_\Upsilon)^* \eta dt.$

As a consequence of this proposition we have the equivariant version of Poincaré Lemma.

Proposition 2.4.14 *Assume that the $\mathbb{S}^1$-action with infinitesimal generator Υ is free on M and let $\rho : M \to M/\mathbb{S}^1$ be the canonical projection. Then for every closed k-form η on M and $m \in M$ there exists a neighborhood U of $\rho(m)$ in $M/\mathbb{S}^1$ such that the restriction of η to the $\mathbb{S}^1-$ invariant domain $\rho^{-1}(U) \subset M$ is exact.*

Proof. By the $\mathbb{S}^1$-invariance and closeness of $\langle \eta \rangle$ we have that $\langle \eta \rangle = \rho^* \gamma$ for a certain closed k-form $\gamma \in \Omega(M/\mathbb{S}^1)$. By the Poincare Lemma, there exists an open neighborhood U of $\rho(m)$ such that $\gamma = d\beta$ for $\beta \in \Omega^{k-1}(U)$ It follows from Proposition 2.4.13

$$\eta|_{\rho^{-1}(U)} = d(\rho^* \beta + i_\Upsilon\ \theta|_{\rho^{-1}(U)})$$

■

2.4.4 The trivial $\mathbb{S}^1$-action

We apply the results on global solvability of homological equations to the smooth manifold

$$M = \mathbb{S}^1 \times \mathbb{R}^n = \{\varphi(\mathrm{mod})2\pi\} \times \{\mathbf{y} = (y_1, y_2, \ldots, y_n)\}.$$

Every vector field Y on $M = \mathbb{S}^1 \times \mathbb{R}^n$ is of the form

$$Y(\varphi, \mathbf{y}) = Y_0(\varphi, \mathbf{y}) \frac{\partial}{\partial \varphi} + Y_i(\varphi, \mathbf{y}) \frac{\partial}{\partial y_i}, \tag{2.4.68}$$

where $Y_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are smooth functions and 2π-periodic in φ. The vector field

$$X = \omega(\mathbf{y}) \frac{\partial}{\partial \varphi}, \tag{2.4.69}$$

where $\omega : \mathbb{R}^n \to \mathbb{R}$ is a smooth function such that $\omega(\mathbf{y}) > 0$. The flow of X is T-periodic with period function $T(\mathbf{y}) := \dfrac{2\pi}{\omega(\mathbf{y})}$. Thus, the vector field $\Upsilon = \frac{\partial}{\partial \varphi}$ is the infinitesimal generator of the $\mathbb{S}^1$-action induced by the flow of X. Let $f \in C^\infty(\mathbb{S}^1 \times \mathbb{R}^n)$. The averaging of f with respect to Υ is given by

$$\begin{aligned} \langle f \rangle(\mathbf{y}) &= \frac{1}{2\pi} \int_0^{2\pi} \left(\mathrm{Fl}_\Upsilon^t \right)^* f(\varphi, \mathbf{y}) \mathrm{d}t \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(\varphi, \mathbf{y}) \mathrm{d}\varphi \end{aligned} \tag{2.4.70}$$

If Y is a vector field on M (2.4.68), then averaging Y is

$$\begin{aligned} \langle Y \rangle(\mathbf{y}) &= \langle Y_0 \rangle(\varphi, \mathbf{y}) \frac{\partial}{\partial \varphi} + \langle Y_i \rangle(\varphi, \mathbf{y}) \frac{\partial}{\partial y_i}, \\ &= \frac{1}{2\pi} \int_0^{2\pi} Y(\varphi, \mathbf{y}) \mathrm{d}\varphi. \end{aligned} \tag{2.4.71}$$

And, if $\eta = \eta_0 \mathrm{d}\varphi + \eta_i \mathrm{d}x_i$ is a 1-form on M, then the averaging of η is

$$\begin{aligned} \langle \eta \rangle(\mathbf{y}) &= \langle \eta_0 \rangle(\mathbf{y}) \mathrm{d}\varphi + \langle \eta_i \rangle(\mathbf{y}) \mathrm{d}x_i, \\ &= \frac{1}{2\pi} \int_0^{2\pi} \eta(\varphi, \mathbf{y}) \mathrm{d}\varphi. \end{aligned} \tag{2.4.72}$$

Now, we study the homological equations for vector fields and 1-forms on $\mathbb{S}^1 \times \mathbb{R}^n$. Corollary 2.4.6, asserts that for a given vector field $W = Y = Y_0(\varphi, \mathbf{y}) \frac{\partial}{\partial \varphi} + Y_i(\varphi, \mathbf{y}) \frac{\partial}{\partial y_i}$ on M, there exist vector fields Z, and $\bar{W}$ satisfying the equations (2.4.32), (2.4.33) if and only if ω and Y are compatible by $\mathcal{L}_{\langle Y \rangle} \omega = 0$. This condition is equivalent to $\langle Y_i(\varphi, \mathbf{y}) \rangle \dfrac{\partial \omega}{\partial y_i} = 0$. Explicit formulas of the solutions Z and $\bar{W}$ are given by

$$\bar{W} = \left(\langle Y_0 \rangle + c_i \cdot \frac{\partial \omega}{\partial Y_i} \right) \frac{\partial}{\partial \varphi} + \langle Y_i \rangle \frac{\partial}{\partial y_i}, \tag{2.4.73}$$

$$Z = \left(\frac{1}{\omega} \mathcal{S}(Y_0) + \frac{1}{\omega^2} \mathcal{S}^2(Y_i) \frac{\partial \omega}{\partial Y_i} + b \right) \frac{\partial}{\partial \varphi} + \left(\frac{1}{\omega} \mathcal{S}(Y_i) + c_i \right) \frac{\partial}{\partial y_i},$$

where $b, c_i \in \mathbb{C}^\infty(M)$ are $\mathbb{S}^1$-invariant smooth functions.

For a given 1-form $\eta = \eta_0 \mathrm{d}\varphi + \eta_i \mathrm{d}x_i$ on M, Corollary 2.4.12 says that equations (2.4.64), (2.4.65) are solvable for α and $\bar{\eta}$ if and only if the condition $i_X \langle \eta \rangle = 0$ holds. But, η and X satisfy this condition if and only if $\langle \eta_0 \rangle = 0$. Formulas of the solutions are

$$\bar{\eta} = (\langle \eta_i \rangle - \bar{a} \frac{\partial \omega}{\partial x_i}) \mathrm{d}x_i, \tag{2.4.74}$$

$$\alpha = (\frac{1}{\omega} \mathcal{S}(\eta_0) + \bar{a}) \mathrm{d}\varphi + (\mathcal{S}(\eta_i) - \frac{1}{\omega^2} \mathcal{S}^2(a) \frac{\partial \omega}{\partial x_i} + \bar{\xi}_i) \mathrm{d}\xi_i, \tag{2.4.75}$$

for arbitrary $\mathbb{S}^1$-invariant functions $\bar{a}$, $\bar{\xi}_i$.

2.5 The Compatibility Condition From Period-Energy Relation.

Here, we show that in the Hamiltonian case the compatibility condition (2.4.32) always holds because of the so called period-energy relation, [2, 10, 29]. This leads to the well known fact: the homological equation (2.4.32) and the condition (2.4.33) are solvable in the Hamiltonian case [2, 6, 13, 15].

First, follow [29], we recall some facts concerning to period energy relation.

Proposition 2.5.1 (The Period-Energy relation) *Let X be a vector field on a manifold M with periodic flow and a smooth period function $T : M \to \mathbb{R}$. If there exists a closed 2-form σ on M such that*

$$i_X\sigma = -\mathrm{d}H \tag{2.5.1}$$

for a certain function $H \in C^\infty(M)$, then

$$\mathrm{d}T \wedge \mathrm{d}H = 0 \quad on \quad M. \tag{2.5.2}$$

Proof. It follows from (2.5.1) that

$$\mathcal{L}_X\sigma = i_X d\sigma + d i_X\sigma = -d^2H = 0. \tag{2.5.3}$$

On the other hand, we have

$$\mathcal{L}_X\sigma = \mathcal{L}_{\omega\Upsilon}\sigma = \omega\mathcal{L}_\Upsilon\sigma + d\omega \wedge i_\Upsilon\sigma = \omega\mathcal{L}_\Upsilon\sigma - \frac{1}{\omega}d\omega \wedge dH, \tag{2.5.4}$$

where $\omega = \frac{2\pi}{T}$ is the frequency function. Consider the $\mathbb{S}^1$-action on M with infinitesimal generator $\Upsilon = \frac{1}{\omega}X_H$. Remark that ω and H are first integrals of X and hence $\mathbb{S}^1$-invariant. Thus, applying the averaging operator to equality (2.5.4) and taking into account (2.5.3) and that $\langle\mathcal{L}_\Upsilon\sigma\rangle = 0$, we get

$$0 = \langle\mathcal{L}_X\sigma\rangle = -\frac{1}{\omega}d\omega \wedge dH.$$

■

Corollary 2.5.2 *The 2-form σ is invariant with respect to the $\mathbb{S}^1$-action associated with the periodic flow of X, $\mathcal{L}_\Upsilon\sigma = 0$.*

Proposition 2.5.3 *Assume that in addition to the hypothesis of Proposition* 2.5.1 *the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = \frac{1}{\omega}X_H$ is free on M and consider the canonical projection $\rho : M \to M/\mathbb{S}^1$. Then, for every $m \in M$, there exists an open neighborhood U of $\rho(m)$ in the orbit space $M/\mathbb{S}^1$ such that the restriction of the 2-form σ to the $\mathbb{S}^1$-invariant domain $\rho^{-1}(U)$ is exact, $\sigma = d\mu$ and the period function satisfies the relation*

$$dJ = TdH \tag{2.5.5}$$

where $J \in C^\infty(\rho^{-1}(U))$ is given by

$$J := T\langle i_X\mu\rangle. \tag{2.5.6}$$

Proof. The existence of an open domain U with desired properties follows from equivariant Poincare Lemma (see Proposition 2.4.14). Then, $\sigma = d\mu$ on the $\mathbb{S}^1$-invariant domain $\varrho^{-1}(U) \subset M$. Taking into account that $\mathcal{L}_\Upsilon \langle \mu \rangle = 0$ and dH is $\mathbb{S}^1$-invariant, we get

$$dJ = 2\pi \langle d i_\Upsilon \mu \rangle = 2\pi \langle \mathcal{L}_\Upsilon \mu \rangle - 2\pi \langle i_\Upsilon d\mu \rangle = -T \langle i_X d\sigma \rangle = T dH.$$

■

Corollary 2.5.4 *If the 2-form σ is exact, then relation (2.5) holds on the whole M.*

Now, suppose we start with the Hamiltonian vector field X_H of a function $H \in C^\infty(M)$ on a symplectic manifold (M, σ). Assuming that X_H has periodic flow with period function T, consider the corresponding $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon := \frac{1}{\omega} X_H$, where $\omega = \frac{2\pi}{T}$ is the frequency function. For a given function $F \in C^\infty(M)$, we consider the homological problem

$$\mathcal{L}_{X_H} Z = X_F - \overline{W}, \tag{2.5.7}$$

$$\mathcal{L}_{X_H} \overline{W} = 0. \tag{2.5.8}$$

Proposition 2.5.5 *If the regular set* $\mathrm{Reg}(X_H) = \{m \in M | d_m H \neq 0\}$ *is dense in M, then the compatibility condition*

$$\mathcal{L}_{\langle X_F \rangle} \omega = 0. \tag{2.5.9}$$

holds and homological problem (2.5.7), (2.5.8) is solvable on M. Every solution is of the form

$$\overline{W} = X_{\langle F \rangle} + \frac{1}{\omega} (\mathcal{L}_Y \omega) X_H, \tag{2.5.10}$$

$$Z = X_{\frac{1}{\omega} S(F)} + Y, \tag{2.5.11}$$

where Y is arbitrary $\mathbb{S}^1$-invariant vector field.

Proof. First, let us show that the period energy relation (2.5.2) implies the compatibility condition (2.5.9). By Corollary 2.5.2, the symplectic form σ is invariant with respect to the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = \frac{1}{\omega} X_H$. This implies that

$$\langle X_F \rangle = X_{\langle F \rangle}. \tag{2.5.12}$$

Then, we have

$$i_{\langle X_F \rangle} \mathrm{d}H = \mathcal{L}_{X_{\langle F \rangle}} H = -\mathcal{L}_{X_H} \langle F \rangle = -\omega \mathcal{L}_\Upsilon \langle F \rangle = 0.$$

It follows from the identity that $\mathrm{d}\omega \wedge \mathrm{d}H = 0$ that

$$0 = i_{\langle X_F \rangle} (\mathrm{d}\omega \wedge \mathrm{d}H) = (i_{\langle X_F \rangle} \mathrm{d}\omega) \mathrm{d}H - (i_{\langle X_F \rangle} \mathrm{d}H) \mathrm{d}\omega = (\mathcal{L}_{\langle X_F \rangle} \omega) dH \tag{2.5.13}$$

and hence condition (2.5.9) holds on $\mathrm{Reg}(X_H)$. But by assumption the regular set is dense and hence (2.5.9) is satisfies on the whole M. This implies the global solvability of the problem (2.5.7), (2.5.8). By Corollary 2.4.6, the corresponding general solutions are by (2.4.28), (2.4.29). Since the $\mathbb{S}^1$-action is symplectic relative to σ. for the Hamiltonian vector field X_F we have the equality $(\mathrm{Fl}_\Upsilon^t)^* X_F = X_{F \circ \mathrm{Fl}_\Upsilon^t}$ which implies the following property of operator $\mathcal{S}$:

$$S(X_{GF}) = X_{GS(F)} = S(F) X_G + G X_{S(F)},$$

for every $\mathbb{S}^1$-invariant function G. In particular, $S(X_F) = X_{S(F)}$ Using this property and (2.5.12), by direct computation, we verify that formulas (2.4.28), (2.4.29) lead to (2.5.10), (2.5.11). ■

Corollary 2.5.6 *In terms of the Poisson bracket on (M, σ), the compatibility condition (2.5.9) reads: the $\mathbb{S}^1$-average $\langle F \rangle$ Poisson commutes with the frequency function ω.*

In the exact case $\sigma = d\mu$, Proposition 2.5.3 implies that the $\mathbb{S}^1$-action associated to the periodic flow X_H is Hamiltonian with momentum map J given by (2.5.6).

Taking $Y = 0$ in (2.5.10), (2.5.11), we get that the solutions $\bar{W}$ and Z are Hamiltonian vector fields. This fact can be also derived from the standard Hamiltonian approach, [15].

Chapter 3

Global Normal Forms and The Geometric Averaging Theorem

3.1 Global Normal Forms.

In this section, we formulate some results on Deprit normalization and $\mathbb{S}^1$-invariant normalization. Let $\mathbf{A}_\varepsilon = A_0 + \varepsilon A_1 + \frac{\varepsilon^2}{2!} A_2 + \ldots + \frac{\varepsilon^k}{k!} A_k$. Given an $\mathbb{S}^1$-action on M and assuming the $\mathbb{S}^1$-invariance of A_0, we say that $\mathbf{A}_\varepsilon$ is in $\mathbb{S}^1$-invariant normal form of order k if $A_1, A_2, \ldots, A_k$ are $\mathbb{S}^1$-invariant vector fields.

3.1.1 First order normalization

Let $\mathbf{A}_\varepsilon = A_0 + \varepsilon A_1 + O(\varepsilon^2)$ be an ε-dependent vector field on M whose unperturbed part A_0 has periodic flow with frequency function $\omega : M \to \mathbb{R}$. Consider the $\mathbb{S}^1$-action on M with infinitesimal generator $\Upsilon = \frac{A_0}{\omega}$.

Theorem 3.1.1 *Let*

$$\Phi_\varepsilon = \mathrm{Fl}_Z^t \mid_{t=\varepsilon} \tag{3.1.1}$$

be the time-ε flow of the vector field

$$Z = \frac{1}{\omega}\mathcal{S}(A_1) + \frac{1}{\omega^3}\mathcal{S}^2(\mathcal{L}_{A_1}\omega)A_0 + Y. \tag{3.1.2}$$

where Y is an $\mathbb{S}^1$-invariant vector field. Then, for a given open domain $N \subset M$ with compact closure, there exists a constant $\delta > 0$ such that formula (3.1.1) defines a near identity transformation $\Phi_\varepsilon : N \to M$ with $\varepsilon \in (-\delta, \delta)$ which takes $\mathbf{A}_\varepsilon$ into the $\mathbb{S}^1$-invariant normal form

$$(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon = A_0 + \varepsilon(\langle A_1 \rangle + (\mathcal{L}_Y \ln \omega) A_0) + O(\varepsilon^2). \tag{3.1.3}$$

If the perturbed vector field A_1 and the frequency function $\omega : M \to \mathbb{R}$ are compatible by the condition

$$\mathcal{L}_{\langle A_1 \rangle}\omega = 0, \tag{3.1.4}$$

then Φ_ε is a normalization transformation of first order for $\mathbf{A}_\varepsilon$ relative to A_0;

$$[A_0, \langle A_1 \rangle] = 0. \tag{3.1.5}$$

Proof. Let Z be the vector field on N given by (3.1.2). By Proposition 1.1.1, there exists a constant $\delta > 0$ such that the mapping (3.1.1) defines a near identity transformation on N for $\varepsilon \in (-\delta, \delta)$. Let

$$\widetilde{\mathbf{A}}_\varepsilon \stackrel{\text{def}}{=} \Phi_\varepsilon^* \mathbf{A}_\varepsilon = A_0 + \varepsilon \widetilde{A}_1 + O(\varepsilon^2). \tag{3.1.6}$$

From Proposition 1.2.6, we get that vector fields Z and $\widetilde{A}_1$ satisfy the homological equation

$$\mathcal{L}_{A_0} Z = A_1 - \widetilde{A}_1. \tag{3.1.7}$$

Since the flow of A_0 is periodic, Proposition 2.4.5 tell us that general solution $(Z_0, \widetilde{A}_1)$ of (3.1.7) is given by (3.1.2) and

$$\widetilde{A}_1 = \langle A_1 \rangle + \mathcal{L}_Y(\ln \omega) A_0, \tag{3.1.8}$$

where Y is an $\mathbb{S}^1$-invariant vector field. In addition, Corollary 2.4.6 asserts that

$$[\widetilde{A}_1, A_0] = 0$$

if and only if vector field A_1 and ω are compatible by (3.1.4). In this case, Φ_ε is a normalization transformation of first order for $\mathbf{A}_\varepsilon$. ■

Remark 3 *It follows form the formula (*3.1.3*) that a normal form of first order of* $\mathbf{A}_\varepsilon$ *is uniquely determines by* $\langle A_1 \rangle$ *modulo* $(\mathcal{L}_Y \ln \omega) A_0$.

Suppose we are given a smooth function $I_0 : M \to \mathbb{R}$ which is invariant with respect to the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = \frac{A_0}{\omega}$. We observe that if I_0 is a first integral of the averaged vector field $\langle A_1 \rangle$,

$$\mathcal{L}_{\langle A_1 \rangle} I^0 = 0,$$

then the original perturbed vector field $\mathbf{A}_\varepsilon$ has an approximate first integral of the form

$$I_\varepsilon = I^0 - \frac{\varepsilon}{\omega} \mathcal{L}_{\mathcal{S}(A_1)} I_0.$$

So, we have

$$\mathcal{L}_{\mathbf{A}_\varepsilon} I = O(\varepsilon^2).$$

Now, let us see how, in the context of the normalization procedure, one can use a freedom in the definition of Φ_ε given by the $\mathbb{S}^1$-invariant Y in (3.1.2) . Consider the perturbed vector field $P_\varepsilon = X + \varepsilon W$ and assume that the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = \frac{1}{\omega} X$ is free on M. Then, the orbit space $\mathcal{O} = M/\mathbb{S}^1$ is a smooth manifold and the projection $\rho : M \to \mathcal{O}$ is a $\mathbb{S}^1$-principal bundle. In this case, the frequency function is of the form $\omega = \omega_{\mathcal{O}} \circ \rho$ for a certain $\omega_{\mathcal{O}} \in C^\infty(\mathcal{O})$. Let $\mathrm{Ver} = \mathrm{Span}\{\Upsilon\}$ be the vertical subbundle and $\mathcal{D} \subset TM$ an arbitrary subbundle which is complimentary to Ver. Then, for every vector field $u \in \mathfrak{X}(\mathcal{O})$ there exists a unique $e \in \mathrm{Sec}(\mathcal{D})$ descending to u, $d\rho \circ e = u \circ \rho$. It follows that $[\Upsilon, e] = b\Upsilon$, where $b \in C^\infty(M)$ with $\langle b \rangle = 0$. Defining $\mathrm{hor}(u) := e - \mathcal{S}(b)e$, by property (2.3.9), we get that $[\Upsilon, \mathrm{hor}(u)] = 0$. Therefore, we have the splitting $TM = \mathrm{Hor} \oplus \mathrm{Ver}$ (a principal connection on M), where the horizontal subbundle $\mathrm{Hor} = \mathrm{Span}\{\mathrm{hor}(u) \mid u \in \mathfrak{X}(\mathcal{O})\}$ is invariant with respect to the $\mathbb{S}^1$-action (for more details, see Section 3.2 and [47]). According to this splitting, the vector field $\bar{W}$ has the decomposition $\bar{W} = \bar{W}^{\mathrm{hor}} + \bar{W}^{\mathrm{ver}}$ into horizontal and vertical parts. The following statement shows that under an appropriate choice of $Y \in \mathfrak{X}_{\mathrm{inv}}(M)$, we can get $\bar{W}^{\mathrm{ver}} = 0$.

Proposition 3.1.2 *If*

$$d\omega \neq 0 \ \text{on } M, \tag{3.1.9}$$

then one can choose an $\mathbb{S}^1$-invariant vector field Y (3.1.12) in a such way that the near identity transformation Φ_ε (3.1.1) brings the perturbed vector field $P_\varepsilon = X+\varepsilon W$ into the form $\tilde{P}_\varepsilon = (\Phi_\varepsilon)^* P_\varepsilon = \tilde{P}_\varepsilon^{\mathrm{hor}} + \tilde{P}_\varepsilon^{\mathrm{ver}}$ with

$$\tilde{P}_\varepsilon^{\mathrm{ver}} = X + O(\varepsilon^2), \tag{3.1.10}$$

$$\tilde{P}_\varepsilon^{\mathrm{hor}} = \varepsilon\, \mathrm{hor}(w) + O(\varepsilon^2), \tag{3.1.11}$$

where $w \in \mathfrak{X}(\mathcal{O})$ is a unique vector field such that $d\rho \circ \langle W \rangle = w \circ \rho$.

Proof. First, let us assume that $\mathcal{O}$ is parallelizable and pick a basis of global vector fields $u_1, ..., u_n$ on $\mathcal{O}$. Then, we have the basis of global $\mathbb{S}^1$-invariant vector fields Υ, $\mathrm{hor}(u_1), ..., \mathrm{hor}(u_n)$ on M. For the perturbation vector field W, we have the decomposition $W = W^{\mathrm{hor}} + W^{\mathrm{ver}}$, where $W^{\mathrm{hor}} = \sum_{i=1}^n c_i \,\mathrm{hor}(u_i)$ and $W^{\mathrm{ver}} = c_0 \Upsilon$ for some $c_i \in C^\infty(M)$. Then, its $\mathbb{S}^1$-average is given by

$$\langle W \rangle = \sum_{i=1}^n \langle c_i \rangle \,\mathrm{hor}(u_i) + \langle c_0 \rangle \Upsilon$$

It follows that the condition $\bar{W}^{\mathrm{ver}} = 0$ is equivalent to the algebraic equation $\mathbf{i}_Y d\omega = -\langle c_0 \rangle$ for $Y \in \mathfrak{X}_{\mathrm{inv}}(M)$. Under assumption (3.1.9), a solution to this equation is given by

$$Y = -\frac{\langle c_0 \rangle}{a^2} \sum_{i=1}^n a_i \,\mathrm{hor}(u_i), \tag{3.1.12}$$

where $a_i = \mathbf{i}_{\mathrm{hor}(u_i)} d\omega$ are $\mathbb{S}^1$-invariant functions on M and $a^2 = \sum_{i=1}^n a_i^2$. In the general case, the statement follows from the partition of unity argument. ∎

Note that, in terms of the averaged vector field w, the normalization condition (3.1.4) reads $\mathcal{L}_w \omega_{\mathcal{O}} = 0$ on $\mathcal{O}$. In this case, $[X, \mathrm{hor}(w)] = 0$.

3.1.2 Second order normalization

Let $\mathbf{A}_\varepsilon = A_0 + \varepsilon A_1 + \frac{\varepsilon^2}{2} A_2 + O(\varepsilon^3)$ be ε-dependent vector field on M whose unperturbed part A_0 has periodic flow with frequency function $\omega : M \to \mathbb{R}$.

Theorem 3.1.3 *Let*

$$\Phi_\varepsilon := \mathrm{Fl}^t_{Z_0 + tZ_1}\big|_{t=\varepsilon} \tag{3.1.13}$$

be the time-ε flow of the vector field $Z_0 + \varepsilon Z_1$, where

$$
\begin{aligned}
Z_0 &= \frac{1}{\omega}\mathcal{S}(A_1) + \frac{1}{\omega^3}\mathcal{S}^2(\mathcal{L}_{A_1}\omega)A_0 + Y_1, && (3.1.14)\\
Z_1 &= \frac{1}{\omega}\mathcal{S}(A_2 + R_1^D) + \frac{1}{\omega^3}\mathcal{S}^2(\mathcal{L}_{A_2+R_1^D}\omega)A_0 + Y_2, && (3.1.15)
\end{aligned}
$$

*with $R_1^D = \mathcal{L}_{Z_0}^2 A_0 + 2\mathcal{L}_{Z_0}A_1$ and Y_1, Y_2 are $\mathbb{S}^1$-invariant vector fields. Then, for a given open domain $N \subset M$ with compact closure, there exists a constant $\delta > 0$ such that formula (*3.1.13*) defines a near identity transformation $\Phi_\varepsilon : N \to M$ with $\varepsilon \in (-\delta, \delta)$ which takes $\mathbf{A}_\varepsilon$ into the $\mathbb{S}^1$-invariant normal form*

$$
\begin{aligned}
\Phi_\varepsilon^* \mathbf{A}_\varepsilon &= A_0 + \varepsilon(\langle A_1\rangle + (\mathcal{L}_{Y_1}\ln\omega)A_0) + \\
&\quad \frac{\varepsilon^2}{2}\left(\langle A_2\rangle + (\mathcal{L}_{Y_2}\ln\omega)A_0 + B(A_0, A_1) + C(Y_1)\right) + O(\varepsilon^3), \quad (3.1.16)
\end{aligned}
$$

where

$$
\begin{aligned}
B(A_0, A_1) &= \langle \mathcal{L}_{\frac{1}{\omega}S(A_1)}A_1 + \mathcal{L}_{\frac{1}{\omega^3}S^2(\mathcal{L}_{A_1}\omega)A_0}A_1\rangle, && (3.1.17)\\
C(Y_1) &= 2\mathcal{L}_{Y_1}\langle A_1\rangle + \left(\mathcal{L}_{Y_1}^2\ln\omega + (\mathcal{L}_{Y_1}\ln\omega)^2\right)A_0. && (3.1.18)
\end{aligned}
$$

Moreover, if there exists an $\mathbb{S}^1$-invariant vector field Y_1 such that the following conditions

$$
\begin{aligned}
\mathcal{L}_{\langle A_1\rangle}\omega &= 0, && (3.1.19)\\
\frac{1}{2}\mathcal{L}_{\langle A_2\rangle + B(A_0,A_1)}\omega &= \mathcal{L}_{[\langle A_1\rangle, Y_1]}\omega && (3.1.20)
\end{aligned}
$$

hold, then $\Phi_\varepsilon^*\mathbf{A}_\varepsilon$ is in normal form of second order relative to A_0.

Proof. Proposition 1.1.1 asserts that for every vector field Z on N , there exists a constant $\delta > 0$ such that the mapping (3.1.13) defines a near identity transformation on N for $\varepsilon \in (-\delta, \delta)$. Let

$$\widetilde{\mathbf{A}}_\varepsilon \stackrel{\text{def}}{=} \Phi_\varepsilon^*\mathbf{A}_\varepsilon = A_0 + \varepsilon\widetilde{A}_1 + \frac{\varepsilon^2}{2}\widetilde{A}_2 + O(\varepsilon^3). \quad (3.1.21)$$

From Proposition 1.2.6, we get that vector fields Z_0, Z_1, $\widetilde{A}_2$ and $\widetilde{A}_1$ satisfy the homological equations

$$\mathcal{L}_{A_0}Z_0 = A_1 - \widetilde{A}_1. \quad (3.1.22)$$

and

$$\mathcal{L}_{A_0}Z_1 = A_2 + R_1^D - \widetilde{A}_2, \quad (3.1.23)$$

where $R_1^D = \mathcal{L}_{Z_0}^2 A_0 + 2\mathcal{L}_{Z_0}A_1$. Since the flow of A_0 is periodic, Proposition 2.4.5 tell us that general solution $(Z_0, \widetilde{A}_1)$ of (3.1.22) is given by (3.1.2) and

$$\widetilde{A}_1 = \langle A_1\rangle + \mathcal{L}_{Y_1}(\ln\omega)A_0, \quad (3.1.24)$$

where Y_1 is an $\mathbb{S}^1$-invariant vector field. Also, by Proposition 2.4.5 , general solution $(Z_1, \widetilde{A}_2)$ is given by (3.1.15) and

$$\widetilde{A}_2 = \langle A_2\rangle + \mathcal{L}_{Y_2}(\ln\omega)A_0 + \langle R_1^D\rangle, \quad (3.1.25)$$

with Y_2 an $\mathbb{S}^1$-invariant vector field. By direct computations, we have

$$\begin{aligned}\langle \mathcal{L}^2_{Z_0} A_0\rangle &= \langle \mathcal{L}_{Z_0}(\mathcal{L}_{Z_0} A_0)\rangle,\\ &= \langle \mathcal{L}_{Z_0}(\langle A_1\rangle + \mathcal{L}_{Y_1}(\ln\omega)A_0 - A_1)\rangle,\\ &= \mathcal{L}_{Y_1}\langle A_1\rangle + \mathcal{L}_{Y_1}((\mathcal{L}_{Y_1}\ln\omega)A_0) - \langle \mathcal{L}_{Z_0} A_1\rangle,\\ &= \mathcal{L}_{Y_1}\langle A_1\rangle + \left(\mathcal{L}^2_{Y_1}\ln\omega + (\mathcal{L}_{Y_1}\ln\omega)^2\right)A_0 - \langle \mathcal{L}_{Z_0} A_1\rangle.\end{aligned}$$

and

$$\langle \mathcal{L}_{Z_0} A_1\rangle = \langle \mathcal{L}_{\frac{1}{\omega}S(A_1)} A_1 + \mathcal{L}_{\frac{1}{\omega^3}S^2(\mathcal{L}_{A_1}\omega)A_0} A_1\rangle + \mathcal{L}_{Y_1}\langle A_1\rangle.$$

Thus, we get

$$\begin{aligned}\widetilde{A}_2 \;=\; & \langle A_2\rangle + (\mathcal{L}_{Y_2}\ln\omega)A_0 + \langle \mathcal{L}_{\frac{1}{\omega}S(A_1)} A_1 + \mathcal{L}_{\frac{1}{\omega^3}S^2(\mathcal{L}_{A_1}\omega)A_0} A_1\rangle\\ & +2\mathcal{L}_{Y_1}\langle A_1\rangle + \left(\mathcal{L}^2_{Y_1}\ln\omega + (\mathcal{L}_{Y_1}\ln\omega)^2\right)A_0.\end{aligned}$$

Finally, Theorem 3.1.1 claims that $\Phi^*_\varepsilon \mathbf{A}_\varepsilon$ is in normal form of first order if and only if A_1 and ω are compatible by

$$\mathcal{L}_{\langle A_1\rangle}\omega = 0.$$

It follows from Corollary 2.4.6 that $\Phi^*_\varepsilon \mathbf{A}_\varepsilon$ is in normal form of second order if and only if there exist $\mathbb{S}^1$-invariant vector field Y_1 such that

$$\mathcal{L}_{\langle A_2\rangle}\omega + \mathcal{L}_{B(A_0,A_1)}\omega + \mathcal{L}_{C(Y_1)}\omega = 0. \tag{3.1.26}$$

Since $\mathcal{L}_{\left(\mathcal{L}^2_{Y_1}\ln\omega + (\mathcal{L}_{Y_1}\ln\omega)^2\right)A_0}\omega = 0$, equation (3.1.26) is reduced to (3.1.20) ■

Actually, the ε-dependent vector field $\mathbf{A}_\varepsilon = A_0 + \varepsilon A_1 + \ldots + O(\varepsilon^k)$ admits an $\mathbb{S}^1$-invariant normalization of arbitrary order.

3.1.3 The regular Hamiltonian case

Let (M,σ) be a symplectic manifold. Assume that a perturbed vector field $\mathbf{A}_\varepsilon = X_{H_\varepsilon} = X_{H_0} + \varepsilon X_{H_1} + \ldots$ is Hamiltonian relative to the symplectic form σ. Assume that the flow of X_{H_0} is periodic with frequency function ω and the regular set $\mathrm{reg}(X_{H_0})$ is dense in M. In this case, we have the following classical result [15].

Proposition 3.1.4 *A canonical near identity transformation Φ_ε on (M,σ) brings the perturbed Hamiltonian vector field V_{H_ε} to the Hamiltonian normal form relative to X_{H_0} of arbitrary order in ε. In particular, the second order normal form is*

$$H_\varepsilon\Phi_\varepsilon = H_0 + \varepsilon\langle H_1\rangle + \frac{\varepsilon^2}{2}\left(\langle H_2\rangle + \langle\{S\left(\frac{H_1}{\omega}\right), H_1\}\rangle\right) + O(\varepsilon^3).$$

The corresponding infinitesimal generator of Φ_ε is a Hamiltonian vector field relative to σ and the function

$$\frac{1}{\omega}\mathcal{S}(H_1) + \varepsilon\left(\frac{1}{\omega}\mathcal{S}(H_2 + \{\frac{1}{\omega}\mathcal{S}(H_1), H_1 + \langle H_1\rangle\})\right).$$

Here $\{\ ,\ \}$ denotes the Poisson bracket on M associated to the symplectic form σ.

3.2 The Averaging Theorem on Riemannian Manifolds

Here, we generalize the classical averaging theorem [7, 62, 66] to the case of general Riemannian manifolds. Consider the so-called the one-frequency system on the cylinder $M = \mathbb{S}^1 \times \mathbb{R}^n = \{\varphi(\mathrm{mod} 2\pi)\} \times \{\mathbf{y} = (y_1, y_2, \dots, y_n)\}$ (equipped with the (flat) product Riemannian metric):

$$\begin{aligned} \dot{\varphi} &= \omega(x) + \varepsilon A^V(\varphi, x), \\ \dot{x} &= \varepsilon A^H(\varphi, x). \end{aligned}$$

Let $(\varphi(t), x(t))$ be a solution of this system. The classical averaging theorem [66] asserts that for small enough ε, the dynamics of the slow variable $x(t)$ is ε-close to the dynamics of the averaged system

$$\dot{y} = \varepsilon \langle A^H \rangle (y),$$

$$\langle A^H \rangle (y) = \frac{1}{2\pi} \int_0^{2\pi} A^H(\varphi, y) d\varphi$$

on the long time scale $t \sim \frac{1}{\varepsilon}$. The proof of this fact is based on the following arguments: (i) a near identity transformation whose infinitesimal generator is a solution of the homological equation; (ii) the triangle inequality and (iii) the Gronwall lemma. Geometrically, the last argument is related to the fact that the curves $((\varphi(0), x(t)))$ and $((\varphi(0), y(t)))$ can be connected by a minimal geodesic in $\mathbb{S}^1 \times \mathbb{R}^n$. In the case of general Riemannian manifolds this property is not necessarily hold. Our approach is based on the idea of including the "perturbed" trajectory and the averaged trajectory into a parameterized surface which consists of the trajectories of a family of vector fields.

3.2.1 Basic facts from Riemannian geometry

Arc Length of Curves and Distance Function. Let $(M, <,>)$ be a smooth connected Riemannian manifold, that is, a smooth manifold M equipped with a (Riemannian) metric $<,>$ on M assigning an interior product $<,>_m$ in each tangent space $T_m M$ at $m \in M$. Recall that the arc length of every smooth curve $c : [0, 1] \to M$ is defined by

$$L(c) := \int_0^1 (< \dot{c}(s), \dot{c}(s) >)^{\frac{1}{2}} ds = \int_0^1 \|\dot{c}(s)\| ds,$$

where $\dot{c}(s) = \frac{d}{ds} c(s) \in T_{c(s)} M$ is the tangent vector. For piecewise smooth curves, the length is defined by taking it for the smooth pieces and then by summing up over all the pieces. For some $p, q \in M$, consider the set $\mathrm{path}(p, q)$ of all piecewise smooth curves on M that begin at p and end at q. Then the distance function $\mathrm{dist} : M \times M \to \mathbb{R}$ is given by

$$\mathrm{dist}(p, q) = \inf\{l \in \mathbb{R} \mid l = L(c) \text{ and } c \in \mathrm{path}(p, q)\}$$

Remark that, in general, the distance $\mathrm{dist}(p, q)$ is not necessarily realized as the length of a curve in $\mathrm{path}(p, q)$. An important fact is that ([52]), (M, dist) is a metric space and the induced topology coincides with the manifold topology on M. Moreover, by the Hopf-Rinov theorem, (M, dist) is a complete metric space (Cauchy sequences converge) if and only if each closed and bounded subset of M is compact. On the complete connected Riemannian manifold any two points can be connected by a geodesic of minimal length.

For a submanifold N of M, we denote by dist_N the distance function on N induced by the restriction of the Riemannian metric $<,>$ to N.

The basis fact in Riemannian geometry, says that for a given Riemannian manifold $(M, <,>)$ there exists the Levi-Civita connection ∇ on $(M, <,>)$, that is, a $\mathbb{R}$-linear map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the conditions

$$\nabla_{fY} X = f \nabla_Y X, \tag{3.2.1}$$

$$\nabla_Y fX = (\mathcal{L}_Y f)X + f \nabla_Y X, \tag{3.2.2}$$

$$\mathcal{L}_Z \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle, \tag{3.2.3}$$

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{3.2.4}$$

for any $X, Y, Z \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

From the basic properties of Levi-Civita connection, $(\nabla_Y X)(m)$ depends only on the value of Y at m and not on the variations of Y around m. Therefore, to each vector field X on M and a point $m \in M$ one can associate the $\mathbb{R}$-linear operator $(\nabla X)_m : T_m M \to T_m M$ given by

$$(\nabla X)_m(v) = (\nabla_Y X)(m)$$

for every $v \in T_m M$. Here, Y is an arbitrary vector field such that $Y(m) = v$. By $\| (\nabla X)_m \|$ we will denote the operator norm on $(T_m M, <,>_m)$. Therefore, we have the vector bundle morphism $\nabla X : TM \to TM$. The covariant derivative ∇_Y along Y is a differential operator of local type which is related with the given Riemannian metric and torsion free.

For every diffeomorphism $g : M \to M$ one can defined the push-forward $g_* \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ of the covariant derivative ∇ by

$$(g_* \nabla)_{g_* Y} g_* X = g_*(\nabla_Y X)$$

for all $X, Y \in \mathfrak{X}(M)$. Remark that if g is an isometry, then g preserves the connection ∇, that is, $g_* \nabla = \nabla$.

Lemma 3.2.1 *Let g be an isometry on $(M, <,>)$ and $X \in \mathfrak{X}(M)$ a vector field. Then, $\| g_* X \|_m = \|X\|_{g^{-1}(m)}$ and for the vector bundle morphisms $\nabla(g_* X)$ and ∇X we have*

$$\|\nabla(g_* X)\|_m = \|\nabla X\|_{g^{-1}(m)}. \tag{3.2.5}$$

Proof. Since g is an isometry, we have that $\nabla_{g_* Y} g_* X = g_*(\nabla_Y X)$ and hence

$$\nabla_v g_* X = (d_{g^{-1}(m)} g)(\nabla_{d_m g^{-1}(v)} X)$$

for every $v \in T_m M$. It follows that

$$\| (\nabla g_* X)(v) \|_m = \| \nabla X(d_m g^{-1}(v)) \|_{g^{-1}(m)} .$$

This equality together with $\| d_m g^{-1}(v) \|_{g^{-1}(m)} = \| v \|_m$ implies (3.2.5). ■

The *curvature* of the Levi-Civita connection ∇ is given by

$$\mathrm{Curv}(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$. A Riemannian manifold $(M, <,>)$ is called *flat* if $\mathrm{Curv} = 0$.

Locally, the linear connection ∇ is described by the connection form. Let $\{e_i\}$ be a basis of local vector fields on $U \subset M$. Then, a matrix-valued 1-form $\theta = (\theta_i^j)$ on U defined by

$$\nabla e_i = \theta_i^j e_j$$

is called the connection form.

Let $[0,1] \ni s \mapsto \gamma(s) \in M$ be a smooth parameterized curve. A smooth function $\mathbf{v} : [0,1] \ni s \mapsto \mathbf{v}(s) \in T_{\gamma(s)} M$ is called a *vector field along* γ. Locally,

$$\mathbf{v}(s) = v^j(s) e_j(\gamma(s)).$$

One can associate to $\mathbf{v}$ another vector field $\nabla_{\dot\alpha} \mathbf{v}$ along γ given by

$$(\nabla_{\frac{d\gamma}{ds}} \mathbf{v})(s) = (\frac{dv^j(s)}{ds} + \theta_i^j \left(\frac{d\gamma}{ds}\right) v^i(s)) e_j(\gamma(s)). \tag{3.2.6}$$

Using the transition rule for the connection form, one can show that this definition is independent of the choice of local trivialization $\{e_i\}$. Remark that , for any smooth function $f : [0,1] \ni t \mapsto f(t) \in \mathbb{R}$, we have $\nabla_{\frac{d\gamma}{ds}} f\mathbf{v} = \frac{df}{ds}\mathbf{v} + f \nabla_{\frac{d\gamma}{ds}} \mathbf{v}$. Moreover, if $Y : M \to TM$ is a smooth vector field on M, then the composition $Y \circ \gamma$ is a vector field along γ and

$$(\nabla_{\frac{d\gamma}{ds}} (Y \circ \gamma))(s) = (\nabla Y)_{\gamma(s)} \left(\frac{d\gamma}{ds}\right). \tag{3.2.7}$$

The *parallel transport* is an isomorphism $\mathcal{P}_\gamma(s) : T_{\gamma(0)} M \to T_{\gamma(s)} M$ given by

$$\mathcal{P}_\gamma(s) e_i(\gamma(0)) = \sum_j P_i^j(s) e_j(\gamma(s))$$

where the matrix function $P(s) = (P_i^j(s))$ is a solution of the problem

$$\frac{dP}{ds} + \theta\left(\frac{d\gamma}{ds}\right) P = 0, \tag{3.2.8}$$

$$P(0) = I. \tag{3.2.9}$$

The parallel transport is an isometry of tangent spaces

$$\langle \mathcal{P}_\gamma(s) a, \mathcal{P}_\gamma(s) b \rangle_{\gamma(s)} = \langle a, b \rangle_{\gamma(0)} \tag{3.2.10}$$

for all $a, b \in T_{\gamma(0)} M$. Let $\mathbf{w} : [0,1] \ni s \mapsto \mathbf{w}(s) \in T_{\gamma(s)} M$ be a vector field along γ given by

$$\mathbf{w} = \nabla_{\frac{d\gamma}{ds}} \mathbf{v} \tag{3.2.11}$$

Let us think of this relation as an equation of $\mathbf{v}$ for a given $\mathbf{w}$.

Lemma 3.2.2 *The solution to (*3.2.11*) is given by the formula*

$$\mathbf{v}(s) = \mathcal{P}_\gamma(s)\left(\int_0^s \mathcal{P}_\gamma^{-1}(\tau)\mathbf{w}(\tau)d\tau\right) + \mathcal{P}_\gamma(s)\mathbf{v}(0). \tag{3.2.12}$$

Proof. In coordinates, the equation (3.2.11) is written as

$$\frac{dv(s)}{ds} + \theta\left(\frac{d\gamma}{ds}\right)v(s) = w(s). \tag{3.2.13}$$

It follows from (3.2.8),(3.2.9) that the solution to this equation is given by the formula

$$v(s) = P(s)\int_0^s P^{-1}(\tau)w(\tau)d\tau + P(s)v(0). \tag{3.2.14}$$

■

Recall that a vector field $\mathbf{v}$ along γ is called parallel if $\nabla_{\frac{d\gamma}{ds}}\mathbf{v} = 0$. It follows from (3.2.12), the vector field is of the form $\mathbf{v}(s) = \mathcal{P}_\gamma(s)\mathbf{v}(0)$. The curve γ is said to be *geodesic* if the tangent vector field $\frac{d\gamma}{ds}$ is parallel, $\nabla_{\frac{d\gamma}{ds}}\frac{d\gamma}{ds} = 0$.

Lemma 3.2.3 *Let* $\mathbf{w}$ *be an arbitrary vector field along* γ *and* $s \mapsto \mathbf{v}\ (s) \in T_{\gamma(s)}M$ *be the solution of the problem*

$$\nabla_{\frac{d\gamma}{ds}}\mathbf{v} = \mathbf{w},$$

$$\mathbf{v}(0) = \mathbf{v}^0 \in T_{\gamma(0)}M.$$

Then,

$$\| \mathbf{v}(s) \|_{\gamma(s)} \leq \| \mathbf{v}^0 \|_{\gamma(0)} + \int_0^s \| \mathbf{w}(\tau) \|_{\gamma(\tau)}\, d\tau. \tag{3.2.15}$$

Proof. Using (3.2.10) and (3.2.12), we get

$$\begin{aligned}
\| \mathbf{v}(s) \|_{\gamma(s)} &\leq \| \mathcal{P}_\gamma(s)\int_0^s \mathcal{P}_\alpha^{-1}(\tau)\mathbf{w}(\tau)d\tau \|_{\gamma(s)} + \| \mathcal{P}_\gamma(s)\mathbf{v}^0 \|_{\gamma(s)}, \\
&\leq \int_0^s \| \mathcal{P}_\gamma^{-1}(\tau)\mathbf{w}(\tau) \|_{\gamma(0)}\, d\tau + \| \mathbf{v}^0 \|_{\gamma(0)}, \\
&= \int_0^s \| \mathbf{w}(\tau) \|_{\gamma(\tau)}\, d\tau + \| v^0 \|_{\gamma(0)}\ .
\end{aligned}$$

■

Corollary 3.2.4 *Let $[0,1] \ni s \mapsto \gamma(s) \in M$ be a smooth parameterized curve and X be a vector field on M. Then,*

$$\| X(\gamma(s)) - \mathcal{P}_\gamma(s) X(\gamma(0)) \|_{\gamma(s)} \leq \int_0^s \| (\nabla X)_{\gamma(s')} \| \| \frac{d\gamma}{ds'} \| \, ds'. \tag{3.2.16}$$

Proof. Consider the following vector fields along γ:

$$\mathbf{v}(s) = X(\gamma(s)) - \mathcal{P}_\gamma(s) X(\gamma(0),$$

$$\mathbf{w}(s) = (\nabla X)_{\gamma(s)}(\frac{d\gamma}{ds}).$$

Taking into account that the vector field $\mathcal{P}_\gamma(s)X(\gamma(0))$ is parallel and using (3.2.7), we get that $\nabla_{\frac{d\gamma}{ds}} \mathbf{v} = \mathbf{w}$ with $\mathbf{v}(0) = 0$. Applying (3.2.15) leads to inequality (3.2.16). ■

The Arc Length on Parameterized Surfaces. Now suppose we are given smooth function

$$\gamma : [0,T] \times [0,1] \ni (t,s) \mapsto \gamma(t,s) \in M, \tag{3.2.17}$$

called a *parameterized surface*. Introduce the following s-dependent vector fields along the curve $t \mapsto \gamma_s(t) = \gamma(t,s)$

$$\mathbf{v}_s(t) := \frac{\partial \gamma(t,s)}{\partial s} \tag{3.2.18}$$

and

$$\mathbf{w}_s(t) := \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}. \tag{3.2.19}$$

By the torsion free condition (4), we have the identity

$$\nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} = \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma}{\partial s} \tag{3.2.20}$$

which is rewritten as

$$\nabla_{\frac{\partial \gamma}{\partial t}} \mathbf{v}_s(t) = \mathbf{w}_s(t). \tag{3.2.21}$$

It follows from here and Lemma 3.2.3 that

$$\left\| \frac{\partial \gamma(t,s)}{\partial s} \right\|_{\gamma(t,s)} \leq \left\| \frac{\partial \gamma(0,s)}{\partial s} \right\|_{\gamma(0,s)} + \int_0^t \left\| \nabla_{\frac{\partial \gamma}{\partial t}} \frac{\partial \gamma(\tau,s)}{\partial s} \right\|_{\gamma(\tau,s)} d\tau. \tag{3.2.22}$$

For every $t \in [0,T]$, denote by $L(t)$ the length of the curve $[0,1] \ni s \mapsto \gamma_t(s) = \gamma(t,s)$,

$$L(t) = \int_0^1 \left\| \frac{\partial \gamma(t,s)}{\partial s} \right\|_{\gamma_t(s)} ds.$$

Integrating both sides of (3.2.22) in s , we get the following result.

Lemma 3.2.5 *(Basic Inequality) For all $t \in [0, T]$, the length $L(t)$ of the s-curve $s \mapsto \gamma_t(s)$ satisfies the inequality*

$$L(t) \leq L(0) + \int_0^t \int_0^1 \left\| \nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t'} \right\|_{\alpha(t',s)} ds dt'. \tag{3.2.23}$$

We will need also the following technical fact. Let Y_s ($s \in [0,1]$) be a s-dependent vector field on M. Then, we associate to Y_s the vector field $(t,s) \mapsto \mathbf{y}(t,s) \in T_{\gamma(t,s)}M$ along the parameterized surface $\gamma(t,s)$ given as

$$\mathbf{y}(t,s) := Y_s(\gamma(t,s)).$$

Recall that $\frac{dY_s}{ds}$ denotes the s-dependent vector field on M defined by the condition $\mathcal{L}_{\frac{dY_s}{ds}} f = \frac{d}{ds}(\mathcal{L}_{Y_s} f)$ for any $f \in C^\infty(M)$.

Lemma 3.2.6 *For every $t \in [0,T]$, the vector field $s \mapsto \mathbf{y}_t(s) = \mathbf{y}(t,s)$ along the s-curve $s \mapsto \gamma_t(s)$ satisfies the relation*

$$\nabla_{\frac{\partial \gamma}{\partial s}} \mathbf{y}_t = \frac{dY_s}{ds}(\gamma(t,s)) + (\nabla Y_s)_{\gamma(t,s)}(\frac{\partial \gamma}{\partial s}). \tag{3.2.24}$$

Here $(\nabla Y_s)_m : T_m M \to T_m M$ is the linear operator associated to the vector field Y_s.

Proof. Locally, $Y_s = Y_s^i e_i$ and

$$\nabla Y_s = (dY_s^i + Y_s^j \theta_j^i) \otimes e_i. \tag{3.2.25}$$

On the other hand, $\mathbf{y}_t(s) = Y_s^i(\alpha(t,s)) e_i(\alpha(t,s))$ and by definition (3.2.6), we have

$$\nabla_{\frac{\partial \gamma}{\partial s}} \mathbf{y}_t = (\frac{dY_s^i}{ds} + dY_s^i \left(\frac{\partial \gamma}{\partial s} \right) + Y_s^j \theta_j^i (\frac{\partial \gamma}{\partial s})) e_i.$$

Comparing this identity with (3.2.25) leads to (3.2.24). ∎

Flat parameterized surfaces of geodesics. Suppose that we are given a parameterized surface

$$\gamma : [0,T] \times [0,1] \ni (t,s) \mapsto \gamma(t,s) \in M,$$

such that, for every $t \in [0,T]$, the curve $[0,1] \ni s \mapsto \gamma_t(s) = \gamma(t,s)$ is a *geodesic.* Then, the vector field $\nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t}$ along the surface satisfies the Jacobi equation

$$\nabla_{\frac{\partial \gamma}{\partial s}} \left(\nabla_{\frac{\partial \gamma}{\partial s}} \frac{\partial \gamma}{\partial t} \right) = \operatorname{Curv} \left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t} \right) \frac{\partial \gamma}{\partial s}. \tag{3.2.26}$$

Denote by $\mathcal{P}_{s_1}^{s_2}(\gamma_t) : T_{\gamma_t(s_1)} M \to T_{\gamma_t(s_2)} M$ the parallel transport along the segment γ_t $([s_1, s_2])$.

Assume also that there exist two vector fields X_0 and X_1 on M such that

$$\frac{\partial \gamma(t,0)}{\partial t} = X_0(\gamma(t,0)), \tag{3.2.27}$$

and

$$\frac{\partial \gamma(t,1)}{\partial t} = X_1(\gamma(t,1)). \tag{3.2.28}$$

Lemma 3.2.7 *If the curvature of the Riemannian connection ∇ on M vanishes along the parameterized surface γ,*

$$\mathrm{Curv}\left(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}\right) = 0, \tag{3.2.29}$$

then $\nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial t}$ is parallel vector field along the geodesic $s \mapsto \gamma_t(s)$ which is given by the formula

$$\nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial t} = (\mathcal{P}_s^1(\gamma_t))^{-1}\left[X_1(\gamma_t(1)) - \mathcal{P}_0^1(\gamma_t)X_0(\gamma_t(0))\right]. \tag{3.2.30}$$

Proof. It follows from assumption (3.2.29) and equation (3.2.26) that

$$\nabla_{\frac{\partial\gamma}{\partial s}}\left(\nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial t}\right) = 0$$

and hence

$$\frac{\partial\gamma}{\partial t} = s\mathbf{a}(t,s) + \mathbf{b}(t,s),$$

where $\mathbf{a}$ and $\mathbf{b}$ are vector field along the surface γ which are parallel along the curve $s \mapsto \gamma_t(s)$,

$$\nabla_{\frac{\partial\gamma}{\partial s}} \mathbf{a} = 0 \text{ and } \nabla_{\frac{\partial\gamma}{\partial s}} \mathbf{b} = 0.$$

In terms of the parallel transport we have the representations

$$\mathbf{a}(t,s) = (\mathcal{P}_s^1(\gamma_t))^{-1}\mathbf{a}(t,1),$$

$$\mathbf{b}(t,s) = \mathcal{P}_0^s(\gamma_t)\mathbf{b}(t,0) = (\mathcal{P}_s^1(\gamma_t))^{-1}\mathbf{b}(t,1).$$

From here and conditions (3.2.27), (3.2.28) we derive

$$\mathbf{b}(t,0) = X_0(\gamma(t,0)),$$

$$\mathbf{a}(t,1) + \mathcal{P}_0^1(\gamma_t)\mathbf{b}(t,0) = X_1(\gamma(t,1)).$$

Therefore,

$$\nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial t} = \mathbf{a}(t,s) = (\mathcal{P}_s^1(\gamma_t))^{-1}\left[X_1(\gamma(t,1)) - \mathcal{P}_0^1(\gamma_t)X_0(\gamma(t,0))\right].$$

■

Corollary 3.2.8 *Under hypotheses (3.2.27), (3.2.28), the tangent vector to the geodesic $s \mapsto \gamma_t(s)$ at $s = 0$ is given by*

$$\frac{\partial\gamma}{\partial s}(t,s) = \mathcal{P}_0^t(\gamma_s)[\int_0^t (\mathcal{P}_0^\tau(\gamma_s))^{-1}\mathbf{a}(\tau,s)d\tau + \frac{\partial\gamma}{\partial s}(0,s)].$$

Corollary 3.2.9 *For every $t \in [0,T]$, $L(t)$ denotes the arc length of the geodesic $s \mapsto \gamma_t(s)$,*

$$L(t) = \int_0^1 \left\| \frac{\partial\gamma(t,s)}{\partial s} \right\| ds = \left\| \frac{\partial\gamma(t,1)}{\partial s} \right\| .$$

The following inequality holds

$$L(t) \leq \| \frac{\partial\gamma(0,0)}{\partial s} \| + \int_0^t \left(\int_0^1 \| \nabla X_0 \|_{\gamma(t',s)} \, ds \right) L(t')dt' \tag{3.2.31}$$
$$+ \int_0^t \| X_1 - X_0 \|_{\gamma(t',1)} \, dt'.$$

Proof. Rewriting representation (3.2.30) in the form

$$\nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial t} = (\mathcal{P}_s^1(\gamma_t))^{-1} \left[X_0(\gamma(t,1)) - (\mathcal{P}_0^1(\gamma_t))X_0(\gamma(t,0)) + (X_1 - X_0)(\gamma(t,1)) \right] .$$

and using (3.2.16), we get

$$\| \nabla_{\frac{\partial\gamma}{\partial s}} \frac{\partial\gamma}{\partial t} \| \leq \int_0^1 \| \nabla X_0 \|_{\gamma(t,s)}) \| \, ds \, \| \frac{\partial\gamma(t,1)}{\partial s} \|$$
$$+ \| X_1 - X_0 \|_{\gamma(t,1)} .$$

This together with inequality (3.2.23) leads to (3.2.31). ∎

3.2.2 Gronwall's type estimates for flows on Riemannian manifolds

The Gronwall type estimates play an important role in the perturbation theory for dynamical systems. Recall that the (specific) Gronwall lemma is formulated as follows (see, for example [66]). Suppose that for $t_0 \leq t \leq t_0 + T$, we have

$$\varphi(t) \leq \delta_2(t - t_0) + \delta_1 \int_{t_0}^t \varphi(\tau)d\tau + \delta_3,$$

with $\varphi(t)$ continuous and constants $\delta_1 > 0$, $\delta_2 \geq 0$, and $\delta_3 \geq 0$. Then,

$$\varphi(t) \leq \left(\frac{\delta_2}{\delta_1} + \delta_3 \right) e^{\delta_1(t-t_0)} - \frac{\delta_2}{\delta_1}$$

for $t_0 \leq t \leq t_0 + T$. Using this fundamental inequality, we get some estimates for the time evolution of the distance between points of trajectories of two vector fields.

First, let us consider the special case where the parameterized surface which comes from trajectories of a parameter dependent vector field . Suppose we start with a 1-parameter family of vector fields X_s on M smoothly depending on the parameter $s \in [0,1]$. Let $[0,1] \ni s \mapsto \beta(s) \in M$ be a smooth curve. Then, one can fix $T > 0$ such that for every $s \in [0,1]$, the trajectory $t \mapsto \mathrm{Fl}^t_{X_s}(\beta(s))$ is defined for all $t \in [0,T]$. Consider the parameterized surface of the form

$$\alpha(t,s) := \mathrm{Fl}^t_{X_s}(\beta(s)). \tag{3.2.32}$$

Therefore, for every t, the s-curve $s \mapsto \alpha_t(s)$ is a result of the time evolution of the "initial" curve $\beta(s)$ under the flow of X_s.

Proposition 3.2.10 *The length $L(t)$ of the s-curve $s \mapsto \alpha_t(s)$ on the parameterized surface (3.2.32) satisfies the Gronwall type estimate*

$$L(t) \leq \left(\frac{C_2}{C_1} + C_3\right) e^{C_1 t} - \frac{C_2}{C_1} \tag{3.2.33}$$

for all $t \in [0, T]$. Here

$$C_1 = \sup_{\substack{m \in \alpha([0,T]\times[0,1]) \\ s \in [0,1]}} \|(\nabla X_s)_m\|,$$

$$C_2 = \sup_{\substack{t \in [0,T] \\ s \in [0,1]}} \left\| \frac{dX_s}{ds} \right\|_{\alpha(t,s)},$$

$$C_3 = L(0).$$

Proof. Formula (3.2.24) implies

$$\left\| \nabla_{\frac{\partial \alpha}{\partial s}} \frac{\partial \alpha}{\partial t} \right\|_{\alpha(t,s)} = \left\| \frac{dX_s}{ds}(\alpha(t,s)) + (\nabla X_s)_{\alpha(t,s)}(\frac{\partial \alpha}{\partial s}) \right\|_{\alpha(t,s)}$$
$$\leq \left\| \frac{dX_s}{ds} \right\|_{\alpha(t,s)} + \left\|(\nabla X_s)_{\alpha(t,s)}\right\| \left\| (\frac{\partial \alpha}{\partial s}) \right\|_{\alpha(t,s)}.$$

Putting this inequality into (3.2.23), we get

$$L(t) \leq L(0) + C_1 \int_0^t L(\tau) d\tau + C_2 t.$$

By the specific Gronwall lemma, this leads to (3.2.33). ■

Theorem 3.2.11 *Let $(M, <,>)$ be a connected Riemannian manifold and* dist : $M \times M \to \mathbb{R}$ *the corresponding distance function. Let X_0 and X_1 be two vector fields on M and $p, q \in M$ some points. Assume that there exists an open subset $N \subset M$ with compact closure such that*

$$p, q \in N, \tag{3.2.34}$$

$$\mathrm{dist}_N(p,q) = \mathrm{dist}(p,q). \tag{3.2.35}$$

Consider the s-dependent vector field

$$X_s = X_0 + s(X_1 - X_0) \tag{3.2.36}$$

and choose $T > 0$ such that for every $s \in [0,1]$ and $m \in \bar{N}$ the trajectory $t \mapsto$ $\mathrm{Fl}^t_{X_s}(m)$ is defined for all $t \in [0, T]$. Then,

$$\mathrm{dist}(\mathrm{Fl}^t_{X_0}(p), \mathrm{Fl}^t_{X_1}(q)) \leq \left(\frac{c_2}{c_1} + c_3\right) e^{c_1 t} - \frac{c_2}{c_1} \tag{3.2.37}$$

for all $t \in [0, T]$. *Here*

$$c_1 = \sup_{\substack{m \in K_T \\ s \in [0,1]}} \|(\nabla X_s)_m\| \,, \tag{3.2.38}$$

$$c_2 = \sup_{m \in K_T} \|X_1 - X_0\|_m \,, \tag{3.2.39}$$

$$c_3 = \operatorname{dist}(p, q) \tag{3.2.40}$$

and $K_T \subset M$ *is a compact subset given by*

$$K_T = \{\mathrm{Fl}^t_{X_s}(m) \mid m \in \bar{N},\ t \in [0, T],\ s \in [0, 1]\}.$$

Proof. Fix a smooth curve $[0, 1] \ni s \mapsto \beta(s) \in N$ joining p and q, $\beta(0) = p, \beta(1) = q$. Using the flow of the s-dependent vector field (3.2.36), define the parameterized surface $\alpha(t, s) = \mathrm{Fl}^t_{X_s}(\beta(s))$. Let $L(t)$ be the length of the s-curve $s \mapsto \alpha_t(s)$ on the parameterized surface α. In particular, $L(0) = L(\beta)$. By hypotheses (3.2.34), (3.2.35), for any $\Delta > 0$, one can choose the curve β in such a way that $L(0) - \Delta < \operatorname{dist}_N(p, q) = \operatorname{dist}(p, q)$. Then, Proposition 3.2.10 implies

$$\operatorname{dist}(\alpha(t, 0), \alpha(t, 1) \le L(t) \le \left(\frac{c_2}{c_1} + \operatorname{dist}(p, q) + \Delta\right) e^{c_1 t} - \frac{c_2}{c_1}.$$

Since Δ is an arbitrary positive number, taking the limit $\Delta \to 0$ in the right hand side of this inequality, we arrive at (3.2.37). ■

Corollary 3.2.12 *In the case when* $X_1 = X_0$*, under assumptions (*3.2.34*), (*3.2.35*), the estimate (*3.2.37*). takes the form*

$$\operatorname{dist}(\mathrm{Fl}^t_{X_0}(p), \mathrm{Fl}^t_{X_0}(q)) \le \operatorname{dist}(p, q) e^{c_1 t}$$

where

$$c_1 = \sup_{m \in K_T} \|(\nabla X_0)_m\| \,,$$

and $K_T = \{\mathrm{Fl}^t_{X_0}(m) \mid m \in \bar{N}, t \in [0, 1]\}$.

This result was obtained in [44].

Remark 4 *In the case when* $p = q$*, assumptions (*3.2.34*), (*3.2.35*) can be omitted and the estimate (*3.2.37*) is written as*

$$\operatorname{dist}(\mathrm{Fl}^t_{X_0}(p), \mathrm{Fl}^t_{X_1}(p)) \le \left(\frac{c_2}{c_1}\right)(e^{c_1 t} - 1).$$

Notice that conditions (3.2.34), (3.2.35) do not hold in general.

3.2.3 Free $\mathbb{S}^1$-actions and Riemannian submersions

To formulate the main results, we need some facts on the invariant metrics about principal $\mathbb{S}^1$-bundles.

$\mathbb{S}^1$-actions. Let M be a connected manifold and A_0 a vector field on M whose flow $\mathrm{Fl}^t_{A_0}$ is periodic with frequency function $\omega : M \to \mathbb{R}$, $\omega > 0$. Consider the action of the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ associated with infinitesimal generator $\Upsilon = \frac{1}{\omega}A_0$ and assume that this action is *free*. This means that each trajectory of Υ is minimally 2π-periodic. Let $\mathcal{O} = M/\mathbb{S}^1$ be the orbit space of the $\mathbb{S}^1$-action and $\rho : M \to \mathcal{O}$ the natural projection. It follows from well-known properties of free actions of compact Lie groups [24, 49] that there exists a unique manifold structure on $\mathcal{O}$ such that ρ is a smooth surjective submersion (a fiber bundle). Moreover, ρ is a principal $\mathbb{S}^1$-bundle over $\mathcal{O}$. For each $\mathbb{S}^1$-invariant vector field Y on M there exists a unique vector field $Y_{\mathcal{O}}$ on $\mathcal{O}$ which is ρ-relative with Y,

$$d\rho \circ Y = Y_{\mathcal{O}} \circ \rho.$$

and called the *reduced vector field*.

Riemannian submersion and horizontal lifts. Recall that a fiber bundle (M, ρ, B, F) consists of the manifolds M, B, E and a surjective submersion $\rho : M \to B$ such that for each $b \in B$ the set $M|_b := \rho^{-1}(b)$ is diffeomorphic to F. M is called the total space, B the base space, ρ the projection and F the standard fiber. The space

$$\mathbb{V} := \ker \mathrm{d}\rho \tag{3.2.41}$$

is called the vertical subbundle of M. If M is a Riemannian manifold, the horizontal subbundle $\mathbb{H}$ of M can be defined as the orthogonal complement of $\mathbb{V}$,

$$\mathbb{H} := \mathbb{V}^{\perp}. \tag{3.2.42}$$

Thus, for each $m \in M$, the linear transformation

$$\mathrm{d}_m\rho : \mathbb{H}_m \to T_{p(m)}B.$$

is an isomorphism. Hence, we have the decomposition

$$TM = \mathbb{H} \oplus \mathbb{V}.$$

The vector fields on M tangent to $\mathbb{H}$ and $\mathbb{V}$ are called horizontal and vertical vector fields, respectively.

If B is also a Riemannian manifold, p is a Riemannian submersions if $\mathrm{d}_m\rho : \mathbb{H}_m \to T_{p(m)}B$ is an isometric isomorphism for all $m \in M$. For every $v \in \mathfrak{X}(B)$ there exists a vector field $\mathrm{hor}(v) \in \mathfrak{X}M$, called the horizontal lift of v satisfying the conditions

- $\mathrm{hor}(v)(m) \in \mathbb{H}_m$,
- $\mathrm{d}\rho \circ \mathrm{hor}(v) = v \circ \rho.$

hor(v) is well-defined and depends on m smoothly. Let $\gamma : [a,b] \to B$ be a smooth curve on B passing trough the point $\gamma(0) = x \in B$. Let $m \in M_a$ be a point in the fiber over x. A *lift* of γ trough m is a smooth curve $\widetilde{\gamma} : [a,b] \to M$ such that

$$\bullet \quad m = \widetilde{\gamma}(a), \tag{3.2.43}$$

$$\bullet \quad \rho \circ \widetilde{\gamma} = \gamma. \tag{3.2.44}$$

A lift curve is *horizontal* if in addition

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\gamma}(t) \in \mathbb{H}_{\widetilde{\gamma}(t)}, \ \forall t \in [a,b]. \tag{3.2.45}$$

Given any smooth curve $\gamma : [a,b] \to B$ there exists always a local horizontal lift of γ. This means that conditions (3.2.44)-(3.2.45) hold in an interval $[a, a+\varepsilon]$ for some $\varepsilon > 0$. If ρ is a proper surjective submersion then for any $\gamma : [a,b] \to B$ there exists a global horizontal lift $\widetilde{\gamma} : [a,b] \to M$, [52].

Riemannian submersions on the $\mathbb{S}^1$-principal bundle $(M, \rho, \mathcal{O})$. Pick a Riemannian metric $<,>$ on M which is *invariant* with respect to the $\mathbb{S}^1$-action, that is, the flow Fl^t_Υ is an isometry on $(M, <,>)$. Such a $\mathbb{S}^1$-invariant Riemannian metric always exists and can be obtained from an arbitrary Riemannian metric on M by applying the averaging procedure . Indeed, if $\tilde{g}$ is the metric tensor of a given Riemannian metric on M, then formula (see Chapter 2)

$$g := \frac{1}{2\pi} \int_0^{2\pi} (\mathrm{Fl}^t_\Upsilon)^* \tilde{g} dt$$

gives the metric tensor of an $\mathbb{S}^1$-invariant Riemannian metric. Consider the $\mathbb{S}^1$-principal bundle $(M, \rho, \mathcal{O})$ where $\mathcal{O} = M \diagup \mathbb{S}^1$ denotes the orbit space and $\rho : M \to \mathcal{O}$ the canonical projection. The line distribution Span$\{\Upsilon\}$ coincides with the vertical subbundle $\mathbb{V}$ of M. Since the Riemannian metric is $\mathbb{S}^1$-invariant, the horizontal subbundle is also invariant with respect to the $\mathbb{S}^1$-action,

$$(d_m \, \mathrm{Fl}^t_\Upsilon)(\mathbb{H}_m) = \mathbb{H}_{\mathrm{Fl}^t_\Upsilon(m)} \quad \forall m \in M.$$

Therefore, we have the $\mathbb{S}^1$-invariant, orthogonal splitting

$$TM = \mathbb{H} \oplus \mathbb{V}, \tag{3.2.46}$$

and for every vector field Y on M there is a the decomposition $Y = Y^{\mathrm{hor}} + Y^{\mathrm{ver}}$ into horizontal and vertical parts. It is clear that the restriction of the differential $d_m\rho : T_mM \to T_{\rho(m)}\mathcal{O}$ to $\mathbb{H}_m$ is an isomorphism. For every vector field v on $\mathcal{O}$ there exists a unique horizontal lift hor(v) on M which is tangent to $\mathbb{H}$ and such that $d\rho \circ \mathrm{hor}(v) = v \circ \rho$.

Moreover, there exists a unique Riemannian metric $<,>^o$ on the orbit space $\mathcal{O}$ such that the projection ρ is a Riemannian submersion (see [52], page 327),

$$\langle u_1, u_2 \rangle_m = \langle (d_m\rho)u_1, (d_m\rho)u_2 \rangle^o_{\rho(m)} \tag{3.2.47}$$

for any $m \in M$ and $u_1, u_2 \in \mathbb{H}_m$. Denote by $\mathrm{dist}^o : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ the distance function associated to the Riemannian metric $<,>^o$ on $\mathcal{O}$. Denote by ∇ and ∇^o the Riemannian connections on $(M, <,>)$ and $(\mathcal{O}, <,>^o)$, respectively.

Lemma 3.2.13 *Let $\gamma : [0,1] \ni s \mapsto \gamma(s) \in M$ be a smooth curve on M and $\alpha := \rho \circ \gamma : [0,1] \ni s \mapsto \rho(\gamma(s)) \in \mathcal{O}$ its projection to the orbit space. Let $(\frac{d\gamma}{ds})^{\rm hor} \in \mathbb{H}_{\gamma(s)}$ and $(\frac{d\gamma}{ds})^{\rm ver} \in \mathbb{V}_{\gamma(s)}$ are horizontal and vertical components in the orthogonal decomposition*

$$\frac{d\gamma}{ds} = \left(\frac{d\gamma}{ds}\right)^{\rm hor} + \left(\frac{d\gamma}{ds}\right)^{\rm ver}. \tag{3.2.48}$$

Then,

(a) the arc lengths of the curves γ and α

$$L(\gamma) = \int_0^1 \left\|\frac{d\gamma}{ds}\right\| ds, \quad L(\alpha) = \int_0^1 \left\|\frac{d\alpha}{ds}\right\|^o ds$$

satisfy the inequalities

$$L(\alpha) \le L(\gamma) \tag{3.2.49}$$

$$L(\gamma) \le L(\alpha) + \int_0^1 \left\|\left(\frac{d\gamma}{ds}\right)^{\rm ver}\right\| ds \le \sqrt{2} L(\gamma). \tag{3.2.50}$$

The equality $L(\alpha) = L(\gamma)$ holds if and only if the curve γ is horizontal, $(\frac{d\gamma}{ds})^{\rm ver} = 0$.

(b) For any $p, q \in M$, we have

$$\mathrm{dist}^o(\rho(p), \rho(q)) \le \mathrm{dist}(p,q). \tag{3.2.51}$$

Proof. The statement (a) is evident and follows from the relation $\frac{d\alpha}{ds} = (d_{\gamma(s)}\rho)(\frac{d\gamma}{ds})^{\rm hor}$, orthogonal decomposition (3.2.48) and the equality

$$\left\|\frac{d\alpha}{ds}\right\|^o = \left\|\left(\frac{d\gamma}{ds}\right)^{\rm hor}\right\|, \tag{3.2.52}$$

which is consequence of the property that ρ is a Riemannian submersion. To prove the item (b), for arbitrary $p, q \in M$ and $\Delta > 0$, let us choose a curve γ on M joining p with q and such that $\mathrm{dist}(p,q) + \Delta \ge L(\gamma)$. Then, by (3.2.49) we get

$$\begin{aligned} \mathrm{dist}^o(\rho(p), \rho(q)) &\le L(\rho \circ \gamma) \le L(\gamma) \\ &\le \mathrm{dist}(p,q) + \Delta. \end{aligned}$$

Since, $\Delta > 0$ is arbitrary, inequality (3.2.51) is true. ■

Remark 5 *One can suppose that a Riemannian metric on the orbit space is arbitrary because of the following fact. For a given Riemannian metric $\langle , \rangle^o$ on $\mathcal{O}$, there exists a $\mathbb{S}^1$-invariant Riemannian metric $\langle , \rangle$ on M such that the projection ρ is a Riemannian submersion.*

The following statement gives us the key property of the horizontal lift.

Proposition 3.2.14 *Let* $X \in \mathfrak{X}(M)$ *be a vector field and* $\gamma : [0,T] \to M$ *the trajectory of* X *through* $m_0 \in M$, $\gamma(t) = \mathrm{Fl}^t_X(m^0)$.*Consider the projection* $\alpha = \rho \circ \gamma$ *and its horizontal lift* $\tilde{\alpha} : [0,T] \ni t \mapsto \tilde{\alpha}(t) \in M$ *through* m^0, $\tilde{\alpha}(0) = m^0$. *Then, there exists a smooth function* $\tau : [0,T] \to \mathbb{R}$ *such that* $\tau(0) = 0$ *and*

$$\tilde{\alpha}(t) = g^t(\gamma(t)) \tag{3.2.53}$$

where

$$g^t = \mathrm{Fl}^{\tau(t)}_{\Upsilon}. \tag{3.2.54}$$

Moreover, the curve $t \mapsto \tilde{\alpha}(t) \in M$ *is the trajectory through* m_0 *of the horizontal* t*-dependent vector field*

$$\tilde{X}_t = (g^t)_* X^{\mathrm{hor}}, \tag{3.2.55}$$

that is,

$$\frac{d\tilde{\alpha}(t)}{dt} = \tilde{X}_t(\tilde{\alpha}(t)).$$

Moreover, the following properties hold

$$\| \tilde{X}_t \|_{\tilde{\alpha}(t)} = \| X \|_{\gamma(t)} \tag{3.2.56}$$

and

$$\| \nabla_v \tilde{X}_t \|_{\tilde{\alpha}(t)} \leq \| \nabla X^{\mathrm{hor}} \|_{\gamma(t)} \cdot \| v \|_{\tilde{\alpha}(t)} \tag{3.2.57}$$

for every $v \in T_{\tilde{\alpha}(t)}M$.

Proof. By definition, the points $\tilde{\alpha}(t)$ and $\gamma(t)$ belongs to the same fiber $\rho^{-1}(\alpha(t))$ and hence they can be joint by a segment of the periodic trajectory of Υ for the time $\tau = \tau(t)$. Differentiating the both sides of (3.2.53) in t and using the decomposition (3.2.48) give

$$\begin{aligned}\frac{d\tilde{\alpha}(t)}{dt} &= (d_{\gamma(t)}g^t)\frac{d\gamma(t)}{dt} + \tau'(t)\Upsilon(\gamma(t)) \\ &= (d_{\gamma(t)}g^t)X^{\mathrm{hor}}(\gamma(t)) + (d_{\gamma(t)}g^t)X^{\mathrm{ver}}(\gamma(t)) \\ &+ \tau'(t)\Upsilon(\gamma(t)).\end{aligned} \tag{3.2.58}$$

Remark that the flow of Υ is an isometry which preserves splitting (3.2.46) of TM into horizontal and vertical subspaces. Hence, the diffeomorphisms g^t have the same properties. From here and the fact that the velocity $\frac{d\tilde{\alpha}(t)}{dt}$ is a horizontal vector, we deduce from (3.2.58) the relations

$$\frac{d\tilde{\alpha}(t)}{dt} = (d_{\gamma(t)}g^t)X^{\mathrm{hor}}(\gamma(t) \tag{3.2.59}$$

and

$$\tau'(t)\Upsilon(\gamma(t)) = -(d_{\gamma(t)}g^t)X^{\mathrm{ver}}(\gamma(t)).$$

The last formula just defines the function $\tau(t)$. Putting $\gamma(t) = (g^t)^{-1}(\tilde{\alpha}(t))$ into (3.2.59) leads to the relation

$$\begin{aligned}\frac{d\tilde{\alpha}(t)}{dt} &= (d_{(g^t)^{-1}(\tilde{\alpha}(t)}g^t)X^{\mathrm{hor}}((g^t)^{-1}(\tilde{\alpha}(t)) \\ &= (g^t)_* X^{\mathrm{hor}}(\tilde{\alpha}(t))\end{aligned}$$

which says that $\tilde{\alpha}(t)$ is the trajectory through m_0 of the vector field $\tilde{X}_t$ in (3.2.55). Equality (3.2.56) follows from the property that the differential of g^t is a linear isometry and the representation $\tilde{X}_t(m) = (d_m g^t)X((g^t)^{-1}m)$. Finally, applying Lemma 3.2.1, we get

$$\| \nabla_v \tilde{X}_t \|_{\tilde{\alpha}(t)} = \| \nabla_{(d_{\tilde{\alpha}(t)}g^t)^{-1}v} X \|_{\gamma(t)} \leq \| \nabla X \|_{\gamma(t)} \cdot \| v \|_{\tilde{\alpha}(t)}$$

■

3.2.4 A geometric proof of the averaging theorem

Suppose we are start with a perturbed vector field on M

$$\mathbf{A}_\varepsilon = A_0 + \varepsilon A_1,$$

where A_0 is a vector field with periodic flow, frequency function $\omega > 0$ and $A_1 \in \mathfrak{X}(M)$ is a certain vector field. We assume that the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = \frac{1}{\omega}A_0$ is free.

Let $\langle A_1 \rangle$ be the $\mathbb{S}^1$-average of the perturbation vector field A_1 and $\langle A_1 \rangle_{\mathcal{O}}$ the reduced averaged vector field on $\mathcal{O}$,

$$\mathrm{d}\rho \circ \langle A_1 \rangle = \langle A_1 \rangle_{\mathcal{O}} \circ \rho.$$

Assume that the orbit space $\mathcal{O}$ is equipped with a certain metric $<,>^o$. The corresponding distance function is denoted by $\mathrm{dist}^o : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$.

Theorem 3.2.15 *Fix $m^0 \in M$ and assume that the trajectory of $\langle A_1 \rangle_{\mathcal{O}}$ through $z^0 = \rho(m^0) \in \mathcal{O}$ is defined for $t \in [0, T]$ and remains in an open domain $\mathcal{D}_0$ with compact closure. Then, there exist some constants $\varepsilon_0 > 0$, $T_0 > 0$ and $c > 0$ such that*

$$\mathrm{dist}^o(\rho \circ \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m^0), \mathrm{Fl}^{\varepsilon t}_{\langle A_1 \rangle_{\mathcal{O}}}(z^0)) \leq c\varepsilon \tag{3.2.60}$$

for all $\varepsilon \in [0, \varepsilon_0]$ and $t \in [0, \frac{T_0}{\varepsilon}]$.

We will proceed the proof of this Theorem in few steps.

First of all we choose an $\mathbb{S}^1$-invariant metric $<,>$ on M such that the projection $\rho : M \to \mathcal{O}$ is a Riemannian submersion

Step 1. (*Fixing* ε_0). Pick another open domain $\mathcal{D}$ in $\mathcal{O}$ with compact closure such that $\mathcal{D}_0 \subset \mathcal{D}$. Then, $N_0 = \rho^{-1}(\mathcal{D}_0)$ and $N = \rho^{-1}(\mathcal{D})$ are open domains in M with compact closure which are invariant with respect to the $\mathbb{S}^1$-action. It is clear that $m_0 \in N_0 \subset N$. By Theorem 3.1.1 there exist $\delta > 0$ and a near identity transformation $\Phi_\varepsilon : N \to M$ with which takes the perturbed vector field $\mathbf{A}_\varepsilon$ into the form

$$(\Phi_\varepsilon)^* \mathbf{A}_\varepsilon = A_0 + \varepsilon\langle A_1\rangle + \varepsilon^2 R_\varepsilon.$$

for all $\varepsilon \in (-\delta, \delta)$. Here, the remainder R_ε is a smooth ε-dependent vector field on N. Without loss of generality, we can assume that R_ε is extended to the closure $\bar{\mathcal{N}}$. The diffeomorphism Φ_ε is given as the time-ε flow of the vector field

$$Z = \frac{1}{\omega}\mathcal{S}(A_1) + \frac{1}{\omega^3}\mathcal{S}^2(\mathcal{L}_{\langle A_1\rangle}\omega)A_0 \tag{3.2.61}$$

which is defined on the whole M.

Since, Φ_ε is a near identity transformation Φ_ε, there exists a constant $\delta_0 \in (0, \delta]$ such that

$$m_0 \in \Phi_\varepsilon(N_0\ )\ \ \forall \varepsilon \in [0, \delta_0]. \tag{3.2.62}$$

Condition (3.2.62) can be rewritten as

$$m_\varepsilon := \Phi_\varepsilon^{-1}(m_0) \in N_0. \tag{3.2.63}$$

Lemma 3.2.16 *Let $[0, \delta_0] \ni \varepsilon \mapsto m_\varepsilon \in N_0$ be the parameterized curve and L_ε its arc length. Then, for all $\varepsilon \in [0, \delta_0]$, we have*

$$\operatorname{dist}(m_0, m_\varepsilon) \le L_\varepsilon \le \varkappa_0\varepsilon$$

where

$$\varkappa_0 = \sup_{m\in N_0} \|Z(m)\|. \tag{3.2.64}$$

Proof. Consider the parameterized curve $[0, \delta_0] \ni \varepsilon \mapsto m_\varepsilon \in N_0$. Taking into account that $\frac{dm_\varepsilon}{d\varepsilon} = -Z(m_\varepsilon)$, we get

$$\begin{aligned} L_\varepsilon &= \int_0^\varepsilon \left\| \frac{dm_{\varepsilon'}}{d\varepsilon'} \right\| d\varepsilon' \\ &= \int_0^\varepsilon \|Z(m_{\varepsilon'})\|\, d\varepsilon' \le \sup_{m\in N_0} \|Z\|_m\, \varepsilon. \end{aligned}$$

■

Corollary 3.2.17 *Let $[0, \varepsilon_0] \ni \varepsilon \mapsto \rho(m_\varepsilon) \in \mathcal{D}_0$ be the parameterized curve on the orbit space and L^o_ε its arc length. Then, the inequality*

$$\operatorname{dist}^o(\rho(m_0), \rho(m_\varepsilon)) \le L^o_\varepsilon \le \varkappa_0\varepsilon$$

holds for $\varepsilon \in [0, \varepsilon_0]$.

Consider the following (s, ε)-dependent vector field on N:

$$\tilde{\mathbf{A}}_{\varepsilon,s} = A_0 + \varepsilon\langle A_1\rangle + s\varepsilon^2 R_\varepsilon \tag{3.2.65}$$

where $s \in [0, 1]$.

Lemma 3.2.18 *There exists* $\varepsilon_0 \in (0, \delta_0]$ *such that for every trajectory of* $\widetilde{\mathbf{A}}_{s,\varepsilon}$ *through* $m_{s\varepsilon}$

$$t \mapsto \gamma_{\varepsilon,s}(t) := \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,s}}(m_{s\varepsilon}) \in N \tag{3.2.66}$$

is defined for $t \in [0, \frac{T_0}{\varepsilon}]$ *if* $\varepsilon \in [0, \varepsilon_0]$ *and* $s \in [0,1]$.

Proof. By standard properties of flows, we have

$$\mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,s}} = \mathrm{Fl}^t_{A_0} \circ \mathrm{Fl}^{\varepsilon t}_{\mathbf{P}_\varepsilon} \tag{3.2.67}$$

where

$$\mathbf{P}_t = (\mathbf{P}_{\varepsilon,s})_t = \langle A_1 \rangle - t(L_{\langle A_1 \rangle}\omega)\Upsilon + \varepsilon s (\mathrm{Fl}^t_{A_0})^* R_\varepsilon$$

is a time-dependent vector field parametercally depending on ε, s in a smooth way. Since the flow of A_0 is periodic, it is enough to show that for small enough ε, the interval $[0, T_0]$ belongs to the interval of definition of the trajectory of $\mathbf{P}_t$ through $m_{s\varepsilon}$. The vector field $(\mathbf{P}_{0,s})_t = \langle A_1 \rangle - t(L_{\langle A_1 \rangle}\omega)\Upsilon$ is $\mathbb{S}^1$-invariant and ρ-related with $\langle A_1 \rangle_{\mathcal{O}}$ and hence by the hypothesis of Theorem 3.2.15, its trajectory through m_0 is defined for $t \in [0, T_0]$. Then, there exists $\varepsilon_0 \in (0, \delta_0]$ such that for every $\varepsilon \in [0, \varepsilon_0]$ and $s \in [0,1]$ the trajectory of $(\mathbf{P}_{\varepsilon,s})_t = (\mathbf{P}_{0,s})_t + \varepsilon s (\mathrm{Fl}^t_{A_0})^* R_\varepsilon$ through $m_{s\varepsilon}$ is also defined for all $t \in [0, T_0]$. Here we use , the following well-known property (see [1], page 222): if $[0, T_0]$ belongs to the domain of definition of the trajectory through m_0, then there exists a neighborhood U of m_0 such that any $m \in U$ has trajectory existing for time $t \in [0, T_0]$. ∎

Step 2 (*Triangle Inequality*) Remark that the perturbed vector field is related with (3.2.65) by the formula

$$\mathbf{A}_\varepsilon = (\Phi_\varepsilon)_* \tilde{\mathbf{A}}_{\varepsilon,1}$$

and hence

$$t \mapsto \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m_0) = (\Phi_\varepsilon \circ \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}})(m_\varepsilon)$$

Fix ε_0 as in Lemma 3.2.18. Then, for each $\varepsilon \in [0, \varepsilon_0]$, the trajectories of the perturbed vector field $\mathbf{A}_\varepsilon$ and the averaged vector field

$$\tilde{\mathbf{A}}_{\varepsilon,0} = A_0 + \varepsilon \langle A_1 \rangle$$

through the point m_0 are defined for all $t \in [0, \frac{T_0}{\varepsilon}]$. To estimate the distance between the points of these trajectories, we start with standard triangle inequality argument [66]

$$\begin{aligned} &\mathrm{dist}(\mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m_0), \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,0}}(m_0)) \\ &\leq \mathrm{dist}(\mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m_0), \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon)) + \mathrm{dist}(\mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon), \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,0}}(m_0)) \end{aligned} \tag{3.2.68}$$

The first term in (3.2.68) has the following estimate

$$\mathrm{dist}(\mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m_0), \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon)) = \mathrm{dist}(\Phi_\varepsilon \circ \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon), \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon)) \leq \varkappa_1 \varepsilon \tag{3.2.69}$$

where

$$\varkappa_1 = \sup_{m \in \bar{\mathcal{N}}} \|Z\|_m \tag{3.2.70}$$

This inequality follows from the same arguments as the proof of Lemma 3.2.16. It follows from (3.2.69) and (3.2.51) that

$$\operatorname{dist}^o(\rho \circ \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m_0),\ \rho \circ \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon)) \le \varkappa_1 \varepsilon$$

This implies the following fact.

Lemma 3.2.19 *For all $\varepsilon \in [0, \varepsilon_0]$ and $t \in [0, \frac{T_0}{\varepsilon}]$ the following estimate holds*

$$\begin{aligned}\operatorname{dist}^o(\rho \circ \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m^0), \mathrm{Fl}^{\varepsilon t}_{\langle A_1 \rangle_{\mathcal{O}}}(z^0)) \le \varkappa_1 \varepsilon \\ + \operatorname{dist}^o(\rho \circ \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon), \rho \circ \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,0}}(m_0)).\end{aligned} \tag{3.2.71}$$

Step 3 (*Gronwall's inequality)* To estimate the second term in (3.2.71), we make some preparation steps. For each fixed ε, denote by $\gamma_\varepsilon : [0, \frac{T_0}{\varepsilon}]. \to N$ the trajectory of the vector field $\tilde{\mathbf{A}}_{\varepsilon,1}$ through m_ε,

$$\gamma_\varepsilon(t) = \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon).$$

Consider its projection to the orbit space $\alpha_\varepsilon = \rho \circ \gamma_\varepsilon : [0, \frac{T_0}{\varepsilon}]. \to \mathcal{D}$ and the horizontal lift $\tilde{\alpha}_\varepsilon : [0, \frac{T_0}{\varepsilon}] \ni t \mapsto \tilde{\alpha}_\varepsilon(t) \in N$ through m_ε of the curve α_ε. Then, by Proposition 3.2.14 for every t, there exists a fiber wise diffeomorphism g^t_ε on N defined by (3.2.54), such that $g^0_\varepsilon = \mathrm{id}$ and

$$\tilde{\alpha}_\varepsilon(t) = g^t_\varepsilon(\gamma_\varepsilon(t)).$$

Moreover, $\tilde{\alpha}_\varepsilon(t)$ is the trajectory of the time-dependent vector field $(g^t_\varepsilon)_* \tilde{\mathbf{A}}^{\mathrm{hor}}_{\varepsilon,1}$ where

$$\tilde{\mathbf{A}}^{\mathrm{hor}}_{\varepsilon,1} = \varepsilon \langle A_1 \rangle^{\mathrm{hor}} + \varepsilon^2 R^{\mathrm{hor}}_\varepsilon.$$

Since g^t_ε is defined as the re parameterized flow of the infinitesimal generator of the $\mathbb{S}^1$-action we have that $(g^t_\varepsilon)_* \langle A_1 \rangle^{\mathrm{hor}} = \langle A_1 \rangle^{\mathrm{hor}}$ and hence

$$(g^t_\varepsilon)_* \tilde{\mathbf{A}}^{\mathrm{hor}}_{\varepsilon,1} = \varepsilon \langle A_1 \rangle^{\mathrm{hor}} + \varepsilon^2 (g^t_\varepsilon)_* R^{\mathrm{hor}}_\varepsilon.$$

For every $\varepsilon \in [0, \varepsilon_0]$ and $s \in [0, 1]$, introduce the following horizontal time-dependent vector field on N:

$$\mathbf{X}_t = (\mathbf{X}_{\varepsilon,s})_t := \varepsilon \langle A_1 \rangle^{\mathrm{hor}} + s\varepsilon^2 (g^t_\varepsilon)_* R^{\mathrm{hor}}_\varepsilon.$$

For each $\varepsilon \in [0, \varepsilon_0]$, using the flow of this vector field, we define a *parameterized surface* in N as

$$\sigma_\varepsilon : [0, \frac{T_0}{\varepsilon}] \times [0, 1] \ni (t, s) \mapsto \sigma_\varepsilon(t, s) := \mathrm{Fl}^t_{\mathbf{X}_t}(m_{\varepsilon s}). \tag{3.2.72}$$

It is clear that

$$\sigma_\varepsilon(t, 0) = \mathrm{Fl}^{\varepsilon t}_{\langle A_1 \rangle^{\mathrm{hor}}}(m_0).$$

Since $(\mathbf{X}_{\varepsilon,1})_t$ coincides with $(g^t_\varepsilon)_*\tilde{\mathbf{A}}^{\mathrm{hor}}_{\varepsilon,1}$ we have

$$\tilde{\alpha}_\varepsilon(t) = \sigma_\varepsilon(t,1)$$

and hence

$$\rho \circ \sigma_\varepsilon(t,1) = \alpha_\varepsilon(t).$$

Therefore, for the second term in (3.2.71) we have the estimation (Proposition 3.2.14)

$$\mathrm{dist}^o(\rho \circ \sigma_\varepsilon(t,1), \rho \circ \sigma_\varepsilon(t,0)) \le \mathrm{dist}(\sigma_\varepsilon(t,1), \sigma_\varepsilon(t,0))$$

which says that it is enough to study the lengths of the s-curves in the surface σ_ε. For a fixed t, consider the horizontal s-curve $s \mapsto \sigma_{\varepsilon,t}(s) = \sigma_\varepsilon(t,s)$ and denote by

$$L_\varepsilon(t) := \int_0^1 \left\| \frac{d\sigma_{\varepsilon,t}(s)}{ds} \right\| ds$$

its arc length.

Lemma 3.2.20 *For all $\varepsilon \in [0,\varepsilon_0]$ and $t \in [0, \frac{T_0}{\varepsilon}]$ the following estimate holds*

$$L_\varepsilon(t) \le \left[\left(\frac{\varkappa_3}{\varkappa_2} + \varkappa_0 \right) e^{\varepsilon \varkappa_2 t} - \frac{\varkappa_3}{\varkappa_2} \right] \varepsilon, \tag{3.2.73}$$

where $\varkappa_0$ is given by (3.2.64) and

$$\varkappa_2 = \sup_{\substack{m \in \bar{\mathcal{N}} \\ \varepsilon \in [0,\varepsilon_0]}} \left(\left\| \nabla \langle A_1 \rangle^{\mathrm{hor}} \right\|_m + \varepsilon \left\| \nabla R^{\mathrm{hor}}_\varepsilon \right\|_m \right), \tag{3.2.74}$$

$$\varkappa_3 = \sup_{\substack{m \in \bar{\mathcal{N}} \\ \varepsilon \in [0,\varepsilon_0]}} \| R^{\mathrm{hor}}_\varepsilon \|_m . \tag{3.2.75}$$

Proof. Applying the basic inequality (3.2.23), we have

$$L_\varepsilon(t) \le L_\varepsilon(0) + \int_0^t \int_0^1 \| \nabla_{\frac{\partial \sigma_\varepsilon}{\partial s}} \frac{\partial \sigma_\varepsilon}{\partial t'} \|_{\sigma_\varepsilon} \, ds dt'. \tag{3.2.76}$$

By definition, the t-curves in σ_ε are horizontal and

$$\frac{\partial}{\partial t} \sigma_\varepsilon(t,s) = \varepsilon \left(\langle A_1 \rangle^{\mathrm{hor}} + s\varepsilon^2 (g^t_\varepsilon)_* R^{\mathrm{hor}}_\varepsilon \right) \circ \sigma_\varepsilon.$$

It follows that

$$\| \nabla_{\frac{\partial \sigma_\varepsilon}{\partial s}} \frac{\partial \sigma_\varepsilon}{\partial t'} \|_{\sigma_\varepsilon} \le \varepsilon \, \| \nabla \langle A_1 \rangle^{\mathrm{hor}} \|_{\sigma_\varepsilon} \cdot \left\| \frac{\partial \sigma_\varepsilon}{\partial s} \right\|_{\sigma_\varepsilon}$$
$$+ s\varepsilon^2 \, \| \nabla_{\frac{\partial \sigma_\varepsilon}{\partial s}} \left((g^t_\varepsilon)_* R^{\mathrm{hor}}_\varepsilon \right) \|_{\sigma_\varepsilon} + \varepsilon^2 \, \| (g^t_\varepsilon)_* R^{\mathrm{hor}}_\varepsilon \|_{\sigma_\varepsilon} .$$

By Lemma 3.2.1, we deduce

$$\begin{aligned}
&\| \nabla_{\frac{\partial \sigma_\varepsilon}{\partial s}} \left((g^t_\varepsilon)_* R^{\mathrm{hor}}_\varepsilon \right) \|_{\sigma_\varepsilon} \leq \| \nabla \left((g^t_\varepsilon)_* R^{\mathrm{hor}}_\varepsilon \right) \|_{\sigma_\varepsilon} \cdot \| \frac{\partial \sigma_\varepsilon}{\partial s} \|_{\sigma_\varepsilon} \\
&= \| \nabla R^{\mathrm{hor}}_\varepsilon \|_{g_\varepsilon^{-t} \circ \sigma_\varepsilon} \cdot \| \frac{\partial \sigma_\varepsilon}{\partial s} \|_{\sigma_\varepsilon}
\end{aligned}$$

and

$$\| (g^t_\varepsilon)_* R^{\mathrm{hor}}_\varepsilon \|_{\sigma_\varepsilon} = \| R^{\mathrm{hor}}_\varepsilon \|_{g_\varepsilon^{-t} \circ \sigma_\varepsilon} .$$

Putting these relations into (3.2.76) we arrive to the inequality

$$L^o_\varepsilon(t) \leq \varepsilon \varkappa_0 + \varepsilon \varkappa_2 \int_0^t L_\varepsilon(t') dt' + \varepsilon^2 \varkappa_3 t.$$

Applying the specific Gronwall lemma leads to (3.2.73). ■

Finally, the proof of Theorem 3.2.15 follows from Lemma 3.2.19, Lemma 3.2.20 and the inequality

$$\mathrm{dist}(\sigma_\varepsilon(t,1), \sigma_\varepsilon(t,0)) \leq L_\varepsilon(t).$$

Corollary 3.2.21 *The ε-independent constant in (3.2.60) can be chosen as follows*

$$c = \varkappa_1 + \left(\frac{\varkappa_3}{\varkappa_2} + \varkappa_0 \right) e^{\varkappa_2 T_0} - \frac{\varkappa_3}{\varkappa_2}, \tag{3.2.77}$$

where the constants $\varkappa_0, \varkappa_1, \varkappa_2, \varkappa_3$ are given by (3.2.64)-(3.2.75).

Remark that the upper estimates for the constants $\varkappa_0$ and $\varkappa_1$ can be expressed directly in terms of vector fields A_0 and A_1 by using the following estimate for the infinitesimal generator (3.2.61) of the near identity transformation:

$$\|Z\|_m \leq \frac{1}{\omega} \|A_1\|_m + \frac{1}{\omega^3} \mid \mathcal{S}^2(\mathcal{L}_{\langle A_1 \rangle} \omega) \mid \|A_0\|_m$$

for every $m \in M$.

Remark 6 *Instead of parameterized surface (3.2.72), one can try to use the surface $\tilde{\sigma}_\varepsilon$ generated by trajectories of the (s,ε)-dependent vector field $\tilde{\mathbf{A}}_{\varepsilon,s}$ (3.2.65) which start at the initial s-curve $s \mapsto m_{s\varepsilon}$. Then,*

$$\mathrm{dist}^o(\rho \circ \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,1}}(m_\varepsilon), \rho \circ \mathrm{Fl}^t_{\tilde{\mathbf{A}}_{\varepsilon,0}}(m_0)) \leq L^1_0(\rho \circ \tilde{\sigma}_{\varepsilon,t}).$$

But we have estimation for the length $L^1_0(\rho \circ \tilde{\sigma}_{\varepsilon,t}) = O(\varepsilon)$ at the time long scale $t \sim \frac{1}{\varepsilon}$, only if $L_{\langle A_1 \rangle} \omega = 0$.

Applications of the Averaging Theorem. Let $\mathbf{A}_\varepsilon = A_0 + \varepsilon A_1$ be a perturbed vector field on a Riemannian manifold $(M, < \cdot >)$. Below we suppose that the flow of the unperturbed vector field A_0 is periodic with frequency function $\omega : M \to \mathbb{R}$ and the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = \frac{1}{\omega} A_0$ is free. Recall that ω is a first integral of A_0 and $\omega = \rho \circ \omega_{\mathcal{O}}$ for a certain smooth function $\omega_{\mathcal{O}} : \mathcal{O} \to \mathbb{R}$. Here $\rho : M \to \mathcal{O}$ is the natural projection.

Proposition 3.2.22 *(Adiabatic Invariant) Assume also that the averaged vector field $\langle A_1\rangle_{\mathcal{O}}$ on the orbit space $\mathcal{O} = M\diagup\mathbb{S}^1$ satisfies the hypothesis of Theorem* 3.2.15 *and admits a smooth first integral $J_{\mathcal{O}} : \mathcal{O} \to \mathbb{R}$,*

$$\mathcal{L}_{\langle A_1\rangle_{\mathcal{O}}} J_{\mathcal{O}} = 0 \tag{3.2.78}$$

Then,

$$J = J_{\mathcal{O}} \circ \rho$$

is an adiabatic invariant of $\mathbf{A}_\varepsilon$, that is,

$$|J \circ \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m^0) - J(m^0)| = O(\varepsilon)$$

for $m^0 \in \mathcal{D}_0$, for small enough ε and $t \in [0, \frac{T_0}{\varepsilon}]$.

Proof. Since the closure of the open domain $\mathcal{D}$ is compact, the function $J_{\mathcal{O}}$ has the Lipschitz property on $\bar{\mathcal{D}}$ (see, for example, [67])

$$|J_{\mathcal{O}}(z) - J_{\mathcal{O}}(y)|^o \leq \lambda_J \|z - y\|^o .$$

Then, by condition (3.2.78) and Theorem 3.2.15 we have

$$\begin{aligned}
&\left\|J \circ \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m^0) - J(m^0)\right\| \\
&= \left\|J_{\mathcal{O}}(\rho \circ \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m^0)) - J_{\mathcal{O}}(\mathrm{Fl}^{\varepsilon t}_{\langle A_1\rangle_{\mathcal{O}}}(\rho(m^0))\right\| \\
&\leq \lambda_J \left\|\rho \circ \mathrm{Fl}^t_{\mathbf{A}_\varepsilon}(m^0) - \mathrm{Fl}^{\varepsilon t}_{\langle A_1\rangle_{\mathcal{O}}}(\rho(m^0)\right\|^o \\
&\leq \lambda_J c\varepsilon,
\end{aligned}$$

where the constant c is given by (3.2.77). ∎

Remark 7 *Another important consequence of Theorem* 3.2.15 *can be formulated as follows (see, for example Moser [58]). If the averaged vector field $\langle A_1\rangle_{\mathcal{O}}$ on $\mathcal{O}$ admits admits a nondegenerate rest point $z^0 \in \mathcal{D}_0$,*

$$\langle A_1\rangle_{\mathcal{O}}(z^0) = 0.$$

then, for small enough ε, the original perturbed vector field $\mathbf{A}_\varepsilon$ admits a periodic trajectory $\gamma_\varepsilon(t)$ whose projection $\rho(\gamma_\varepsilon(t)) \in \mathcal{O}$ is ε-close to z^0 and the period $T(\varepsilon)$ has the representation

$$T(\varepsilon) = \frac{2\pi}{\omega_{\mathcal{O}}(z^0)} + O(\varepsilon).$$

Chapter 4

Periodic Averaging on Slow-Fast Spaces

In this chapter, in the context of normal forms, we study a wide class of perturbed Hamiltonian systems so-called slow-fast phase spaces. This kind of systems appear in the theory of adiabatic approximation [7, 38, 62] and its generalizations [16, 17, 19, 43, 74, 76]. In applications, such perturbed models come from ε-dependent Hamiltonians which are slow or rapidly varying in some degrees of freedom as $\varepsilon \to 0$. Geometrically, the perturbation theory for slow-fast systems deals with phase spaces equipped with symplectic forms (or Poisson brackets) depending on the perturbation parameter ε in a singular way at $\varepsilon = 0$. As a consequence, the main feature of our perturbed model is that, in the limit $\varepsilon \to 0$, the unperturbed system does not inherits any natural Hamiltonian structure. This means that one can not apply directly any results of the regular Hamiltonian perturbation theory.

By a slow-fast phase space we mean a product $M = S_1 \times S_2$ of two symplectic manifolds (S_1, σ_1) and (S_2, σ_2) equipped with a rescaled product symplectic form $\sigma = \sigma_1 \oplus \varepsilon\sigma_2$. We think of M as the total space of the trivial fiber bundle $\pi_1 : M \to S_1$ over the "slow" base with "fast" fiber S_2. On such a phase space we consider a perturbed Hamiltonian system with Hamiltonian $H_\varepsilon = H_0 + \varepsilon H_1$, whose leading term H_0 depends on the slow variables $m_1 \in S_1$ and the fast variables $m_2 \in S_2$ appear only in the perturbation H_1. The corresponding Hamiltonian vector field V_{H_ε} is of the form $V_{H_\varepsilon} = \mathbb{V} + \varepsilon\mathbb{W}$, where the unperturbed vector field $\mathbb{V}$ is no longer Hamiltonian but projects to the Hamiltonian vector field v_f on (S_1, σ_1). In particular, when $H_0 \equiv 0$, we arrive at the adiabatic situation [7, 62].

We are interested in two types of normalization related to $\mathbb{S}^1$-actions. First, we show that in the resonant case, when the flow $\mathrm{Fl}^t_{\mathbb{V}}$ of the unperturbed system is periodic, the perturbed vector field $\mathbb{V}+\varepsilon\mathbb{W}$ admits a first order normalization relative to $\mathbb{V}$. Our main observation is that, although the unperturbed and perturbation vector fields $\mathbb{V}$ and $\mathbb{W}$ are not Hamiltonian, because of a special relationship between $\mathbb{V}$ and $\mathbb{W}$, one can still use the general criterion (Theorem 4.2.1) applying the period-energy relation argument for the Hamiltonian vector field v_f. The term "resonance" is motivated by the following interpretation of the periodicity condition for the flow $\mathrm{Fl}^t_{\mathbb{V}}$. Since the flow $\mathrm{Fl}^t_{\mathbb{V}}$ is a fiber preserving mapping on the trivial symplectic bundle $S_1 \times S_2 \to S_1$, under the periodicity of the flow of v_f, one can introduce the monodromy map $g : S_1 \to \mathrm{Sym}(S_2, \sigma_2)$. Then, the flow $\mathrm{Fl}^t_{\mathbb{V}}$ is periodic if $g^k(m_1) = \mathrm{id}$ for all $m_1 \in S_1$and some integer $k \geq 1$. In the particular case, when $S_2 = \mathbb{R}^{2m}$ and H_1 is a quadratic function in the fast variables, this condition is precisely the resonance condition between the "tangential" and "normal" frequencies of the linearized Hamiltonian dynamics over S_1 . Such perturbed models appear in the study of Hamiltonian dynamics near an invariant symplectic submanifold (S_1, σ_1) [39, 74]. Here, $S_1 \times \mathbb{R}^{2m}$ plays the role of the normal bundle of the submanifold S_1 and the unperturbed vector field $\mathbb{V}$ presents the linearized dynamics around S_1.

The second normalization setting for $V_{H_\varepsilon} = \mathbb{V} + \varepsilon\mathbb{W}$ is motivated by the question on the geometric meaning of the normalization transformation in the proof of the classical adiabatic theorem [7, 62]. In this case, the flow of $\mathbb{V}$ is not necessarily periodic and we only assume that $\mathbb{V}$ admits a circle first integral J. This means that the vertical Hamiltonian vector field V_J is an infinitesimal generator of an $\mathbb{S}^1$-action. Therefore we deal with the situation when the unperturbed vector field $\mathbb{V}$ is invariant with respect to the $\mathbb{S}^1$-action but not the symplectic form σ nor the Hamiltonian H_ε. To correct this "defect", we are looking for a near identity transformation $\mathcal{T}_\varepsilon$ which brings the original perturbed model to a system which is ε^2 close to a $\mathbb{S}^1$-symmetric Hamiltonian system. We show that such a normalization transformation can be defined as a symplectomorphism between the symplectic structure σ and its $\mathbb{S}^1$-average $\langle\sigma\rangle$. In the case of two degrees of freedom, we perform a detailed analysis of the properties of $\mathcal{T}_\varepsilon$ and derive various results concerning nearly integrable Hamiltonian systems and adiabatic invariants. Here, our main tools are the averaging technique on symplectic fibered spaces [28, 47, 55], the notion of weak coupling symplectic structures [30] and the Moser homotopy method [57] (see also [17, 74]).

4.1 General Normalization Settings

Let $M = S_1 \times S_2$ be a product of two symplectic manifolds (S_1, σ_1) and (S_2, σ_2). Let $\pi_1 : M \to S_1$ and $\pi_2 : M \to S_2$ be the canonical projections and d_1 and d_2 the partial exterior derivatives on M along S_1 and S_2, respectively. It is clear that $d = d_1 + d_2$ is the exterior derivative on M and $d_1^2 = d_2^2 = d_1 \circ d_2 + d_2 \circ d_1 = 0$. Denoting $\sigma^{(1)} = \pi_1^*\sigma_1$ and $\sigma^{(2)} = \pi_2^*\sigma_2$, let us consider the following ε-dependent 2-form on M

$$\sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)} \tag{4.1.1}$$

which is a symplectic structure for all $\varepsilon \neq 0$. For $H \in C^\infty(M)$, denote by V_H the Hamiltonian vector field relative to σ. Then, $V_H = V_H^{(1)} + \frac{1}{\varepsilon}V_H^{(2)}$, where $V_H^{(1)}$ and $V_H^{(2)}$ are vector fields on M uniquely defined by the relations

$$\mathbf{i}_{V_H^{(1)}}\sigma^{(1)} = -d_1 H, \tag{4.1.2}$$

$$\mathbf{i}_{V_H^{(1)}}\sigma^{(2)} = 0, \tag{4.1.3}$$

and

$$\mathbf{i}_{V_H^{(2)}}\sigma^{(2)} = -d_2 H, \tag{4.1.4}$$

$$\mathbf{i}_{V_H^{(2)}}\sigma^{(1)} = 0. \tag{4.1.5}$$

It follows that, for all $m_1 \in S_1$ and $m_2 \in S_2$, the vector fields $V_H^{(1)}$ and $V_H^{(2)}$ are tangent to the symplectic slices $S_1 \times \{m_2\}$ and $\{m_1\} \times S_2$, respectively. For every $u \in \mathfrak{X}(S_1)$, denote by $\hat{u} = u \oplus 0 \in \mathfrak{X}(M)$ the lifting associated to the canonical decomposition $TM = TS_1 \oplus TS_2$. By $\{,\}_1$ and $\{,\}_2$ we denote the Poisson brackets on M associated to the presymplectic structures $\sigma^{(1)}$ and $\sigma^{(2)}$, respectively. Then, $\{H,G\}_1 = \Pi^{(1)}(dH,dG) = \mathcal{L}_{V_H^{(1)}}G$ and $\{H,G\}_2 = \Pi^{(2)}(dH,dG) = \mathcal{L}_{V_H^{(2)}}G$. Here, $\Pi^{(1)},\Pi^{(2)} \in \chi^2(M)$ denote the corresponding Poisson tensor fields. In this terms, we have $V_H^{(1)} = \mathbf{i}_{dH}\Pi^{(1)}$ and $V_H^{(2)} = \mathbf{i}_{dH}\Pi^{(2)}$.

On the slow-fast phase space (M,σ), let us consider the following perturbed Hamiltonian model [17, 19, 74, 76]

$$H_\varepsilon = f \circ \pi_1 + \varepsilon F, \tag{4.1.6}$$

for some $f \in C^\infty(S_1)$ and $F \in C^\infty(M)$. The corresponding Hamiltonian vector field takes the form

$$V_{H_\varepsilon} = \mathbb{V} + \varepsilon \mathbb{W}, \tag{4.1.7}$$

where

$$\mathbb{V} = \hat{v}_f + V_F^{(2)} \tag{4.1.8}$$

and

$$\mathbb{W} = V_F^{(1)} \tag{4.1.9}$$

are unperturbed and perturbation vector fields, respectively. Here v_f denotes the Hamiltonian vector field on (S_1,σ_1) of f. It is clear that $\hat{v}_f = V^{(1)}_{f\circ\pi_1}$ and we have the relations

$$[\hat{v}_f, V_F^{(1)}] = V^{(1)}_{\mathcal{L}_{\hat{v}_f}F}, \tag{4.1.10}$$

$$[\hat{v}_f, V_F^{(2)}] = V^{(2)}_{\mathcal{L}_{\hat{v}_f}F}. \tag{4.1.11}$$

In the bracket form, the Hamiltonian system of (4.1.7) is written as follows

$$\begin{aligned} \dot{\xi}^i &= \{f\circ\pi_1, \xi^i\}_1 + \varepsilon\{F,\xi^i\}_2, \\ \dot{x}^\alpha &= \{F, x^\alpha\}_2, \end{aligned}$$

where $\xi = (\xi^i) \in S_1$ and $x = (x^\alpha) \in S_2$.

The Deprit Normalization. The first normalization question we address for Hamiltonian vector field (4.1.7) is the standard one: put V_{H_ε} into a normal form relative to $\mathbb{V}$ up to desired order. Since the vector field in (4.1.8) and (4.1.9) are related through the dependence of F, first of all, it is natural to see under which conditions perturbed vector field (4.1.8) is in normal form relative to $\mathbb{V}$. We need the following useful technical fact.

Lemma 4.1.1 *For every $G \in C^\infty(M)$, the Lie brackets between the vector fields $V_G^{(1)}$, $V_G^{(2)}$ and $\mathbb{V}$ (4.1.8) are given by the relations*

$$\mathbf{i}_{[V_G^{(1)},\mathbb{V}]}(\sigma^{(1)} + \sigma^{(2)}) = \mathcal{L}_{\mathbb{V}} d_1 G - \mathcal{L}_{V_G^{(1)}} d_2 F, \tag{4.1.12}$$

$$[V_G^{(2)}, \mathbb{V}] = -V^{(2)}_{\mathcal{L}_{\mathbb{V}}G}. \tag{4.1.13}$$

Proof. From relations (4.1.2)-(4.1.4) and Cartan formula $\mathcal{L}_X = \mathbf{i}_X \circ d + d \circ \mathbf{i}_X$, we get

$$\mathcal{L}_{V_G^{(1)}} \sigma^{(1)} = -d_2 \circ d_1 G, \qquad \mathcal{L}_{V_G^{(1)}} \sigma^{(2)} = 0, \tag{4.1.14}$$

$$\mathcal{L}_{V_F^{(2)}} \sigma^{(1)} = 0, \qquad \mathcal{L}_{V_F^{(2)}} \sigma^{(2)} = -d_1 \circ d_2 F. \tag{4.1.15}$$

It follows that $\mathcal{L}_{V_G^{(1)}}(\sigma^{(1)} + \sigma^{(2)}) = -d_2 \circ d_1 G$ and $\mathbf{i}_{V_F^{(2)}}(\sigma^{(1)} + \sigma^{(2)}) = -d_2 F$. Next, using the identity $\mathbf{i}_{[V_G^{(1)}, V_F^{(2)}]} = \mathcal{L}_{V_G^{(1)}} \circ \mathbf{i}_{V_F^{(2)}} - \mathbf{i}_{V_F^{(2)}} \circ \mathcal{L}_{V_G^{(1)}}$, we derive the following formula for the Lie bracket between $V_G^{(1)}$ and $V_F^{(2)}$

$$\begin{aligned} \mathbf{i}_{[V_G^{(1)}, V_F^{(2)}]}(\sigma^{(1)} + \sigma^{(2)}) &= -\mathcal{L}_{V_G^{(1)}} d_2 F + \mathbf{i}_{V_F^{(2)}}(d_2 \circ d_1 G), \\ &= -\Big(\mathbf{i}_{V_G^{(1)}}(d_1 \circ d_2 F) + \mathbf{i}_{V_F^{(2)}}(d_1 \circ d_2 G)\Big). \end{aligned} \tag{4.1.16}$$

Finally, (4.1.10) implies the equality $\mathbf{i}_{[V_G^{(1)}, \hat{v}_f]}(\sigma^{(1)} + \sigma^{(2)}) = d_1(\mathcal{L}_{\hat{v}_f} G)$ which together with (4.1.16) leads to the formula

$$\mathbf{i}_{[V_G^{(1)}, \hat{v}_f + V_F^{(2)}]}(\sigma^{(1)} + \sigma^{(2)}) = d_1(\mathcal{L}_{\hat{v}_f} G) - \mathbf{i}_{V_F^{(2)}}(d_1 \circ d_2 G) - \Big(\mathbf{i}_{V_G^{(1)}}(d_1 \circ d_2 F)\Big).$$

Taking into account the property $d_1 \circ \mathcal{L}_{\hat{v}_f} = \mathcal{L}_{\hat{v}_f} \circ d_1$ and the equalities

$$\mathbf{i}_{V_F^{(2)}}(d_1 \circ d_2 G) = -\mathcal{L}_{V_F^{(2)}} d_1 G \quad \text{and} \quad \mathbf{i}_{V_G^{(1)}}(d_1 \circ d_2 F) = \mathcal{L}_{V_G^{(1)}} d_2 F,$$

we derive the identity (4.1.12). Formula (4.1.13) follows directly from (4.1.11) and the identity $[V_G^{(2)}, V_F^{(2)}] = -V^{(2)}_{\mathcal{L}_{V_F^{(2)}} G}$. ■

The canonical decomposition $TM = TS_1 \oplus TS_2$ induces decomposition of every of 1-forms on M into horizontal and vertical components which vanish the vector fields tangent to the slices $\{m_1\} \times S_2$ and $S_1 \times \{m_2\}$, respectively. Taking into account that $d_1 G$ and $d_2 F$ are horizontal and vertical 1-forms respectively, we observe that the first and the second terms in the right hand side of (4.1.12) belong to the subspaces vertical and horizontal, respectively. This leads to the following consequence of Lemma 4.1.1.

Corollary 4.1.2 *The perturbed vector field $\mathbb{V} + \varepsilon \mathbb{W}$ (4.1.7) is in normal form relative to $\mathbb{V}$,*

$$[\mathbb{V}, \mathbb{W}] = 0,$$

if and only if the functions f and F are related by the conditions

$$\mathcal{L}_{\mathbb{V}} d_1 F = 0 \qquad \textit{and} \qquad \mathcal{L}_{\mathbb{W}} d_2 F = 0. \tag{4.1.17}$$

Therefore, in general, the vector fields $\mathbb{V}$ and $\mathbb{W}$ do not commute. This fact gives rise to the normalization question.

The Hamiltonization Problem. We observe that in general, the unperturbed vector field $\mathbb{V}$ is not Hamiltonian relative to the symplectic structure (4.1.1). Indeed, it follows from (4.1.14), (4.1.15) that

$$\mathcal{L}_{\mathbb{V}} \sigma = -\varepsilon d_1 \circ d_2 F \tag{4.1.18}$$

and hence $\mathbb{V}$ is Hamiltonian relative to σ for $\varepsilon \neq 0$, only in the case when $F = \pi_1^* f_1 + \pi_2^* f_2$, for some $f_1 \in C^\infty(S_1)$ and $f_2 \in C^\infty(S_2)$. This feature of our unperturbed system comes from the singular dependence of the symplectic form σ on the perturbation parameter at $\varepsilon = 0$. In the limit $\varepsilon \to 0$, the 2-form σ becomes degenerate and one can think of $\mathbb{V}$ as a Hamiltonian vector field only relative to the pre-symplectic structure $\sigma^{(1)}$.

To correct this "defect" of the unperturbed dynamics one can try to search a Hamiltonian structure for $\mathbb{V}$ by deforming the symplectic structure σ. The following result [17, 74] shows that it can be done under some appropriate conditions.

Proposition 4.1.3 *If there exists a horizontal 1-form θ satisfying the homological equation*

$$\mathcal{L}_{\mathbb{V}}\theta = d_1 F, \tag{4.1.19}$$

then the vector field $\mathbb{V}$ is Hamiltonian relative to the pre-symplectic structure

$$\tilde{\sigma} = \sigma - \varepsilon d\theta \tag{4.1.20}$$

and the function

$$\tilde{H}_\varepsilon = f \circ \pi_1 + \varepsilon \tilde{F}, \tag{4.1.21}$$

where

$$\tilde{F} := F - \mathbf{i}_{\hat{v}_f}\theta. \tag{4.1.22}$$

Moreover, the functions $f \circ \pi$ and $\tilde{F}$ are first integrals of $\mathbb{V}$.

Proof. Let θ be an arbitrary horizontal 1-form. Taking into account that $\mathbf{i}_{V_F^{(2)}}\theta = 0$, we get the relation

$$\mathbf{i}_{\mathbb{V}}(\sigma^{(1)} + \varepsilon\sigma^{(2)}) = -d_1(f \circ \pi_1) - \varepsilon d_2 F = -d(f \circ \pi_1 + \varepsilon F) + \varepsilon d_1 F$$

and $\mathbf{i}_{\mathbb{V}} \circ d\theta = \mathcal{L}_{\mathbb{V}}\theta - d(\mathbf{i}_{\hat{v}_f}\theta)$. It follows from here that

$$\mathbf{i}_{\mathbb{V}}(\sigma^{(1)} + \varepsilon\sigma^{(2)} - \varepsilon d\theta) = -d(f \circ \pi_1 + \varepsilon(F - \mathbf{i}_{\hat{v}_f}\theta)) - \varepsilon(\mathcal{L}_{\mathbb{V}}\theta - d_1 F).$$

Therefore, if θ satisfies (4.1.19), then formulas (4.1.20),(4.1.21) give a Hamiltonian structure for $\mathbb{V}$. Finally, it is easy to see that $f \circ \pi$ is a first integral of $\mathbb{V}$. Then, by the representation (4.1.21) for the Hamiltonian function of (4.1.19), we conclude that $\tilde{F}$ is also a first integral. ■

One can show [20] also that the solvability of (4.1.19) is necessary in some sense for the Hamiltonization of $\mathbb{V}$. There is also a geometric interpretation of the homological equation related with the notion of invariant connections [17, 20, 74].

In general, the solvability criteria for homological equation (4.1.19) is a nontrivial question [20, 74]. But, if this equation is solvable, one can get a normalization of the following type [16, 17]: there exists a near identity transformation $\mathcal{T}_\varepsilon$ such that the unperturbed vector field $\mathbb{V}$ and the transformed vector field $(\mathcal{T}_\varepsilon)^* V_{H_\varepsilon}$ are Hamiltonian relative to one and the same symplectic form $\mathcal{T}_\varepsilon^*\sigma$ for small $\varepsilon \neq 0$.

$\mathbb{S}^1$-invariant Normalization. Given an action of the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ on M and assuming the $\mathbb{S}^1$-invariance of the unperturbed vector field $\mathbb{V}$, we are looking for a near identity transformation which brings V_{H_ε} to a $\mathbb{S}^1$-invariant vector field up to desired order. The infinitesimal generator Z of a first order normalization transformation must satisfy the equations

$$\begin{aligned} \mathcal{L}_{\mathbb{V}} Z &= \mathbb{W} - \bar{\mathbb{W}}, \\ \mathcal{L}_{\Upsilon} \bar{\mathbb{W}} &= 0, \end{aligned}$$

where Υ is the infinitesimal generator of the $\mathbb{S}^1$-action. We distinguish two situations, the first one when the $\mathbb{S}^1$-action comes from the periodic flow of $\mathbb{V}$. In this case normalization results depend on the properties of Hamiltonian vector field v_f. In the second case, the flow of $\mathbb{V}$ is not necessarily periodic.

4.2 Normalization Relative to Periodic Skew Flows

Our point is to study normal forms of perturbed model (4.1.7)-(4.1.9) in the periodic case, when the flow of the unperturbed vector field $\mathbb{V}$ is periodic.

4.2.1 The first order normalization

The vector field $\mathbb{V}$ is π_1-related with v_f and hence the trajectories of $\mathbb{V}$ are projected onto trajectories of the Hamiltonian vector field v_f, $\pi_1 \circ \mathrm{Fl}^t_{\mathbb{V}} = \varphi^t \circ \pi_1$. Here, φ^t denotes the flow of v_f. Therefore, $\mathrm{Fl}^t_{\mathbb{V}}$ is the skew-product flow,

$$\mathrm{Fl}^t_{\mathbb{V}}(m_1, m_2) = (\varphi^t(m_1), \mathcal{G}^t_{m_1}(m_2)), \tag{4.2.1}$$

where $\mathcal{G}^t_{m_1}$ is a smooth family of symplectomorphisms on (S_2, σ_2) determining as the solution of the time-dependent Hamiltonian system

$$\frac{d\mathcal{G}^t_{m_1}(m_2)}{dt} = V_F^{(2)}(\varphi^t(m_1), \mathcal{G}^t_{m_1}(m_2)), \tag{4.2.2}$$

$$\mathcal{G}^0_{m_1} = \mathrm{id}_{S_2}\,. \tag{4.2.3}$$

Assume that the flow $\mathrm{Fl}^t_{\mathbb{V}}$ is periodic with frequency function $\omega = \frac{2\pi}{T}$. Then,

$$\varphi^{t+T(m_1,m_2)}(m_1) = \varphi^t(m_1) \tag{4.2.4}$$

for all $m_1 \in S_1, m_2 \in S_2$ and $t \in \mathbb{R}$. If $v_f \neq 0$ on S_1, then differentiating equality (4.2.4) along S_2 says that the period function T is independent of m_2 and hence $\omega = \varpi \circ \pi_1$, for a certain smooth positive function ϖ on S_1. Therefore, the Hamiltonian flow φ^t of v_f is also periodic with frequency function ϖ.

Let $V_{H_\varepsilon} = \mathbb{V} + \varepsilon\mathbb{W}$ be the Hamiltonian vector field on $(M = S_1 \times S_2, \sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)})$.

Theorem 4.2.1 *Assume that the flow of vector field* $\mathbb{V} = \hat{v}_f + V_F^{(2)}$ *(*4.1.8*) is periodic with frequency function* $\omega : M \to \mathbb{R}$ *and the set* $\mathrm{Reg}(v_f) = \{m_1 \in S_1 | v_f(m_1) \neq$

$0\}$ *is dense in* S_1. *Then, the vector field* $\mathbb{W} = V_F^{(1)}$ *(4.1.9) satisfies the compatibility condition*

$$\mathcal{L}_{\langle \mathbb{W} \rangle} \omega = 0 \text{ on } M \tag{4.2.5}$$

where $\langle \cdot \rangle$ *denotes the averaging with respect to* $\mathbb{S}^1$*-action on* M *with infinitesimal generator* $\Upsilon = \frac{1}{\omega}\mathbb{V}$.

Proof. It is sufficient to show that (4.2.5) holds on the domain $\pi^{-1}(\mathrm{Reg}(v_f))$ which is dense in M. By Proposition (2.5.1), the period-energy relation for v_f says that $d\varpi \wedge df = 0$ on $\mathrm{Reg}(v_f)$ and hence

$$d\omega \wedge d(f \circ \pi_1) = 0 \tag{4.2.6}$$

on $\pi_1^{-1}(\mathrm{Reg}(v_f))$. The hypotheses of the theorem imply that $d(f \circ \pi_1) \neq 0$ on $\pi_1^{-1}(\mathrm{Reg}(v_f))$. On the other hand, taking into account (4.1.2), (4.1.4), (4.1.9) and the identity $\hat{v}_f = V_{f \circ \pi_1}^{(1)}$, we get

$$\begin{aligned} \mathcal{L}_{\mathbb{V}} F = \mathcal{L}_{\hat{v}_f} F &= \pi_1^* \sigma_1 (V_{f\circ\pi_1}^{(1)}, V_F^{(1)}) \\ &= -\pi_1^* \sigma_1 (V_F^{(1)}, V_{f\circ\pi_1}^{(1)}) = -\mathcal{L}_{\mathbb{W}}(f \circ \pi_1). \end{aligned}$$

It follows from here and the $\mathbb{S}^1$-invariance of $f \circ \pi_1$ that

$$\mathcal{L}_{\langle \mathbb{W} \rangle}(f \circ \pi_1) = \langle \mathcal{L}_{\mathbb{W}}(f \circ \pi_1) \rangle = -\langle \mathcal{L}_{\mathbb{V}} F \rangle = -\mathcal{L}_{\mathbb{V}} \langle F \rangle = 0.$$

Finally, using these relations and applying the interior product with $\langle \mathbb{W} \rangle$ to both sides of (4.2.6), we get the equality

$$0 = (\mathbf{i}_{\langle \mathbb{W} \rangle} d\omega) d(f \circ \pi_1) - (\mathbf{i}_{\langle \mathbb{W} \rangle} d(f \circ \pi_1)) d\omega = -(\mathcal{L}_{\langle \mathbb{W} \rangle}\omega) d(f \circ \pi_1)$$

which implies (4.2.5). ∎

Now, on the phase space $(M = S_1 \times S_2, \sigma = \sigma^{(1)} + \varepsilon \sigma^{(2)})$, let us consider the perturbed Hamiltonian is of the form

$$H_\varepsilon = f \circ \pi_1 + \varepsilon F + \frac{\varepsilon^2}{2} G + O(\varepsilon^3) \tag{4.2.7}$$

for a certain $G \in C^\infty(M)$. For $\varepsilon \neq 0$, the corresponding Hamiltonian vector field is represented as

$$V_{H_\varepsilon} = \mathbb{V} + \varepsilon(\mathbb{W} + \frac{1}{2} V_G^{(2)}) + O(\varepsilon^2), \tag{4.2.8}$$

Theorem 4.2.2 *Suppose that the unperturbed vector field* $\mathbb{V}$ *satisfies the hypothesis of Theorem* 4.2.1. *Then, the perturbed Hamiltonian vector field* V_{H_ε} *(*4.2.8*) admits a normalization of first order with respect to* $\mathbb{V}$*, that is, for every open domain* $N \subset M$ *with compact closure and small enough* ε*, there exists a (noncanonical) near identity transformation* $\Phi_\varepsilon : N \to M$ *such that*

$$\Phi_\varepsilon^* V_{H_\varepsilon} = \mathbb{V} + \varepsilon(\langle \mathbb{W} \rangle + \frac{1}{2}\langle V_G^{(2)} \rangle) + O(\varepsilon^2) \tag{4.2.9}$$

and

$$[\mathbb{V}, \langle \mathbb{W} \rangle] = [\mathbb{V}, \langle V_G^{(2)} \rangle] = 0. \tag{4.2.10}$$

Proof. It is clear that $\mathcal{L}_{V_G^{(2)}}\omega = 0$ and hence $\mathcal{L}_{\langle V_G^{(2)}\rangle}\omega = 0$. Then, by (4.2.5), we conclude that the perturbation vector field satisfies the compatibility condition

$$\mathcal{L}_{\langle \mathbb{W}+\frac{1}{2}V_G^{(2)}\rangle}\omega = \langle \mathcal{L}_{\mathbb{W}+\frac{1}{2}V_G^{(2)}}\omega\rangle = \mathcal{L}_{\langle\mathbb{W}\rangle}\omega + \frac{1}{2}\mathcal{L}_{\langle V_G^{(2)}\rangle}\omega = 0$$

It follows from here and (4.2.5) that conditions (4.2.10) are satisfied. Finally, according to Theorem 3.1.1, the normalization transformation Φ_ε in (4.2.10) is defined as the time-ε flow of the vector field

$$Z = \frac{1}{\omega}\mathcal{S}(V_F^{(1)} + \frac{1}{2}V_G^{(2)}) + \frac{1}{\omega^3}\mathcal{S}^2(\mathcal{L}_{V_F^{(1)}+\frac{1}{2}V_G^{(2)}}\omega)\mathbb{V}.$$

■

By this theorem and Proposition 3.2.22, we derive the following fact.

Corollary 4.2.3 *The frequency function $\omega = \varpi\circ\pi_1$ is an adiabatic invariant of Hamiltonian system (*4.1.7*).*

Proposition 4.2.4 *Under the hypothesis of Theorem* 4.2.2*, we have the following representation*

$$\langle V_G^{(2)}\rangle = V_{\langle G\rangle}^{(2)}, \tag{4.2.11}$$

$$\langle\mathbb{W}\rangle = P^{(1)} + P^{(2)} \tag{4.2.12}$$

where

$$P^{(1)} = \mathbf{i}_{\langle d_1 F\rangle}\ \Pi^{(1)}, \quad P^{(2)} = \mathbf{i}_\alpha\ \Pi^{(2)}, \tag{4.2.13}$$

$$\alpha := \mathbf{i}_{P^{(1)}} d_2\beta - \langle \mathbf{i}_{V_F^{(1)}}\beta\rangle, \quad \beta := \mathcal{S}(d_1\left(\frac{F}{\omega}\right)). \tag{4.2.14}$$

For the proof of this proposition, we need the following fact.

Lemma 4.2.5 *The Poisson tensor $\Pi^{(2)}$ and the presymplectic 2-forms $\sigma^{(1)}$ and*

$$\sigma^{(2)} - d\beta \tag{4.2.15}$$

*are invariant with respect to the $\mathbb{S}^1$-action on M with infinitesimal generator $\Upsilon = \frac{\mathbb{V}}{\omega}$. Here, the 1-form β is given by (*4.2.14*).*

Proof. By the hypotheses, we have $\omega = \varpi\circ\pi_1$, where $\varpi : S_1 \to \mathbb{R}$ is the frequency function of the period flow of the Hamiltonian vector field v_f. The period energy relation says that $d\varpi \wedge df = 0$. Computing the Lie derivative of $\Pi^{(2)}$ and $\sigma^{(1)}$ along the infinitesimal generator $\Upsilon = \frac{1}{\omega}\hat{v}_f + V_{\frac{F}{\omega}}^{(2)}$ gives

$$\mathcal{L}_\Upsilon \Pi^{(2)} = \mathcal{L}_{V_{\frac{F}{\omega}}^{(2)}}\Pi^{(2)} = 0,$$

$$\mathcal{L}_\Upsilon\sigma^{(1)} = -d(\frac{1}{\omega}d_1(f\circ\pi_1)) = \frac{1}{\omega^2}d_1(\varpi\circ\pi_1)\wedge d_1(f\circ\pi_1) = 0.$$

Now, using the property of averaging operator and the identity $\mathbf{i}_\Upsilon \sigma^{(2)} = -d_2(\frac{F}{\omega})$, we get

$$0 = \langle \mathcal{L}_\Upsilon \sigma^{(2)} \rangle = \langle d(\mathbf{i}_{\boldsymbol{\Upsilon}} \sigma^{(\mathbf{2})}) \rangle = -\langle d_1 \circ d_2 \left(\frac{F}{\omega}\right) \rangle.$$

Taking into account this equality and the identity $d\beta = -\mathcal{S}(d_1 \circ d_2 \left(\frac{F}{\omega}\right))$ together with the properties of the operator $\mathcal{S}$, we verify the invariance of 2-form (4.2.15)

$$\begin{aligned} \mathcal{L}_\Upsilon \sigma^{(2)} + \mathcal{L}_\Upsilon \circ \mathcal{S}\left(d_1 \circ d_2 \left(\frac{F}{\omega}\right)\right) &= d(\mathbf{i}_\Upsilon \sigma^{(2)}) + d_1 \circ d_2 \left(\frac{F}{\omega}\right) \\ &= -d_1 \circ d_2 \left(\frac{F}{\omega}\right) + d_1 \circ d_2 \left(\frac{F}{\omega}\right). \end{aligned}$$

■

Proof of Proposition 4.2.4. The equality (4.2.11) follows from the representation $V_G^{(2)} = \mathbf{i}_{dG} \Pi^{(2)}$ and the $\mathbb{S}^1$-invariance of the Poisson tensor $\Pi^{(2)}$. According to splitting $TM = TS_1 \oplus TS_2$, we have the decomposition $\langle \mathbb{W} \rangle = P^{(1)} + P^{(2)}$. By the $\mathbb{S}^1$-invariance of $\sigma^{(1)}$, we have

$$\mathbf{i}_{\langle V_F^{(1)} \rangle} \sigma^{(1)} = \langle \mathbf{i}_{V_F^{(1)}} \sigma^{(1)} \rangle = -\langle d_1 F \rangle. \tag{4.2.16}$$

This says that in terms of Poisson tensor $\Pi^{(1)}$, the vector field $P^{(1)}$ in (4.2.12) has the representation (4.2.13). Next,

$$\mathbf{i}_{V_F^{(1)}}(\sigma^{(2)} - d\beta) = -\mathbf{i}_{V_F^{(1)}} d_1 \beta - \mathbf{i}_{V_F^{(1)}} d_2 \beta,$$

and hence

$$\langle \mathbf{i}_{V_F^{(1)}}(\sigma^{(2)} - d\beta) \rangle = -\langle \mathbf{i}_{V_F^{(1)}} d_1 \beta \rangle - \langle \mathbf{i}_{V_F^{(1)}} d_2 \beta \rangle. \tag{4.2.17}$$

On the other hand, using the $\mathbb{S}^1$-invariance of 2-form (4.2.15), we get

$$\begin{aligned} \langle \mathbf{i}_{V_F^{(1)}}(\sigma^{(2)} - d\beta) \rangle &= \mathbf{i}_{\langle V_F^{(1)} \rangle}(\sigma^{(2)} - d\beta) = \mathbf{i}_{P^{(1)}}(\sigma^{(2)} - d\beta) + \mathbf{i}_{P^{(2)}}(\sigma^{(2)} - d\beta), \\ &= -\mathbf{i}_{P^{(1)}} d_1 \beta - \mathbf{i}_{P^{(1)}} d_2 \beta + \mathbf{i}_{P^{(2)}} \sigma^{(2)} - \mathbf{i}_{P^{(2)}} d_1 \beta - \\ &\quad \mathbf{i}_{P^{(2)}} d_2 \beta. \end{aligned} \tag{4.2.18}$$

Comparing (4.2.17) and (4.2.18) gives

$$\mathbf{i}_{P^{(2)}} \sigma^{(2)} = \mathbf{i}_{P^{(2)}} d_1 \beta + \mathbf{i}_{P^{(2)}} d_2 \beta,$$

$$\mathbf{i}_{P^{(1)}} d_1 \beta + \mathbf{i}_{P^{(1)}} d_2 \beta = \langle \mathbf{i}_{V_F^{(1)}} d_1 \beta \rangle + \langle \mathbf{i}_{V_F^{(1)}} d_2 \beta \rangle.$$

Rewriting the last equation in terms of Poisson tensor $\Pi^{(2)}$, we get (4.2.12). ■

The Adiabatic Case. In the situation when $f \equiv 0$ and $G \equiv 0$ in (4.2.7), we arrive at a perturbed Hamiltonian model

$$V_{H_\varepsilon} = V_F^{(2)} + \varepsilon V_F^{(1)}$$

which appears in the theory of adiabatic approximation [6, 62]. Suppose that the flow of $\mathbb{V} = V_F^{(2)}$ is periodic with frequency function ω. In this case, the periodicity of the flow of $\mathbb{V}$ does not imply that the perturbation vector field $\mathbb{W} = V_F^{(1)}$ satisfies compatibility condition (4.2.5) and hence Theorem 4.2.2 does not provide the existence of the first order normalization of V_{H_ε} relative to $V_F^{(2)}$. The period-energy relation for the restriction of $V_F^{(2)}$ to the symplectic slices $\{m_1\} \times S_2$ implies only that $d_2\omega \wedge d_2 F = 0$. On other hand by the general result of Theorem 4.2.2, we conclude that under the near identity transformation $\mathcal{T}_\varepsilon$ (4.2.9), the perturbed vector field V_{H_ε} is transformed to normal form of first order

$$\mathcal{T}_\varepsilon^*(V_F^{(2)} + \varepsilon V_F^{(1)}) = V_F^{(2)} + \varepsilon \mathbf{i}_{dF} \langle \Pi^{(1)} \rangle + O(\varepsilon^2),$$

which is invariant relative to the $\mathbb{S}^1$-action on M associated to the periodic flow of $V_F^{(2)}$. Then, the compatibility condition (4.2.5) reads

$$\langle \Pi^{(1)} \rangle (dF, d\omega) = 0.$$

Remark that the $\mathbb{S}^1$-average $\langle \Pi^{(1)} \rangle$ of $\Pi^{(1)}$ is not a Poisson tensor in general.

In contrast to the regular case (see Section 3.1.3), vector field $V_F^{(1)}$ is not Hamiltonian on (M, σ) in general. As consequence, the normalization transformation Φ_ε is not necessarily canonical. This "defect" of the normalization transformation comes from the following feature: the $\mathbb{S}^1$-action associated to the periodic flow of the unperturbed vector field $\mathbb{V}$ does not preserve the symplectic form σ (see condition (4.1.18)).

4.2.2 Periodicity criteria and resonances

The periodicity of the flow of vector field $\mathbb{V}$ (4.1.8) can be formulated as a resonance relation. Suppose that

- $\mathbb{V}$ is a complete vector field;
- the flow φ^t of v_f is periodic with frequency function $\varpi : S_1 \to \mathbb{R}$;
- the regular set $\mathrm{Reg}(v_f)$ is dense in S_1 and the orbit $t \mapsto \varphi^t(m_1)$ through every point $m_1 \in \mathrm{Reg}(v_f)$ is $\tau(m_1)$-minimally periodic, where $\tau(m_1) = \frac{2\pi}{\varpi(m_1)}$.

It follows from these conditions that v_f satisfies all hypotheses of Theorem 4.2.1 and the $\mathbb{S}^1$-action associated to the periodic flow φ^t is free on $\mathrm{Reg}(v_f)$. The group property of the flow $\mathrm{Fl}_{\mathbb{V}}^t$ implies the relations

$$\mathcal{G}_{m_1}^{t_1+t_2} = \mathcal{G}_{\varphi^{t_2}(m_1)}^{t_1} \circ \mathcal{G}_{m_1}^{t_2} = \mathcal{G}_{\varphi^{t_1}(m_1)}^{t_2} \circ \mathcal{G}_{m_1}^{t_1}$$

for any $m_1 \in S_1$ and $m_2 \in S_2$. In particular, we have $\mathcal{G}_{m_1}^{t+\tau(m_1)} = \mathcal{G}_{m_1}^t \circ g_{m_1}$, where

$$g_{m_1} := \mathcal{G}_{m_1}^{\tau(m_1)}. \tag{4.2.19}$$

Definition 4.2.1 *The symplectomorphism* $g_{m_1}: S_2 \to S_2$ *in* (4.2.19) *is called the monodromy of the flow* $\mathrm{Fl}^t_{\mathbb{V}}$ *over a point* $m_1 \in S_1$.

Proposition 4.2.6 *The flow* $\mathrm{Fl}^t_{\mathbb{V}}$ *is periodic if and only if there exists an integer* $k \geq 1$ *such that*

$$g^k_{m_1} = \mathrm{id} \qquad \forall m_1 \in \mathrm{Reg}(v_f). \tag{4.2.20}$$

In this case, the corresponding frequency and period functions can be defined as

$$\omega = \frac{1}{k}\varpi \circ \pi_1, \qquad \textit{and} \qquad T = k\tau \circ \pi_1.$$

Proof. We assume that flow of $\mathbb{V}$ is periodic with period function T. Since $\mathbb{V}$ and v_f are π_1-related, there exists a positive integer k such that $T = k\tau \circ \pi_1$. Conversely, we assume that there exists and integer $k \geq 1$ such that condition (4.2.20) holds. Let $T = k\tau \circ \pi_1$. By the group property of $\mathcal{G}$, equation (4.2.1) and periodicity of φ^t, we have

$$\mathrm{Fl}^{t+T}_{\mathbb{V}}(m_1, m_2) = (\varphi^{t+T}(m_1), \mathcal{G}^t_{m_1} \circ g^k_{m_1}(m_2)) = (\varphi^t(m_1), \mathcal{G}^t_{m_1}(m_2)).$$

Therefore, $\mathrm{Fl}^t_{\mathbb{V}}$ is periodic with period function T. ■

It is naturally to separate the resonance condition (4.2.20) into two hypotheses. First, we assume that the monodromy mapping does not depend on the points in S_1 up to conjugation, that is, for any $m_1, \tilde{m}_1 \in S_1$ there exists a diffeomorphism $\mathcal{U}: S_2 \to S_2$ such that

$$g_{\tilde{m}_1} = \mathcal{U} \circ g_{m_1} \circ \mathcal{U}^{-1}. \tag{4.2.21}$$

Then, the resonance condition (4.2.20) reads

$$g^k_{m^0_1} = \mathrm{id}, \tag{4.2.22}$$

where $m^0_1 \in \mathrm{Reg}(v_f)$ is fixed.

In the linear case, condition (4.2.21) is known as the "isospectral deformation" property.

Example 4.2.1 *Consider perturbed model* (4.1.6) *in the case when* $S_1 = \mathbb{S}^1 \times \mathbb{R} = \{(s, \alpha (\mathrm{mod}\, 2\pi))\}$ *is a cylinder and* $S_2 = \mathbb{R}^2 = \{x = (x_1, x_2)\}$ *is a plane equipped with canonical symplectic forms* $\sigma_1 = ds \wedge d\alpha$ *and* $\sigma_2 = dx_1 \wedge dx_2$*, respectively. Suppose that* $f = f(s)$ *and the perturbation term in the Hamiltonian* H_ε *is a quadratic function in the fast variables,*

$$F(s, \alpha, x) = \frac{1}{2}\left\langle \mathbf{J}\mathbf{V}(s, \alpha)x, x \right\rangle, \tag{4.2.23}$$

where $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $\mathbf{V}: \mathbb{S}^1 \times \mathbb{R} \to \mathrm{sl}(2;\mathbb{R})$ *is a smooth matrix-valued function. In this case, the dynamical system of the unperturbed vector field* $\mathbb{V}$ *is of the form*

$$\frac{ds}{dt} = 0, \quad \frac{d\alpha}{dt} = \varpi(s), \tag{4.2.24}$$

$$\frac{dx}{dt} = \mathbf{V}(s,\alpha))x, \tag{4.2.25}$$

where $\varpi(s) = \frac{\partial f}{\partial s} > 0$. It is clear that the flow of this system satisfies all conditions above and is given by

$$\mathrm{Fl}^t_{\mathbb{V}}(s,\alpha,x) = (s, \varpi(s)t + \alpha, \mathbf{G}_{s,\alpha}x)$$

where $\mathbf{G}_{s,\alpha} \in \mathrm{Sp}(1;\mathbb{R})$ is the fundamental solution of the s-dependent periodic linear Hamiltonian system:

$$\frac{d}{d\alpha}\mathbf{G}_{s,\alpha} = \frac{1}{\varpi(s)}\mathbf{V}(s,\alpha)\mathbf{G}_{s,\alpha},$$
$$\mathbf{G}_{s,0} = \mathbf{I}.$$

Since $\mathbf{V}(s,\alpha+2\pi) = \mathbf{V}(s,\alpha)$, we have that $\mathbf{G}_{s,\alpha+2\pi} = \mathbf{G}_{s,\alpha}\cdot\mathbf{G}_{s,2\pi}$ and the monodromy of $\mathbb{V}$ is a linear symplectic mapping $g_s : \mathbb{R}^2 \to \mathbb{R}^2$ given by $g_s = \mathbf{G}_{s,2\pi}$. Remark that the minimal period of $\mathbf{V}$ in α is not necessarily equals 2π. We have the following fact [26] ; the linear monodromy g_s possesses the property (4.2.21) if and only if there exists a vector field on $(\mathbb{S}^1\times\mathbb{R})\times\mathbb{R}^2$ of the form $\mathbb{U} = \frac{\partial}{\partial s} + < \mathbf{U}((s,\alpha)x, \frac{\partial}{\partial x} >$ which commutes with $\mathbb{V}$, that is, the following "zero curvature" condition holds

$$\frac{\partial\mathbf{U}}{\partial\alpha} - \frac{\partial\mathbf{V}}{\partial s} + \frac{1}{\varpi}[\mathbf{U},\mathbf{V}] = 0.$$

Under this hypothesis, the periodicity condition (4.2.22) is formulated as follows. Fix $s_0 \in \mathbb{R}$, then the flow of system (4.2.24), (4.2.25) is periodic if and only if $g_{s_0} = \mathbf{I}, -\mathbf{I}$ or

$$\mathrm{tr}\, g_{s_0} = 2\cos\left(2\pi\frac{m}{k}\right)$$

for some coprime integers $m,k \in \mathbb{Z}$ such that $0 < m < \frac{k}{2}$. In each case, the corresponding period functions are $T(s) = \frac{2\pi}{\varpi(s)}, \frac{4\pi}{\varpi(s)}, \frac{2\pi k}{\varpi(s)}$. This result follows from the Floquet theory for linear periodic Hamiltonian systems [26, 79] and says that the periodicity condition for the flow of $\mathbb{V}$ coincides with the resonance condition for the frequencies of the quasiperiodic motion of system (4.2.24), (4.2.25) which are defined by ϖ and the Floquet exponent.

Condition (4.2.21) can be also derived from the following homogeneity arguments. Let Λ be a smooth manifold which will be play the role of a parameter space. Suppose we have two smooth mappings

$$\varrho_1 : \Lambda\times S_1 \to S_1, \quad \varrho_2 : \Lambda\times S_2 \to S_2$$

of the form $\varrho_1(\lambda,m_1) = \varrho_1^\lambda(m_1)$ and $\varrho_2(\lambda,m_2) = \varrho_2^\lambda(m_2)$ where $\varrho_1^\lambda : S_1 \to S_1$ and $\varrho_2^\lambda : S_2 \to S_2$ are diffeomorphisms for every $\lambda \in \Lambda$. Let ϱ^λ be the λ-dependent diffeomorphism on $M = S_1 \times S_2$ defined as as the direct product $\varrho^\lambda(m_1,m_2) = (\varrho_1^\lambda(m_1), \varrho_2^\lambda(m_2))$. Assume that the family $\{\varrho^\lambda\}_{\lambda\in\Lambda}$ of diffeomorphisms gives a "conformal symmetry " for the unperturbed vector field in the sense that

$$(\varrho^\lambda)^*\mathbb{V} = \kappa(\lambda)\mathbb{V} \tag{4.2.26}$$

for some nowhere vanishing smooth function $\kappa : \Lambda \to \mathbb{R}$. It follows that $(\varrho_1^\lambda)^* v_f = \kappa(\lambda) v_f$. In terms of the flows these relations read

$$\varrho^\lambda \circ \mathrm{Fl}_{\mathbb{V}}^{\kappa(\lambda)t} = \mathrm{Fl}_{\mathbb{V}}^t \circ \varrho^\lambda \tag{4.2.27}$$

and

$$\varrho_1^\lambda \circ \varphi^{\kappa(\lambda)t} = \varphi^t \circ \varrho_1^\lambda. \tag{4.2.28}$$

By (4.2.1), we have

$$\mathrm{Fl}_{\mathbb{V}}^t(\varrho_1^\lambda(m_1), \varrho_2^\lambda(m_2)) = \left(\varphi^t(\varrho_1^\lambda(m_1), \mathcal{G}^t_{\varrho_1^\lambda(m_1)}(\varrho_2^\lambda(m_2))\right).$$

On the other hand,

$$\begin{aligned} \varrho^\lambda(\mathrm{Fl}_{\mathbb{V}}^{\kappa(\lambda)t}(m_1, m_2)) &= \varrho^\lambda\left(\varphi^{\kappa(\lambda)t}(m_1), \mathcal{G}_{m_1}^{\kappa(\lambda)t}(m_2)\right) \\ &= \left(\varrho_1^\lambda \circ \varphi^{\kappa(\lambda)t}(m_1), \varrho_2^\lambda \circ \mathcal{G}_{m_1}^{\kappa(\lambda)t}(m_2)\right). \end{aligned}$$

By property (4.2.27), the left hand sides of the last two equalities coincide and hence we get

$$\mathcal{G}^t_{\varrho_1^\lambda(m_1)} = \varrho_2^\lambda \circ \mathcal{G}_{m_1}^{\kappa(\lambda)t} \circ (\varrho_2^\lambda)^{-1} \tag{4.2.29}$$

for any $m_1 \in S_1$. Now, let $\tau : S_1 \to \mathbb{R}$ be the period function of the flow φ^t of v_f. Then, identity (4.2.28) implies $\tau(\varrho_1^\lambda(m_1))$

$$\varphi^{\kappa(\lambda)\tau(\varrho_1^\lambda(m_1))}(m_1) = m_1$$

for all $m_1 \in S_1, \lambda \in \Lambda$. Since $v_f \neq 0$ on S_1, this identity says that $\kappa(\lambda)\tau(\varrho_1^\lambda(m_1))$ is independent of λ. Suppose that S_1 is connected and there exists $\lambda_0 \in \Lambda$ such that

$$\kappa(\lambda_0) = 1 \text{ and } \varrho_1^{\lambda_0} = \mathrm{id}\,. \tag{4.2.30}$$

Then, we get the equality

$$\kappa(\lambda) = \frac{\tau(m_1)}{\tau(\varrho_1^\lambda(m_1))}.$$

Putting this relation into identity (4.2.29) for $t = \tau(\varrho_1^\lambda(m_1))$, we arrive at the following fact.

Proposition 4.2.7 *Assume that $\mathbb{V}$ is complete and the flow φ^t of v_f is periodic with period function $\tau : S_1 \to \mathbb{R}$. Under hypotheses (4.2.26), (4.2.30), we have the following variation of parameters formula for the monodromy mapping $g_{m_1} : S_2 \to S_2$*

$$g_{\varrho_1^\lambda(m_1)} = \varrho_2^\lambda \circ g_{m_1} \circ (\varrho_2^\lambda)^{-1} \tag{4.2.31}$$

for any $m_1 \in S_1, \lambda \in \Lambda$.

In the case when the $\mathbb{S}^1$-action associated with flow φ^t is free, the orbit space $\mathrm{Orb}(v_f)$ is a smooth manifold. If the hypotheses of Proposition 4.2.7 hold for the parameter space $\Lambda = \mathrm{Orb}(v_f)$, then the "isospectral deformation" condition (4.2.21) is satisfied.

Example 4.2.2 *On the standard slow-fast space* $(\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2, \sigma = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2)$*, consider the Hamiltonian system* $H_\varepsilon = f \circ \pi_1 + \varepsilon F$*, where the functions* f *and* F *are homogeneous in the weighted degree with a weight* n*,*

$$f(\lambda^s p_1, \lambda^l q_1) = \lambda^n f(p_1, q_1),$$

$$F(\lambda^s p_1, \lambda^l q_1, \lambda^s p_2, \lambda^l q_2) = \lambda^n F(p_1, q_1, p_2, q_2)$$

for all $\lambda > 0$ *and some nonzero integers* s *and* l*. If the level set* $\{f(p_1, q_1) = 1\}$ *is bounded in* $\mathbb{R}^2$*, the open domain* S_1 *is foliated by periodic trajectories of the Hamiltonian vector field* $v_f = \frac{\partial f}{\partial p_1}\frac{\partial}{\partial q_1} - \frac{\partial f}{\partial q_1}\frac{\partial}{\partial p_1}$*. In this case,* $\Lambda = \mathrm{Orb}(v_f) = \mathbb{R}_+$ *and conditions (*4.2.26*), (*4.2.30*) hold for scaling map*

$$\varrho^\lambda(p_1, q_1, p_2, q_2) = (\lambda^s p_1, \lambda^l q_1, \lambda^s p_2, \lambda^l q_2)$$

where $\lambda_0 = 1$ *and formula (*4.2.26*) reads* $\kappa(\lambda) = \lambda^{n-s-l}$*.*

4.3 $\mathbb{S}^1$-Invariant Hamiltonian Normalization

In this section, we study a perturbed Hamiltonian system of the form (4.1.1), (4.1.6) in the case when the flow of the unperturbed system is not necessarily periodic but this system possesses a $\mathbb{S}^1$-symmetry. We formulate some results on the approximation of the original perturbed model by an ε-dependent Hamiltonian system with $\mathbb{S}^1$-symmetry. This setting is not standard in the Hamiltonian perturbation theory [6, 50] because of the singular dependence of the symplectic structure on the small parameter ε.

4.3.1 Hamiltonian systems with rapidly varying perturbations

On the standard phase space $(\mathbb{R}^4, dp_1 \wedge dq_1 + dP_2 \wedge dQ_2)$, let us consider a Hamiltonian system of the form [19]

$$H = f(p_1, q_1) + \varepsilon F\left(p_1, q_1, \frac{P_2}{\varepsilon^\kappa}, \frac{Q_2}{\varepsilon^{1-\kappa}}\right), \tag{4.3.1}$$

where $\varepsilon \ll 1$ is a small parameter and $s \in [0, 1]$ is a constant. After rescaling $p_2 = \frac{P_2}{\varepsilon^\kappa}$, $q_2 = \frac{Q_2}{\varepsilon^{1-\kappa}}$, we get the ε-dependent symplectic form

$$\sigma = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2, \tag{4.3.2}$$

and the Hamiltonian

$$H_\varepsilon = f(p_1, q_1) + \varepsilon F(p_1, q_1, p_2, q_2), \tag{4.3.3}$$

depending regularly on ε. According to the appearance of the small parameter ε in the corresponding Poisson bracket

$$\{p_1, q_1\} = 1,$$

$$\{p_2, q_2\} = \frac{1}{\varepsilon},$$

one can separate the space coordinates into slow (p_1, q_1) and fast (p_2, q_2) variables. For $\varepsilon \neq 0$, the equations of motion of Hamiltonian system (4.3.2), (4.3.3) are of the form

$$\dot{p}_1 = -\frac{\partial f}{\partial q_1} - \varepsilon \frac{\partial F}{\partial q_1}, \quad \dot{q}_1 = \frac{\partial f}{\partial p_1} + \varepsilon \frac{\partial F}{\partial p_1}, \tag{4.3.4}$$

$$\dot{p}_2 = -\frac{\partial F}{\partial q_2}, \quad \dot{q}_2 = \frac{\partial F}{\partial p_2}, \tag{4.3.5}$$

and viewed as a perturbed dynamical system. As we have mentioned above, for $\varepsilon = 0$, the unperturbed system

$$\dot{p}_1 = -\frac{\partial f}{\partial q_1}, \quad \dot{q}_1 = \frac{\partial f}{\partial p_1}, \tag{4.3.6}$$

$$\dot{p}_2 = -\frac{\partial F}{\partial q_2}, \quad \dot{q}_2 = \frac{\partial F}{\partial p_2}, \tag{4.3.7}$$

is not Hamiltonian relative to σ, in general. According to the notations introduced in Section 4.1, the total space $M = \mathbb{R}^4$ is product of the symplectic planes $S_1 = \mathbb{R}^2_{p_1,q_1}$ and $S_2 = \mathbb{R}^2_{p_2,q_2}$ which is equipped with symplectic structure $\sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)}$, where $\sigma^{(1)} = dp_1 \wedge dq_1$ and $\sigma^{(2)} = dp_2 \wedge dq_2$. Moreover, the unperturbed vector field of system (4.3.6), (4.3.7) is represented in the form $\mathbb{V} = \hat{v}_f + V_F^{(2)}$, where

$$\hat{v}_f = -\frac{\partial f}{\partial q_1}\frac{\partial}{\partial p_1} + \frac{\partial f}{\partial p_1}\frac{\partial}{\partial q_1} \ , \tag{4.3.8}$$

$$V_F^{(2)} = -\frac{\partial F}{\partial q_2}\frac{\partial}{\partial p_2} + \frac{\partial F}{\partial p_2}\frac{\partial}{\partial q_2}.$$

Circle first integrals. As is usual in the perturbation theory, we need some good properties of the unperturbed dynamics. We make the following assumption:

- (*Symmetry Hypothesis*). In an invariant open domain $N \subseteq \mathbb{R}^4$, the unperturbed system (4.3.6), (4.3.7) admits a first integral $J = J(p_1, q_1, p_2, q_2)$,

$$\mathcal{L}_{\mathbb{V}} J \equiv \mathcal{L}_{\hat{v}_f} J - \frac{\partial F}{\partial q_2}\frac{\partial J}{\partial p_2} + \frac{\partial F}{\partial p_2}\frac{\partial J}{\partial q_2} = 0 \tag{4.3.9}$$

  such that the flow $\mathrm{Fl}^t_{V_J^{(2)}}$ of the vector field

$$V_J^{(2)} := \frac{\partial J}{\partial p_2}\frac{\partial}{\partial q_2} - \frac{\partial J}{\partial q_2}\frac{\partial}{\partial p_2} \tag{4.3.10}$$

  is *periodic* and the corresponding period function is equal to 2π,

$$\mathrm{Fl}^{t+2\pi}_{V_J^{(2)}} = \mathrm{Fl}^t_{V_J^{(2)}} . \tag{4.3.11}$$

This means that $\Upsilon = V_J^{(2)}$ is an infinitesimal generator of the $\mathbb{S}^1$-action on $N \subseteq \mathbb{R}^4 = \mathbb{R}^2_{p_1,q_1} \times \mathbb{R}^2_{p_2,q_2}$. The circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ acts along the slices $\{p_1, q_1\} \times \mathbb{R}^2_{p_2,q_2} \cap N$ in a Hamiltonian fashion, with momentum map J_{p_1,q_1} parametrically depending on the slow variables p_1, q_1. We admit that this $\mathbb{S}^1$-action is not necessarily free. Therefore, the phase portrait of the Hamiltonian system with one degree of freedom $(dp_2 \wedge dq_2, J_{p_1,q_1})$ consists of periodic orbits and rest points.

Definition 4.3.1 *A smooth function $J : N \to \mathbb{R}$ satisfying (4.3.9)-(4.3.11) is said to be a circle first integral of $\mathbb{V}$.*

Under above assumptions, the unperturbed vector field $\mathbb{V}$ has two first integrals $f \circ \pi_1$ and J. Here, as usual, $\pi_1 : \mathbb{R}^2_{p_1,q_1} \times \mathbb{R}^2_{p_2,q_2} \to \mathbb{R}^2_{p_1,q_1}$ denotes the canonical projection and throughout this section, by $\langle\cdot\rangle$ we will denote the average with respect the $\mathbb{S}^1$-action on N associated to the infinitesimal generator $V_J^{(2)}$.

It is clear that the first integrals $f \circ \pi_1$ and J are invariant under the $\mathbb{S}^1$-action. Moreover, the unperturbed vector field $\mathbb{V}$ is also $\mathbb{S}^1$-invariant, $\langle\mathbb{V}\rangle = \mathbb{V}$.

Indeed by (4.3.9) and formula (4.1.13), we get $[\mathbb{V}, V_J^{(2)}] = V^{(2)}_{\mathcal{L}_{\mathbb{V}}J} = 0$. On the other hand, it follows from (4.1.15) that

$$\mathcal{L}_{V_J^{(2)}}\sigma = -\varepsilon d_1 \circ d_2 J,$$

and then, for $\varepsilon \neq 0$, symplectic form (4.3.2) is not $\mathbb{S}^1$-invariant, and hence the $\mathbb{S}^1$-action is noncanonical, in general.

Original perturbed Hamiltonian system (4.3.2), (4.3.3) is nonintegrable, that is, does not admit an additional first integral. In what follows, a natural question is to construct an approximation of (4.3.2), (4.3.3) by a completely integrable system as $\varepsilon \to 0$. Remark that our setting is slightly unusual in Hamiltonian perturbation theory because of the singular dependence of the symplectic structure in the small parameter ε.

4.3.2 Approximate Hamiltonian models with $\mathbb{S}^1$-symmetry

Here we formulate the results on $\mathbb{S}^1$-invariant normal forms of perturbed Hamiltonian system (4.3.2), (4.3.3).

Suppose that the symmetry hypothesis holds and we are given a circle first integral $J : N \to \mathbb{R}$ of the unperturbed vector field $\mathbb{V}$ defined on an open subset $N \subset \mathbb{R}^4$ with compact closure. Consider the $\mathbb{S}^1$-action on N with infinitesimal generator $V_J^{(2)}$. Recall that by a near identity transformation we mean a smooth family of mappings $\mathcal{T}_\varepsilon : N \to \mathbb{R}^4, \varepsilon \in (-\delta, \delta)$ such that $\mathcal{T}_0 = \mathrm{id}$ and $\mathcal{T}_\varepsilon$ is a diffeomorphism onto its image.

Theorem 4.3.1 *For small enough $\varepsilon \neq 0$, the following assertions are true:*
(a) the $\mathbb{S}^1$-average $\langle\sigma\rangle$ of the original symplectic structure σ (4.3.2) is a nondegenerate closed 2-form on N;
(b) the $\mathbb{S}^1$-action is Hamiltonian relative to $\langle\sigma\rangle$,

$$\mathbf{i}_{V_J^{(2)}}\langle\sigma\rangle = -\varepsilon dJ^0,$$

where the function $J^0 : N \to \mathbb{R}$ is given by

$$J^0 := \mathbf{i}_{V_J^{(2)}}\langle p_2 dq_2\rangle \tag{4.3.12}$$

and related with J by

$$J - J^0 = g \circ \pi_1 \tag{4.3.13}$$

for a certain function $g \in C^\infty(\pi_1(N))$*;*
(c) there exits a near identity transformation $\mathcal{T}_\varepsilon : N \to \mathbb{R}^4$ *which gives a symplectomorphism between* σ *and* $\langle\sigma\rangle$*,*

$$\langle\sigma\rangle = \mathcal{T}_\varepsilon^* \sigma; \tag{4.3.14}$$

(d) the pull-back by $\mathcal{T}_\varepsilon$ *of the original Hamiltonian system (*4.3.2*), (*4.3.3*)*

$$(N, \langle\sigma\rangle,\ H_\varepsilon \circ \mathcal{T}_\varepsilon) \tag{4.3.15}$$

is ε^2*-close to the Hamiltonian system with* $\mathbb{S}^1$*-symmetry*

$$(N, \langle\sigma\rangle, \langle H_\varepsilon\rangle = f \circ \pi_1 + \varepsilon\langle F\rangle) \tag{4.3.16}$$

in the sense that

$$H_\varepsilon \circ \mathcal{T}_\varepsilon = \ \langle H_\varepsilon\rangle + O(\varepsilon^2). \tag{4.3.17}$$

(e) J^0 *is a first integral of the Hamiltonian vector field* $X_{\langle H_\varepsilon\rangle}$ *of* $\langle H_\varepsilon\rangle$ *on* $(N, \langle\sigma\rangle)$ *and*

$$X_{\langle H_\varepsilon\rangle} = \mathbb{V} + O(\varepsilon). \tag{4.3.18}$$

Remark 8 *Relations (*4.3.13*), (*4.3.18*) mean that* $V_J^{(2)} = V_{J^0}^{(2)}$ *and* J^0 *is also a circle first integral of* $\mathbb{V}$*. In general,* J *is not a first integral of (*4.3.16*), we have only that*

$$\mathcal{L}_{X_{\langle H_\varepsilon\rangle}} J = O(\varepsilon).$$

Remark 9 *The symmetry hypothesis can be reformulated as follows: the unperturbed vector field* $\mathbb{V}$ *admits a first integral* $G : N \to \mathbb{R}$ *of* $\mathbb{V}$ *such that the flow of* $V_G^{(2)}$ *is periodic with period function* $T : N \to \mathbb{R}$*. Then, the statement of Theorem* 4.3.1 *remains true, where formula (*4.3.12*) reads*

$$J^0 := \frac{T}{2\pi} \mathbf{i}_{V_G^{(2)}} \langle p_2 dq_2\rangle. \tag{4.3.19}$$

Therefore, Theorem 4.3.1 states that under the symmetry hypothesis, the original perturbed Hamiltonian model (4.3.2), (4.3.3) can be transformed by a near identity transformation to a system which is approximated by Hamiltonian system with $\mathbb{S}^1$-symmetry (4.3.16). The approximate Hamiltonian system is defined as the averaged system where the "new" symplectic form $\langle\sigma\rangle$ is a ε-deformation of the original one. The next result says that under natural additional assumptions the approximate Hamiltonian system (4.3.16) is completely integrable.

Proposition 4.3.2 *Assume that in addition to the symmetry hypothesis the following condition holds*
(i) there exists an open dense domain $\mathcal{U}$ *in* $\pi_1(N)$ *such that*

$$df \neq 0 \ \text{on}\ \mathcal{U}. \tag{4.3.20}$$

Then, the first integral J^0 *is functionally independent with Hamiltonian* $\langle H_\varepsilon\rangle$ *on* $N_0 = \pi_1^{-1}(\mathcal{U})$*. Moreover, if* *(ii) the flow of v_f is periodic on $\mathcal{U}$ with frequency function $\varpi : \mathcal{U} \to \mathbb{R}$ and the $\mathbb{S}^1$-action associated to infinitesimal generator $V^{(2)}_{J^0}$ is free on $N_0 = \pi_1^{-1}(\mathcal{U})$, then:*

(a) the flow of the unperturbed vector field $\mathbb{V}$ is quasiperiodic on N_0,

$$\mathbb{V} = \omega_1 \tilde{\Upsilon} + \omega_2 V^{(2)}_{J^0} \tag{4.3.21}$$

where $\tilde{\Upsilon}$ is a vector field with 2π-periodic flow on N_0 such that $[\tilde{\Upsilon}, V^{(2)}_{J^0}] = 0$ and $\omega_1 = \varpi \circ \pi_1, \omega_2 : N_0 \to \mathbb{R}$ are frequency functions;
(b) for small enough $\varepsilon \neq 0$, the averaged Hamiltonian system $(N, \langle\sigma\rangle,\ \ \langle H_\varepsilon\rangle = f \circ \pi_1 + \varepsilon\langle F\rangle)$ is completely integrable and the Liouville 2-tori are connected components of the level sets of two first integrals,

$$\mathbb{T}^2_{c_1,c_2}(\varepsilon) = \{f \circ \pi_1 + \varepsilon\langle F\rangle = c_1;\quad J^0 = c_2\}^{\mathrm{cc}} \tag{4.3.22}$$

which carry a quasiperiodic motion with frequencies $\omega_1 + O(\varepsilon)$ and $\omega_2 + O(\varepsilon)$.

Proof. By hypothesis of Proposition we have

- vector fields $\mathbb{V}$ and $V^{(2)}_{J^0}$ are independent on N and commute;
- the functions $f \circ \pi_1$ and J^0 are functionally independent on N and give mutual first integrals of $\mathbb{V}$ and $V^{(2)}_{J^0}$;
- the connected components of $f \circ \pi_1$ and J^0 are compact.

Then, it is well-known (see, for example [6]) that $\{f \circ \pi_1 = c_1;\ J^0 = c_2\}^{\mathrm{cc}}$ is a 2-torus $\mathbb{T}^2_{c_1,c_2}(0)$ carrying a quasiperiodic motion along the trajectories of $\mathbb{V}$. It remains to derive some information about the corresponding frequency functions. Remark that the vector field $\frac{1}{\omega_1}\mathbb{V}$ is projectable relative to $\pi_1 : M \to S_1$ and descends to the vector field $\frac{1}{\varpi} v_f$ with 2π-periodic flow . Moreover, each torus $\mathbb{T}^2_{c_1,c_2}(0)$ is invariant with respect to the $\mathbb{S}^1$-action with infinitesimal generator $V^{(2)}_{J^0}$. It follows from these properties that for every $m \in \mathbb{T}^2_{c_1,c_2}(0)$ there exists a $\alpha = \alpha(m) \neq 0$ such that

$$\mathrm{Fl}^{2\pi}_{\frac{1}{\omega_1}\mathbb{V}}(m) = \mathrm{Fl}^{\alpha(m)}_{V^{(2)}_{J^0}}(m).$$

If we take another point $m' \in \mathbb{T}^2_{c_1,c_2}(0)$ such that $m' = \mathrm{Fl}^{t}_{\frac{1}{\omega_1}\mathbb{V}}(m)$, then $\mathrm{Fl}^{2\pi}_{\frac{1}{\omega_1}\mathbb{V}}(m') = \mathrm{Fl}^{2\pi+t}_{\frac{1}{\omega_1}\mathbb{V}}(m) = \mathrm{Fl}^{2\pi}_{\frac{1}{\omega_1}\mathbb{V}}(m)$ and hence $\alpha(m) = \alpha(m')$. In other words , the function $m \mapsto \alpha(m)$ is first integral of $\mathbb{V}$. Then, we can define the vector field $\tilde{\Upsilon}$ in (4.3.21) as

$$\tilde{\Upsilon} := \frac{1}{\omega_1}\mathbb{V} - \frac{\alpha}{2\pi} V^{(2)}_{J^0}.$$

Its flow is 2π-periodic,

$$\begin{aligned}\mathrm{Fl}^{2\pi}_{\tilde{\Upsilon}}(m) &= \mathrm{Fl}^{2\pi}_{\frac{1}{\omega_1}\mathbb{V}} \circ \mathrm{Fl}^{2\pi}_{\frac{-\alpha}{2\pi}V^{(2)}_{J^0}}(m) \\ &= \mathrm{Fl}^{2\pi}_{\frac{1}{\omega_1}\mathbb{V}} \circ \mathrm{Fl}^{-\alpha(m)}_{V^{(2)}_{J^0}}(m) = m.\end{aligned}$$

Therefore, the second frequency of the quasiperiodic motion is defined as

$$\omega_2 = \frac{\alpha}{2\pi}\varpi \circ \pi_1.$$

■

Notice that for $\varepsilon = 0$, the connected components

$$\mathbb{T}^2_{c_1,c_2}(0) = \{f \circ \pi_1 = c_1;\ J^0 = c_2\}^{\mathrm{cc}}$$

are just the quasiperiodic 2-tori of the unperturbed vector field $\mathbb{V}$.

Remark 10 *One can use Proposition* 4.3.2 *to apply the KAM arguments [6, 14] to the nearly integrable Hamiltonian system* $(N, \langle\sigma\rangle,\ H_\varepsilon \circ \mathcal{T}_\varepsilon)$ *to state the persistence of quasiperiodic tori* $\mathbb{T}^2_{c_1,c_2}(\varepsilon)$*. For more applications see also [18, 19]*

Adiabatic Models. The adiabatic situation appears in the case when the leading term in Hamiltonian (4.3.3) is zero, $f \equiv 0$. The dynamics due to the adiabatic model

$$(\sigma = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2, \quad H_\varepsilon = \varepsilon F(p_1, q_1, p_2, q_2)) \tag{4.3.23}$$

is described by the system

$$\dot{p}_1 = -\varepsilon\frac{\partial F}{\partial q_1}, \qquad \dot{q}_1 = \varepsilon\frac{\partial F}{\partial p_1}, \tag{4.3.24}$$

$$\dot{p}_2 = -\frac{\partial F}{\partial q_2}, \qquad \dot{q}_2 = \frac{\partial F}{\partial p_2}, \tag{4.3.25}$$

which is known as a *slow fast Hamiltonian system* [62].

Remark 11 *System (*4.3.24*), (*4.3.25*) can be also derive starting from a slow varying Hamiltonian*

$$H = F(\varepsilon^\kappa P_1, \varepsilon^{1-\kappa} Q_1, p_2, q_2)$$

on the standard phase space $(\mathbb{R}^4,\ dP_1 \wedge dQ_1 + dp_2 \wedge dq_2)$*. After the scaling* $p_1 = \varepsilon^\kappa P_1$*,* $q_1 = \varepsilon^{1-\kappa} Q_1$ *we arrive at the adiabatic model*

$$\left(\tilde{\sigma} = \frac{1}{\varepsilon}dp_1 \wedge dq_1 + dp_2 \wedge dq_2, H_\varepsilon = F(p_1, q_1, p_2, q_2)\right)$$

*whose dynamical system coincides with (*4.3.23*), (*4.3.24*).*

The unperturbed vector field of (4.3.24), (4.3.25) is of the form $\mathbb{V} = V_F^{(2)}$ and the corresponding dynamics is describes by the 1-dimensional Hamiltonian system $(\mathbb{R}^2, dp_2 \wedge dq_2, F_{p_1,q_1})$ for frozen value of the slow variables p_1, q_1:

$$\dot{p}_2 = -\frac{\partial F_{p_1,q_1}}{\partial q_2}, \qquad \dot{q}_2 = \frac{\partial F_{p_1,q_1}}{\partial p_2}, \tag{4.3.26}$$

where $F_{p_1,q_1}(p_2, q_2) = F(p_1, q_1, p_2, q_2)$. In this case, the symmetry hypothesis is formulated as follows: there exists an open domain N in $\mathbb{R}^4 = \mathbb{R}^2_{p_1,q_1} \times \mathbb{R}^2_{p_2,q_2}$ with compact closure which is invariant with respect to the flow of $\mathbb{V}$ and $\mathrm{Fl}^t_{V_F^{(2)}} : N \to N$ is periodic with period function $T : N \to \mathbb{R}$. This condition is reduced to the verification of the compactness of the level sets $\{F_{p_1,q_1} = \mathrm{const}\}$ in the slice $N_{p_1,q_1} = N \cap \{p_1, q_1\} \times \mathbb{R}^2_{p_2,q_2}$ for every $(p_1, q_1) \in S_1 = \pi_1(N)$. Consider the $\mathbb{S}^1$-action on N defined by the 2π-periodic flow of the vector field $\Upsilon = \frac{T}{2\pi} V_F^{(2)}$. The circle first integral in (4.3.12) is given by

$$J^0 := \frac{T}{2\pi} \mathbf{i}_{V_F^{(2)}} \langle p_2 dq_2 \rangle.$$

Then, we have the relationship

$$d_2 J^0 = \frac{T}{2\pi} d_2 F,$$

which says that the infinitesimal generator of the $\mathbb{S}^1$-action is just $\Upsilon = V_{J^0}^{(2)}$. In a open domain $N_0 \subseteq N$ where the $\mathbb{S}^1$-action is free, the circle first integral J^0 is defined as the *standard action* [5, 42] of Hamiltonian system (4.3.26) with one degree of freedom:

$$J^0_{p_1,q_1}(p_2, q_2) = \frac{1}{2\pi} \operatorname{Area}(D_{p_1,q_1}) = \frac{1}{2\pi} \oint\limits_{\gamma_{p_1,q_1}} p_2 dq_2$$

Here, $\gamma_{p_1,q_1}(t) = \mathrm{Fl}^t_{V_F^{(2)}}(p_2, q_2)$ and D_{p_1,q_1} is a domain in N_{p_1,q_1} bounded by the periodic trajectory γ_{p_1,q_1}. It is clear that F is invariant with respect to the $\mathbb{S}^1$-action and hence $\langle F \rangle = F$. But, if $d_1 \circ d_2 F \neq 0$, then the $\mathbb{S}^1$-action is not Hamiltonian relative to σ. Theorem 4.3.1 leads to the following normalization result.

Proposition 4.3.3 *Under the symmetry hypothesis, for small enough ε, a symplectomorphism $\mathcal{T}_\varepsilon : N \to \mathbb{R}^4$ between the original symplectic structure σ and its $\mathbb{S}^1$-average $\langle\sigma\rangle$ brings slow -fast Hamiltonian system (*4.3.24*), (*4.3.25*) to another one which is approximated* $\operatorname{mod} \varepsilon^2$ *by the Hamiltonian system with $\mathbb{S}^1$-symmetry*

$$(N, \langle\sigma\rangle, H_\varepsilon = \varepsilon F).$$

The $\mathbb{S}^1$-action associated to the infinitesimal generator $\frac{T}{2\pi} V_F^{(2)}$ is Hamiltonian on $(N, \langle\sigma\rangle)$ with momentum map εJ^0.

As we will show in Subsection 4.3 (Theorem 4.3.20), in the adiabatic case, the symplectomorphism $\mathcal{T}_\varepsilon$ can be viewed as a free coordinate normalization step in the proof of the classical adiabatic theorem [6, 62] (the usual method uses action-angle variables and generating functions).

4.3.3 The averaging procedure and the homotopy method

Here we give a proof of Theorem 4.3.1 in few steps which carry a general character and are based on the averaging technique on symplectic fibered spaces [28, 47, 55], the notion of weak coupling symplectic structures [56] and the Moser homotopy method [38].

Given 2-dimensional symplectic manifolds S_1 and S_2, we start with a Hamiltonian system $H_\varepsilon = f \circ \pi_1 + \varepsilon F$ on the 4-dimensional symplectic manifold $(M = S_1 \times S_2, \sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)})$. Through this section we will assume that the symmetry hypothesis holds and a circle first integral $J : N \to \mathbb{R}$ of the unperturbed vector field $\mathbb{V}$ is fixed. The corresponding $\mathbb{S}^1$-action on N is given by the infinitesimal generator $\Upsilon = V_J^{(2)}$ satisfying the relations $\mathbf{i}_{V_J^{(2)}}\sigma^{(2)} = -d_2 J$ and $\mathbf{i}_{V_J^{(2)}}\sigma^{(1)} = 0$. By (p_1, q_1) and (p_2, q_2) we will denote (local) Darboux coordinates on the symplectic surfaces S_1 and S_2 , respectively.

The $\mathbb{S}^1$-Average of the Symplectic Form. Recall that the $\mathbb{S}^1$-average of the ε-dependent symplectic form $\sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)}$ is defined by the formula

$$\langle\sigma\rangle := \frac{1}{2\pi}\int_0^{2\pi} (\mathrm{Fl}^t_{V_J^{(2)}})^*\sigma dt.$$

Since the exterior differential commutes with the averaging operator, the 2-form $\langle\sigma\rangle$ is closed but it is not necessarily nondegenerate for all $\varepsilon \neq 0$.

Lemma 4.3.4 *The $\mathbb{S}^1$-average $\langle\sigma\rangle$ has the representation*

$$\langle\sigma\rangle = \sigma - \varepsilon d\theta^0, \tag{4.3.27}$$

where the 1-form $\theta^0 = \theta^0_1 dp_1 + \theta^0_2 dq_1$ is given by

$$\theta^0 := \mathcal{S}(d_1 J) \equiv \frac{1}{2\pi}\int_0^{2\pi} (t-\pi)(\mathrm{Fl}^t_{V_J^{(2)}})^* d_1 J dt. \tag{4.3.28}$$

and has zero $\mathbb{S}^1$-average,

$$\langle\theta^0\rangle = 0. \tag{4.3.29}$$

Here $d_1 J = \frac{\partial J}{\partial p_1}dp_1 + \frac{\partial J}{\partial q_1}dq_1$. Moreover, the $\mathbb{S}^1$-average of $d_1 J$ is a closed 1-form which has a representation on N:

$$\langle d_1 J\rangle = \pi_1^*\varsigma \tag{4.3.30}$$

for a certain closed 1-form $\varsigma = \varsigma_1 dp_1 + \varsigma_2 dq_1$ on $\pi_1(N) \subset S_1$.

Proof. By Proposition 2.4.13, the closed 2- form σ splits into a $\mathbb{S}^1$-invariant form and an exact one,

$$\sigma = \langle\sigma\rangle + d(\mathbf{i}_{V_J^{(2)}}\mathcal{S}(\sigma)),$$

where $\mathcal{S}(\sigma) = \frac{1}{2\pi}\int_0^{2\pi}(t-\pi)\left(\mathrm{Fl}^t_{V_J^{(2)}}\right)^*\sigma$. Taking into account that the exterior differential d commutes with operator $\mathcal{S}$ and using the identity $\mathbf{i}_{V_J^{(2)}}\sigma = -\varepsilon d_2 J$, we get

$$\mathbf{i}_{V_{\mathbb{J}}^{(2)}}(\mathcal{S}(\sigma)) = \mathcal{S}(\mathbf{i}_{V_J^{(2)}}\sigma) = -\varepsilon\mathcal{S}(d_2 J) = -\varepsilon d \circ \mathcal{S}(J) + \varepsilon\mathcal{S}(d_1 J)$$

and hence $d(\mathbf{i}_{V_{\mathbb{J}}^{(2)}}\mathcal{S}(\sigma)) = \varepsilon d(\mathcal{S}(d_1 J))$. To prove (4.3.30), first, we remark that

$$0 = \langle \mathcal{L}_{V_J^{(2)}}\sigma^{(2)}\rangle = \langle d(\mathbf{i}_{V_J^{(2)}}\sigma^{(2)})\rangle = -d(\langle d_1 J\rangle) \tag{4.3.31}$$

and hence $\langle d_1 J\rangle$ is closed on N. On the other hand, by properties $(\mathrm{Fl}^t_{V_J^{(2)}})^* dp_1 = dp_1$ and $(\mathrm{Fl}^t_{V_J^{(2)}})^* dq_1 = dq_1$ we have $\langle d_1 J\rangle = \langle \frac{\partial J}{\partial p_1}\rangle dp_1 + \langle \frac{\partial J}{\partial q_1}\rangle dq_1$. It follows from here and (4.3.31) that

$$d_2\langle \frac{\partial J}{\partial p_1}\rangle = d_2\langle \frac{\partial J}{\partial q_1}\rangle = 0 \tag{4.3.32}$$

and hence (4.3.30) holds for $\varsigma_1 = \langle \frac{\partial J}{\partial p_1}\rangle$ and $\varsigma_2 = \langle \frac{\partial J}{\partial q_1}\rangle$. ■

Now, let us associate to the 1-form θ^0 (4.3.28) the following λ-dependent vector fields

$$Y_1^\lambda = \frac{\partial}{\partial p_1} + (1-\lambda)(\frac{\partial \theta_1^0}{\partial p_2}\frac{\partial}{\partial q_2} - \frac{\partial \theta_1^0}{\partial q_2}\frac{\partial}{\partial p_2}), \tag{4.3.33}$$

$$Y_2^\lambda = \frac{\partial}{\partial q_1} + (1-\lambda)(\frac{\partial \theta_2^0}{\partial p_2}\frac{\partial}{\partial q_2} - \frac{\partial \theta_2^0}{\partial q_2}\frac{\partial}{\partial p_2}). \tag{4.3.34}$$

It is clear that , for every λ, the set

$$\{Y_1^\lambda, Y_2^\lambda, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial q_2}\} \tag{4.3.35}$$

defines a basis of vector fields on N. Consider also the dual basis of 1-forms

$$\{dp_1, dq_1, \Gamma_1^\lambda, \Gamma_2^\lambda\}, \tag{4.3.36}$$

where

$$\Gamma_1^\lambda := \ dp_2 + (1-\lambda)\left(\frac{\partial \theta_1^o}{\partial q_2}dp_1 + \frac{\partial \theta_2^0}{\partial q_2}dq_1\right), \tag{4.3.37}$$

$$\Gamma_2^\lambda \ := \ dq_2 + (1-\lambda)\left(\frac{\partial \theta_1^o}{\partial p_2}dp_1 + \frac{\partial \theta_2^0}{\partial p_2}dq_1\right). \tag{4.3.38}$$

Lemma 4.3.5 *There exists $\delta > 0$ such that*

$$\sigma_\lambda = (1-\lambda)\langle\sigma\rangle + \lambda\sigma \tag{4.3.39}$$

is a symplectic form on N for all $\varepsilon \in (0,\delta)$ and $\lambda \in [0,1]$.

Proof. By Lemma 4.3.4, the 2-form σ_λ is represented as

$$\sigma_\lambda = \sigma - (1-\lambda)\varepsilon d\theta^0.$$

It is clear that σ_λ is closed for all values of λ and ε. To study the nondegeneracy of the 2-form σ_λ, let us use bases (4.3.35) and (4.3.36). By a straightforward computation, we obtain

$$\sigma_\lambda(Y_1^\lambda, Y_2^\lambda) = 1 - \varepsilon\Delta_\lambda, \quad \sigma_\lambda(Y_1^\lambda, \frac{\partial}{\partial p_2}) = 0, \quad \sigma_\lambda(Y_1^\lambda, \frac{\partial}{\partial q_2}) = 0,$$

$$\sigma_\lambda(Y_2^\lambda, \frac{\partial}{\partial p_2}) = 0, \quad \sigma_\lambda(Y_2^\lambda, \frac{\partial}{\partial q_2}) = 0, \quad \sigma_\lambda(\frac{\partial}{\partial p_2}, \frac{\partial}{\partial q_2}) = \varepsilon,$$

where

$$\Delta_\lambda = (1-\lambda)\left[\frac{\partial \theta_2^0}{\partial p_1} - \frac{\partial \theta_1^0}{\partial q_1} + (1-\lambda)(\frac{\partial \theta_1^0}{\partial p_2}\frac{\partial \theta_2^0}{\partial q_2} - \frac{\partial \theta_1^0}{\partial q_2}\frac{\partial \theta_2^o}{\partial p_2})\right]. \tag{4.3.40}$$

It follows that the 2-form σ_λ has the representation

$$\sigma_\lambda = (1 - \varepsilon\Delta_\lambda) dp_1 \wedge dq_1 + \varepsilon \Gamma_1^\lambda \wedge \Gamma_2^\lambda \tag{4.3.41}$$

and has the coefficient matrix of σ_λ with respect to basis (4.3.36) is given by

$$\begin{bmatrix} 0 & -(1-\varepsilon\Delta_\lambda) & 0 & 0 \\ 1-\varepsilon\Delta_\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & \varepsilon & 0 \end{bmatrix}.$$

The determinant of this matrix equals $\varepsilon^2(1-\varepsilon\Delta_\lambda)^2$. Since $\bar{N}$ is compact, there exists $\delta > 0$ such that $1 - \varepsilon\Delta_\lambda \neq 0$ on N for all $\varepsilon \in (0, \delta)$ and $\lambda \in [0,1]$. ■

A Normalization Transformation $\mathcal{T}_\varepsilon$. To construct a normalization map $\mathcal{T}_\varepsilon$, one can use a parameter-dependent version of the Moser homotopy method [30]. For every $\varepsilon \in (0, \delta)$, consider the curve of symplectic forms σ_λ (4.3.39) on N joining $\langle\sigma\rangle$ with σ. Introduce the following (ε, λ)-dependent family vector fields on N :

$$Z_\lambda := \frac{1}{1-\varepsilon\Delta_\lambda}[-\theta_2^0 Y_1^\lambda + \theta_1^0 Y_2^\lambda]. \tag{4.3.42}$$

For a fixed ε, denote by Φ^λ the flow of the time-dependent vector field εZ_λ,

$$\frac{d}{d\lambda}\Phi^\lambda = \varepsilon Z_\lambda(\Phi^\lambda).$$

$$\Phi^0 = \mathrm{id}\,.$$

Lemma 4.3.6 *One can choose $\delta > 0$ in Lemma 4.3.5 such that the flow $\Phi^\lambda : N \to M$ is well-defined for all $\varepsilon \in (-\delta, \delta)$ and $\lambda \in [0,1]$. Moreover, we have*

$$(\Phi^\lambda)^*\sigma_\lambda = \langle\sigma\rangle, \quad \forall\lambda \in [0,1]. \tag{4.3.43}$$

Proof. By Lemma 4.3.5, we can fix $\delta > 0$ so that σ_λ is nondegenerate on N if $\varepsilon \in (0, \delta)$, that is, $1 - \varepsilon\Delta_\lambda \neq 0$. Since the closure $\bar{\mathcal{N}}$ is compact, shrinking δ if it is necessarily, we can arrange that the flow Φ^λ of the time-dependent vector field εZ_λ is defined on N for all $\varepsilon \in (-\delta, \delta)$ and $\lambda \in [0,1]$. Next, to verify (4.3.43), we have to show that

$$\frac{\partial}{\partial\lambda}((\Phi^\lambda)^*\sigma_\lambda) = 0. \tag{4.3.44}$$

Because of the identity

$$\frac{\partial}{\partial\lambda}((\Phi^\lambda)^*\sigma_\lambda) = (\Phi^\lambda)^*(\varepsilon\mathcal{L}_{Z_\lambda}\sigma_\lambda + \frac{\partial}{\partial\lambda}\sigma_\lambda),$$

condition (4.3.44) is equivalent to the equation for Z_λ

$$\varepsilon \mathcal{L}_{Z_\lambda} \sigma_\lambda + \frac{\partial}{\partial \lambda} \sigma_\lambda = 0.$$

Taking into account that $\frac{\partial}{\partial \lambda}\sigma_\lambda = \varepsilon d\theta^0$ and the closeness of σ_λ, we conclude that it is sufficient to find a vector field Z_λ satisfying the relation

$$\mathbf{i}_{Z_\lambda} \sigma_\lambda = -\theta^0. \tag{4.3.45}$$

Then, putting

$$Z_\lambda = c_1 Y_1^\lambda + c_2 Y_2^\lambda + c_3 \frac{\partial}{\partial p_2} + c_4 \frac{\partial}{\partial q_2}.$$

and using representation (4.3.41), we get equality

$$\mathbf{i}_{Z_\lambda} \sigma_\lambda = -c_2(1 - \varepsilon \Delta_\lambda) dp_1 + c_1 (1 - \varepsilon \Delta_\lambda) dq_1 - \varepsilon c_4 \Gamma_1^\lambda + \varepsilon c_3 \Gamma_2^\lambda.$$

which says that a solution to (4.3.45) is just given by formula (4.3.42). ■

Corollary 4.3.7 *The time-*1 *flow of time-dependent vector field εZ_λ (*4.3.42*)*

$$\mathcal{T}_\varepsilon = \mathrm{Fl}^\lambda_{\varepsilon Z_\lambda} \mid_{\lambda=1} \tag{4.3.46}$$

gives a symplectomorphism between the original symplectic structure σ and its $\mathbb{S}^1$-average $\langle \sigma \rangle$.

The $\mathbb{S}^1$-Invariant Plane Distribution $\mathcal{H}$. For $\lambda = 0$, the vector fields Y_1^λ and Y_1^λ in (4.3.33), (4.3.34) can be represented as

$$Y_1^0 = \frac{\partial}{\partial p_1} + V^{(2)}_{\theta_1^0}, \tag{4.3.47}$$

$$Y_2^0 = \frac{\partial}{\partial q_1} + V^{(2)}_{\theta_2^0}. \tag{4.3.48}$$

Lemma 4.3.8 *The vector fields Y_1^0 and Y_2^0 and the function*

$$\Delta_0 = \frac{\partial \theta_2^0}{\partial p_1} - \frac{\partial \theta_1^0}{\partial q_1} + \frac{\partial \theta_1^0}{\partial p_2}\frac{\partial \theta_2^0}{\partial q_2} - \frac{\partial \theta_1^0}{\partial q_2}\frac{\partial \theta_2^0}{\partial p_2}. \tag{4.3.49}$$

are $\mathbb{S}^1$-invariant, that is,

$$[Y_1^0, V_J^{(2)}] = [Y_2^0, V_J^{(2)}] = 0, \tag{4.3.50}$$

and

$$\mathcal{L}_{V_J^{(2)}} \Delta_0 = 0. \tag{4.3.51}$$

Proof. It follows from representations (4.3.47) and (4.3.48) that

$$[Y_1^0, V_J^{(2)}] = V^{(2)}_{\mathcal{L}_{Y_1^0} J}, \tag{4.3.52}$$

$$[Y_2^0, V_J^{(2)}] = V^{(2)}_{\mathcal{L}_{Y_2^0} J}. \tag{4.3.53}$$

By definition (4.3.28) of the 1-form θ^0 and properties of the operator $\mathcal{S}$ (see Chapter 2), we have

$$\mathcal{L}_{V_{\mathbb{J}}^{(2)}} \theta^0 = d_1 J - \langle d_1 J \rangle.$$

Taking the interior product of both sides of this equality with vector fields $\frac{\partial}{\partial p_1}$ and $\frac{\partial}{\partial q_1}$, we get

$$\mathcal{L}_{Y_1^0} J = \langle \frac{\partial J}{\partial p_1} \rangle, \quad \mathcal{L}_{Y_2^0} J = \langle \frac{\partial J}{\partial q_1} \rangle. \tag{4.3.54}$$

From here and property (4.3.32) we get the identities

$$d_2(\mathcal{L}_{Y_1^0} J) = d_2(\mathcal{L}_{Y_2^0} J) = 0$$

which together with (4.3.52) and (4.3.53) prove (4.3.50). Property (4.3.51) follows from (4.3.50) and the relation

$$1 - \varepsilon \Delta_0 = \langle \sigma \rangle (Y_1^0, Y_2^0)$$

which is consequence of (4.3.41). ■

Let us associate to every vector field $u = u^1 \frac{\partial}{\partial p_1} + u^2 \frac{\partial}{\partial q_1}$ on $\pi_1(N) \subset S_1$, the following vector field on the total space N :

$$\mathrm{hor}_u := u^1 Y_1^0 + u^2 Y_2^0 \equiv \hat{u} + V^{(2)}_{\mathbf{i}_{\hat{u}} \theta^0}. \tag{4.3.55}$$

By Lemma 4.3.8, hor_u is $\mathbb{S}^1$-invariant. Then, the (horizontal) plane distribution

$$\mathcal{H} := \{\mathrm{hor}_u \mid u \in \mathfrak{X}(\pi_1(N))\} \tag{4.3.56}$$

is also invariant with respect to the $\mathbb{S}^1$-action.

Lemma 4.3.9 *For any vector field u on $\pi_1(N)$ and $G \in C^\infty(N)$, the $\mathbb{S}^1$-average of the vector fields $V_G^{(2)}$ and $\hat{u}$ are given by the formulas*

$$\langle V_G^{(2)} \rangle = V^{(2)}_{\langle G \rangle}, \tag{4.3.57}$$

$$\langle \hat{u} \rangle = \mathrm{hor}_u \,. \tag{4.3.58}$$

Proof. To verify identity (4.3.57), let us consider the degenerate Poisson tensor on M:

$$\Pi^{(2)} = \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}. \tag{4.3.59}$$

Then, in terms of $\Pi^{(2)}$, the definition (4.1.2), (4.1.3) of the vector field $V_G^{(2)}$ reads

$$V_G^{(2)} = \mathbf{i}_{dG}\Pi^{(2)}. \tag{4.3.60}$$

This says that $V_G^{(2)}$ is a Hamiltonian vector field relative to $\Pi^{(2)}$ and hence the flow of $V_G^{(2)}$ preserves $\Pi^{(2)}$. In particular, we conclude that $\Pi^{(2)}$ is invariant with respect to the $\mathbb{S}^1$-action,

$$\mathcal{L}_{V_J^{(2)}}\Pi^{(2)} = 0. \tag{4.3.61}$$

Averaging both sides of (4.3.60) gives the identity $\langle V_G^{(2)}\rangle = \mathbf{i}_{d\langle G\rangle}\Pi^{(2)}$ which proves (4.3.57). Next, by (4.3.55) we have the representation $\hat{u} = \mathrm{hor}_u - V^{(2)}_{\mathbf{i}_{\hat{u}}\theta^0}$. Applying again the averaging operator to this equality and using the $\mathbb{S}^1$-invariance of hor_u, property (4.3.29) and identity (4.3.57), we get

$$\langle \hat{u}\rangle = \langle \mathrm{hor}_u\rangle - V^{(2)}_{\mathbf{i}_{\hat{u}}\langle\theta^0\rangle} = \mathrm{hor}_u\,.$$

■

We have also the follow useful technical result.

Lemma 4.3.10 *For every $\mathbb{S}^1$-invariant function I on N the following identity holds*

$$\mathbf{i}_{\hat{u}}\langle d_1 I\rangle = \mathcal{L}_{\mathrm{hor}(u)}I, \quad \forall u \in \mathfrak{X}(S_1), \tag{4.3.62}$$

where hor(u) *is defined by (* 4.3.55*).*

Proof. First, using the $\mathbb{S}^1$-invariance of dI and formula (4.3.58), we get

$$\langle \mathbf{i}_{\hat{u}} dI\rangle = \mathbf{i}_{\langle\hat{u}\rangle}dI = \mathbf{i}_{\mathrm{hor}(u)}dI = \mathcal{L}_{\mathrm{hor}(u)}I.$$

On other hand, the $\mathbb{S}^1$-invariance of $\mathrm{hor}(u)$ and the fact that $\langle d_1 J\rangle$ is horizontal 1-form imply

$$\langle \mathbf{i}_{\hat{u}} dI\rangle = \langle \mathbf{i}_{\hat{u}} d_1 I\rangle = \langle \mathbf{i}_{\hat{u}+V^{(2)}_{\theta^0(u)}} d_1 I\rangle = \mathbf{i}_{\hat{u}+V^{(2)}_{\theta^0(u)}}\langle d_1 I\rangle = \mathbf{i}_{\hat{u}}\langle d_1 I\rangle.$$

Comparing these identities, we show (4.3.62). ■

For any 1-forms α and β on M denote by $\{\alpha\wedge\beta\}_2$ the 2-form given by $\{\alpha\wedge\beta\}_2(X,Y) = \{\alpha(X),\beta(Y)\}_2 - \{\alpha(Y),\beta(X)\}_2$.

Lemma 4.3.11 *The $\mathbb{S}^1$-invariant distribution $\mathcal{H}$ (* 4.3.56*) is involutive if the* 1*-form θ^o satisfies the equation*

$$d_1\theta^o + \frac{1}{2}\{\theta^o\wedge\theta^o\}_2 = 0. \tag{4.3.63}$$

Proof. Consider the 2-form

$$\mathcal{C} := d_1\theta^o + \frac{1}{2}\{\theta^o\wedge\theta^o\}_2. \tag{4.3.64}$$

It is clear that $\mathcal{C}$ annihilates the vector fields $\frac{\partial}{\partial p_2}$ and $\frac{\partial}{\partial q_2}$ and hence condition (4.3.63) reads $\mathcal{C}(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial q_1}) = 0$. On other hand, we have the equality

$$[Y_1^0, Y_2^0] = \left[\frac{\partial}{\partial p_1} + V^{(2)}_{\theta_1^o}, \frac{\partial}{\partial q_1} + V^{(2)}_{\theta_2^o}\right] = V^{(2)}_{\mathcal{C}(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial q_1})},$$

which says that conditions (4.3.63) implies the involutivity of $\mathcal{H}$, $[Y_1^0, Y_2^0] = 0$. ■
By direct verification one can show that there is the representation $\mathcal{C} = \Delta_0 \sigma^{(2)}$ and hence condition (4.3.63) is equivalent to the following

$$\Delta_0 = 0. \tag{4.3.65}$$

Remark 12 *According to [47] the $\mathbb{S}^1$-invariant splitting $TN = \mathcal{H} \oplus TS_2$ gives the Hannay-Berry connection on a trivial symplectic bundle $\pi_1 : S_1 \times S_2 \to S_1$ equipped with $\mathbb{S}^1$-action. The horizontal distribution $\mathcal{H}$ can be derived by applying the general averaging procedure [47] to the trivial connection associated to the canonical distribution $TS_1 \oplus \{0\}$. The curvature of the Hannay-Berry connection is just the vector valued 2-form $V^{(2)}_{\mathcal{C}}$. Therefore, condition (4.3.63) (respectively (4.3.65)) implies vanishing of the curvature and can be called the zero curvature equation. If (4.3.63) holds, then the distribution is integrable in the sense of Frobenius.*

The ε-expansion of the Hamiltonian $H_\varepsilon \circ \mathcal{T}_\varepsilon$. To end the proof of Theorem 4.3.1 it remains to find the ε-expansion of the transformed Hamiltonian $H_\varepsilon \circ \mathcal{T}_\varepsilon$, where the near identity transformation $\mathcal{T}_\varepsilon$ is given by $\mathcal{T}_\varepsilon = \mathrm{Fl}^\lambda_{\varepsilon Z_\lambda}\big|_{\lambda=1}$. Remark that the vector field Z_λ in (4.3.42) has the following expansion at $\varepsilon = 0$:

$$Z_\lambda(\varepsilon) = \mathcal{X} - \lambda\mathcal{Y} + \varepsilon\Delta_\lambda(\mathcal{X} - \lambda\mathcal{Y}) + O(\varepsilon^2), \tag{4.3.66}$$

where

$$\mathcal{X} = -\theta_2^0 Y_1^0 + \theta_1^0 Y_2^0, \quad \mathcal{Y} = -\theta_2^0 V^{(2)}_{\theta_1^0} + \theta_1^0 V^{(2)}_{\theta_2^0}, \tag{4.3.67}$$

$$\Delta_\lambda = (1-\lambda)\Delta_0 + \lambda(\lambda - 1)\{\theta_1^0, \theta_2^0\}_2. \tag{4.3.68}$$

Lemma 4.3.12 *Under the near identity mapping $\mathcal{T}_\varepsilon$, original perturbed system (4.3.2), (4.3.3) is transformed to a system which is Hamiltonian relative to the symplectic structure $\langle\sigma\rangle$ and the Hamiltonian*

$$H_\varepsilon \circ \mathcal{T}_\varepsilon = f \circ \pi_1 + \varepsilon\langle F\rangle + \frac{\varepsilon^2}{2}K + O(\varepsilon^3), \tag{4.3.69}$$

where the second order term and its $\mathbb{S}^1$-average is given by the formulas

$$K = \frac{1}{3}\mathcal{L}_{\mathcal{X}}(F + 2\langle F\rangle) - \frac{1}{3}\mathbf{i}_{V^{(1)}_{2F+\langle F\rangle}}\theta^0 + \left(\Delta_0 - \frac{1}{3}\{\theta_1^0, \theta_2^0\}_2\right)(\langle F\rangle - F), \tag{4.3.70}$$

$$\langle K\rangle = -\frac{1}{3}\langle \mathbf{i}_{V^{(1)}_{2F+\langle F\rangle}}\rangle - \frac{1}{3}\langle\{\theta_1^0, \theta_2^0\}_2(\langle F\rangle - F)\rangle. \tag{4.3.71}$$

Proof. Let $H = H_0 + \varepsilon H_1$. Using (4.3.66) and the identity

$$H \circ \mathrm{Fl}^{\lambda}_{\varepsilon Z_\lambda} = H + \int_0^\lambda (\mathcal{L}_{Z_{\lambda'}} H) \circ \mathrm{Fl}^{\lambda'}_{\varepsilon Z_{\lambda'}} d\lambda',$$

by direct computation we derive the following representation

$$\begin{aligned}(H_0 + \varepsilon H_1) \circ \mathcal{T}_\varepsilon &= H_0 + \varepsilon \left(H_1 + \frac{1}{2}\mathcal{L}_{2\mathcal{X}-\mathcal{Y}} H_0 \right) \\ &+ \frac{\varepsilon^2}{2} \Bigg((\mathcal{L}^2_{\mathcal{X}} - \frac{1}{3}(\mathcal{L}_{\mathcal{Y}} \circ \mathcal{L}_{\mathcal{X}} + \mathcal{L}_{\mathcal{X}} \circ \mathcal{L}_{\mathcal{Y}}) \\ &\quad + \frac{1}{4}\mathcal{L}^2_{\mathcal{Y}} + c_1 \mathcal{L}_{\mathcal{X}} + c_2 \mathcal{L}_{\mathcal{Y}})(H_0) + \mathcal{L}_{2\mathcal{X}-\mathcal{Y}} H_1 \Bigg) \\ &+ O(\varepsilon^3),\end{aligned} \tag{4.3.72}$$

where

$$c_1 = 2\int_0^1 \Delta_\lambda d\lambda = \Delta_0 - \frac{1}{3}\{\theta^0_1, \theta^0_2\}_2, \tag{4.3.73}$$

$$c_2 = -2\int_0^1 \lambda \Delta_\lambda d\lambda. \tag{4.3.74}$$

Putting $H_0 = f \circ \pi_1$ and $H_1 = F$, que get that $\mathcal{L}_{\mathcal{Y}} H_0 = 0$ and the first order term in the decomposition (4.3.72) takes the form

$$\mathcal{L}_{\mathcal{X}} H_0 + H_1 = -\mathbf{i}_{\hat{v}_f}\theta^0 + F. \tag{4.3.75}$$

By (4.3.57), (4.3.58), the $\mathbb{S}^1$-average of $\mathbb{V} = \hat{v}_f + V^{(2)}_F$ is given by the formula

$$\langle \mathbb{V} \rangle = \hat{v}_f + V^{(2)}_{\langle F \rangle + \mathbf{i}_{\hat{v}_f}\theta^0} \tag{4.3.76}$$

On the other hand , we know that $\langle \mathbb{V} \rangle = \mathbb{V}$. Then, this property together with(4.3.76) implies the relation

$$\langle F \rangle + \mathbf{i}_{v_f}\theta^0 = F + c \circ \pi, \tag{4.3.77}$$

for a certain $c \in C^\infty(\pi(N))$. T0he averaging of both sides of this equality gives $c \circ \pi = -\mathbf{i}_{\hat{v}_f}\langle \theta^0 \rangle = 0$ and hence

$$\langle F \rangle = F - \mathbf{i}_{\hat{v}_f}\theta^0. \tag{4.3.78}$$

Next, the second order term in (4.3.72) is given by the expression

$$(\mathcal{L}_{\mathcal{X}} - \frac{1}{2}\mathcal{L}_{\mathcal{Y}})\mathcal{L}_{\mathcal{X}}(H_0) + c_1 \mathcal{L}_{\mathcal{X}}(H_0) + \mathcal{L}_{2\mathcal{X}-\mathcal{Y}}(H_1),$$

which together with (4.3.73), (4.3.75) and (4.3.78 leads to (4.3.70). Finally, averaging (4.3.70) and using the equalities $\langle \Delta_0 \rangle = \Delta_0$ and $\langle \mathcal{X} \rangle = 0$, we prove (4.3.71). ■

Remark 13 *It follows from (4.3.72) that the near identity transformation $\mathcal{T}_\varepsilon$ can be also represented as the time ε-flow of a time-dependent vector field $\widetilde{Z}_\lambda$ with $\widetilde{Z}_0 = \mathcal{X} - \frac{1}{2}\mathcal{Y}$.*

Hamiltonian $\mathbb{S}^1$-Spaces. It follows from (4.3.27), (4.3.28) that the ε-dependent Poisson tensor associated to the $\mathbb{S}^1$-invariant symplectic form $\langle\sigma\rangle = \sigma - \varepsilon d\theta^0$ is given by the formula (see, also [49])

$$\Pi = \frac{1}{1-\varepsilon\Delta_0} Y_1^0 \wedge Y_2^0 + \frac{1}{\varepsilon}\frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}. \tag{4.3.79}$$

This is just straightforward computation. It is clear that Π is well defined in the invariant domain $\{1-\varepsilon\Delta_0 \neq 0,\, \varepsilon \neq 0\}$ in M. Moreover, Π is $\mathbb{S}^1$-invariant,

$$\mathcal{L}_{V_J^{(2)}} \Pi = 0. \tag{4.3.80}$$

This fact follows directly from (4.3.50), (4.3.51) and (4.3.61). The Hamiltonian vector field X_G relative to Π and a function G is defined by the relation $X_G = \mathbf{i}_{dG}\Pi$ and represented as

$$X_G = \frac{1}{1-\varepsilon\Delta_0}\left((\mathcal{L}_{Y_1^0} G)Y_2^0 - \mathcal{L}_{Y_2^0} G)Y_1^0\right) + \frac{1}{\varepsilon} V_G^{(2)}. \tag{4.3.81}$$

In particular, the Hamiltonian vector field of the approximate model is of the form

$$\begin{aligned} X_{f\circ\pi_1 + \varepsilon\langle F\rangle + \frac{\varepsilon^2}{2}K} &= \frac{1}{1-\varepsilon\Delta_0}\operatorname{hor}_{v_f}^{\theta^0} + V_{\langle F\rangle}^{(2)} + \frac{\varepsilon}{2}V_K^{(2)} \qquad (4.3.82)\\ &+ \frac{\varepsilon}{1-\varepsilon\Delta_0}\left((\mathcal{L}_{Y_1^0}(\langle F\rangle + \frac{\varepsilon}{2}K))Y_2^0 - (\mathcal{L}_{Y_2^0}(\langle F\rangle + \frac{\varepsilon}{2}K))Y_1^0\right)\\ &= \mathbb{V} + \varepsilon\left(\frac{1}{2}V_K^{(2)} + \Delta_0 \operatorname{hor}_{v_f} + (\mathcal{L}_{Y_1^0}\langle F\rangle)Y_2^0 - (\mathcal{L}_{Y_2^0}\langle F\rangle)Y_1^0\right) + O(\varepsilon^2). \end{aligned}$$

The following result says under which conditions the canonical $\mathbb{S}^1$-action is Hamiltonian on $(N, \langle\sigma\rangle)$.

Proposition 4.3.13 *The $\mathbb{S}^1$-action associated to the infinitesimal generator $\Upsilon = V_J^{(2)}$ is Hamiltonian relative the symplectic form $\langle\sigma\rangle$ if and only if the cohomology class of the 1-form ς in (4.3.30) is trivial, that is,*

$$\langle d_1 J\rangle = d(g\circ\pi_1) \tag{4.3.83}$$

for a certain $g \in C^\infty(\pi_1(N))$. Under this condition , the corresponding momentum map is $\varepsilon(J - g\circ\pi_1)$,

$$\mathbf{i}_{V_J^{(2)}}\langle\sigma\rangle = -\varepsilon d(J - g\circ\pi_1)$$

Proof. It follows from representation (4.3.81) that the infinitesimal generator $V_J^{(2)}$ of the $\mathbb{S}^1$-action is Hamiltonian relative to Π (also, relative to $\langle\sigma\rangle$) and a function G, $V_J^{(2)} = X_G$ if and only if

$$V_G^{(2)} = \varepsilon V_J^{(2)}$$

and G is a first integral of the vector fields Y_1^0 and Y_2^0, $\mathcal{L}_{Y_1^0}G = \mathcal{L}_{Y_2^0}G = 0$. These conditions are equivalent to the following: $G = \varepsilon(J - g \circ \pi_1)$ for a certain $g \in C^\infty(\pi_1(N))$ and

$$\mathcal{L}_{Y_1^0}J = \frac{\partial g}{\partial p_1}, \mathcal{L}_{Y_2^0}J = \frac{\partial g}{\partial q_1}. \tag{4.3.84}$$

By (4.3.54) we have the relations $\mathcal{L}_{Y_1^0}J = \langle \frac{\partial J}{\partial p_1} \rangle, \mathcal{L}_{Y_2^0}J = \langle \frac{\partial J}{\partial q_1} \rangle$ which together with (4.3.84) lead to (4.3.83). ■

Corollary 4.3.14 *If the domain $\pi_1(N)$ is simply connected, then the $\mathbb{S}^1$-action is Hamiltonian on $(N, \langle\sigma\rangle)$.*

Lemma 4.3.15 *If condition (4.3.83) holds for a certain $g \in C^\infty(\pi_1(N))$, then the function $J - g \circ \pi_1$ is a mutual first integral of the vector fields Y_1^0, Y_2^0 and the Hamiltonian system $(N, \langle\sigma\rangle, \langle H_\varepsilon\rangle = f \circ \pi_1 + \varepsilon\langle F\rangle)$. Moreover,*

$$\mathcal{L}_{v_f} g = 0.$$

Proof. The fact that $\tilde{J} = J - g \circ \pi_1$ is a first integral of Y_1^0, Y_2^0 is a direct consequence of (4.3.83). The $\mathbb{S}^1$-invariance of $\langle H_\varepsilon\rangle$ and condition (4.3.83) mean that $\mathcal{L}_{V_J^{(2)}}\langle H_\varepsilon\rangle = 0$ and $V_J^{(2)} = \varepsilon X_{\tilde{J}}$. It follows that

$$\mathcal{L}_{X_{\langle H_\varepsilon\rangle}}\tilde{J} = -\mathcal{L}_{X_{\tilde{J}}}\langle H_\varepsilon\rangle = -\frac{1}{\varepsilon}\mathcal{L}_{V_J^{(2)}}\langle H_\varepsilon\rangle = 0. \tag{4.3.85}$$

By (4.3.82) we have $X_{\tilde{H}_\varepsilon} = \mathbb{V} + O(\varepsilon)$. From here and (4.3.85) we deduce that $\mathcal{L}_{\mathbb{V}}\tilde{J} = 0$. ■

In the following special case, condition (4.3.83) is always satisfied.

Lemma 4.3.16 *Consider the $\mathbb{S}^1$-action associated to the infinitesimal generator $\Upsilon = V_J^{(2)}$ and suppose that the presymplectic 2-form $\sigma^{(2)}$ is exact on N,*

$$\sigma^{(2)} = d\eta \tag{4.3.86}$$

for a certain 1-form $\eta \in \Omega^1(N)$. Then, the function

$$J^0 := \mathbf{i}_\Upsilon\langle\eta\rangle \tag{4.3.87}$$

satisfies the relations

$$\mathbf{i}_\Upsilon\sigma^{(2)} = -d_2J^0, \tag{4.3.88}$$

$$\langle d_1J^0\rangle = 0. \tag{4.3.89}$$

Proof. Since the 1-form $\langle\eta\rangle$ is $\mathbb{S}^1$-invariant, we have $\mathcal{L}_{V_J^{(2)}}\langle\eta\rangle = 0$ and hence

$$d\mathbf{i}_{V_J^{(2)}}\langle\eta\rangle = -\mathbf{i}_{V_J^{(2)}}\langle d\eta\rangle.$$

In terms of the presymplectic 2-form $\sigma^{(2)}$, this equality is rewritten as follows

$$\mathbf{i}_{V_J^{(2)}}\langle\sigma^{(2)}\rangle = -dJ^0. \tag{4.3.90}$$

On the other hand, averaging the identity $\mathbf{i}_{V_J^{(2)}}\sigma^{(2)} = -d_2 J$ gives

$$\mathbf{i}_{V_J^{(2)}}\langle\sigma^{(2)}\rangle = -\langle d_2 J\rangle \tag{4.3.91}$$

. Taking into account the identity $\langle d_2 J\rangle = d_1 J - \langle d_1 J\rangle + d_2 J$ and equations (4.3.90), (4.3.91), we deduce the relationship between J and J^0:

$$d_1 J - \langle d_1 J\rangle + d_2 J = d_1 J^0 + d_2 J^0. \tag{4.3.92}$$

It follows that $d_2(J - J^0) = 0$ and hence there exists a function $g \in C^\infty(\pi_1(N))$ such that

$$J - J^0 = g \circ \pi_1. \tag{4.3.93}$$

This implies (4.3.88). Moreover, it follows form (4.3.92) that $d_1 J^0 = d_1 J - \langle d_1 J\rangle$. So, equality (4.3.89) holds. ■

Corollary 4.3.17 *In the exact case (*4.3.86*), condition (*4.3.83*) holds for a function g given by (*4.3.93*). Thus, $\Upsilon = V_{J^0}^{(2)}$ and the $\mathbb{S}^1$-action is Hamiltonian relative to $\langle\sigma\rangle$.*

To complete the proof of Theorem 4.3.1 it remains to apply Lemma 4.3.16 to the case when $S_2 = \mathbb{R}^2$ and $\sigma^{(2)} = d(p_2 dq_2)$.

Remark 14 *Hypothesis (*4.3.18*), called the adiabatic condition was introduced in [47, 55], in the context of the theory of Hannay-Berry connections on symplectic and Poisson fiber bundles.*

4.3.4 The geometric structure of normal forms

Resuming the above results, we will formulate a free coordinate version of Theorem 4.3.1 and clarify the geometric meaning of the corresponding normal forms. Let (S_1, σ_1) and (S_2, σ_2) be two 2-dimensional symplectic surfaces. Consider a perturbed Hamiltonian system with two degrees of freedom

$$(M = S_1 \times S_2, \sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)}, \quad H_\varepsilon = f \circ \pi_1 + \varepsilon F\ ), \tag{4.3.94}$$

for some $f \in C^\infty(S_1)$ and $F \in C^\infty(M)$. Assume that on an open domain $N \subseteq M$, the unperturbed vector field $\mathbb{V}$ admits a circle first integral $J : N \to \mathbb{R}$ and consider the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = V_J^{(2)}$. By Lemma 4.3.4, the cohomology class of the closed 1 form ς on $\pi_1(N) \subset S_1$ given by

$$\langle d_1 J\rangle = \pi_1^*\varsigma \tag{4.3.95}$$

is an intrinsic characteristic of the $\mathbb{S}^1$-action.

By Lemma 4.3.8, the 1-form $\theta^0 = \mathcal{S}(d_1 J)$ induces the horizontal distribution $\mathcal{H} = \{\mathrm{hor}_u \mid u \in \mathfrak{X}(\pi_1(N))\}$ which is invariant with respect to the $\mathbb{S}^1$-action. Here $\mathrm{hor}_u = \hat{u} + V^{(2)}_{\mathbf{i}_{\hat{u}}\theta^o}$. Consider a smooth global function Δ_0 on N which is defined as the density of the horizontal 2-form $\mathcal{C} = d_1\theta^o + \frac{1}{2}\{\theta^o \wedge \theta^o\}_2$ with respect to the pre-symplectic 2-form $\sigma^{(1)} = \pi_1^*\sigma_1$, $\mathcal{C} = \Delta_0 \sigma^{(1)}$. The coordinate representation of Δ_0 is just given by (4.3.49). Let $\Pi_1 \in \chi^2(S_1)$ be the nondegenerate Poisson tensor associated to the symplectic structure σ_1 on S_1. Denote by $\mathrm{hor}(\Pi_1)$ the horizontal lift of Π_1 to N via the connection $\mathcal{H}$, that is, a unique bivector field on N which is tangent to $\mathcal{H}$ and such that

$$\mathrm{hor}(\Pi_1)(\pi_1^* df, \pi_1^* dg) = \Pi_1(df, dg) \circ \pi_1 \equiv \sigma_1(v_f, v_g)$$

for all $f, g \in C^\infty(S_1)$. It is clear that $\mathrm{hor}(\Pi_1)$ is $\mathbb{S}^1$-invariant and locally,

$$\mathrm{hor}(\Pi_1) = Y_1^0 \wedge Y_2^0,$$

where the vector fields Y_1^0 and Y_2^0 are given by (4.3.47), (4.3.48). Moreover, it is easy to see that the global representation of the function K in (4.3.70) is

$$K = \frac{1}{3}\mathcal{L}_{\mathcal{X}}(F + 2\langle F\rangle) - \frac{1}{3}\mathbf{i}_{V^{(1)}_{2F+\langle F\rangle}}\theta^0 + \left(\Delta_0 - \frac{1}{6}\{\theta^0 \wedge \theta^0\}_2\right)(\langle F\rangle - F), \tag{4.3.96}$$

where

$$\mathcal{X} = \mathbf{i}_{\theta^0}\mathrm{hor}(\Pi_1).$$

Theorem 4.3.18 *Suppose that the closure of the domain N is compact. Then,*
(a) for small enough $\varepsilon \neq 0$, the near identity mapping $\mathcal{T}_\varepsilon$ (4.3.46) brings the original perturbed model to a Hamiltonian system of the form

$$\left(N, \langle\sigma\rangle = \sigma - \varepsilon d\theta^0, H_\varepsilon \circ \mathcal{T}_\varepsilon = f \circ \pi_1 + \varepsilon\langle F\rangle + \frac{\varepsilon^2}{2}K + O(\varepsilon^3)\right), \tag{4.3.97}$$

where the Hamiltonian vector field $X_{H_\varepsilon \circ \mathcal{T}_\varepsilon} = \mathcal{T}_\varepsilon^* V_{H_\varepsilon}$ of $H_\varepsilon \circ \mathcal{T}_\varepsilon$ relative to the averaged form $\langle\sigma\rangle$ has the representation

$$X_{H_\varepsilon \circ \mathcal{T}_\varepsilon} = \mathbb{V} + \varepsilon\bar{\mathbb{W}} + O(\varepsilon^2), \tag{4.3.98}$$

where

$$\mathbb{V} = \mathrm{hor}_{v_f} + V^{(2)}_{\langle F\rangle}, \tag{4.3.99}$$

and

$$\bar{\mathbb{W}} = \Delta_0\, \mathrm{hor}_{v_f} + \mathbf{i}_{d\langle F\rangle}\mathrm{hor}(\Pi_1) + \frac{1}{2}V^{(2)}_K. \tag{4.3.100}$$

(b) Moreover, if the cohomology class of the 1-form ς in (4.3.30) is trivial,

$$[\varsigma] = 0,$$

then the $\mathbb{S}^1$-action associated to the infinitesimal generator $V_J^{(2)}$ is Hamiltonian on $(N, \langle\sigma\rangle)$ with momentum map εJ^0, where

$$J^0 = J - g \circ \pi_1 \tag{4.3.101}$$

for a certain $g \in C^\infty(\pi_1(N))$ satisfying (4.3.83). In this case, perturbed system (4.3.97) is ε^2-close to the Hamiltonian system with $\mathbb{S}^1$-symmetry

$$(N, \langle\sigma\rangle, \langle H_\varepsilon\rangle = f \circ \pi_1 + \varepsilon\langle F\rangle\ ) \tag{4.3.102}$$

and the function J^0 is a first integral of this system and the averaged perturbation vector field $\tilde{\mathbb{W}}$ in (4.3.98),

$$\mathcal{L}_{\langle\tilde{\mathbb{W}}\rangle} J^0 = 0. \tag{4.3.103}$$

Proof. The proof of the first part of the theorem is a direct consequence of the results of the previous subsection. Representation (4.3.99) follows from equality (4.3.78). By Lemma 4.3.12, we have $\langle V_K^{(2)}\rangle = V_{\langle K\rangle}^{(2)}$. Then by averaging the both sides of the equality

$$\bar{\mathbb{W}} = \Delta_0 \operatorname{hor}_{v_f} + \mathbf{i}_{d\langle F\rangle} Y_1^0 \wedge Y_2^0 + \frac{1}{2} V_K^{(2)},$$

we get the formula for the $\mathbb{S}^1$-average of the perturbation vector field

$$\langle\bar{\mathbb{W}}\rangle = \Delta_0 \operatorname{hor}_{v_f} + \mathbf{i}_{d\langle F\rangle} Y_1^0 \wedge Y_2^0 + \frac{1}{2} V_{\langle K\rangle}^{(2)}$$

which together with properties $\mathcal{L}_{Y_i^0} J^0 = 0$ (see Lemma 4.3.15) and the identity $\mathcal{L}_{V_{\langle K\rangle}^{(2)}} J^0 = -\mathcal{L}_{V^{(2)}{}_{J^0}} \langle K\rangle = 0$ implies (4.3.103). ■

We observe that condition (4.3.18) holds in each of the following cases:

- the "slow" symplectic manifold S_1 is simply connected;

or

- the symplectic form on "fast" symplectic manifold S_2 is exact.

In the last case,

$$\sigma^{(2)} = d\eta, \quad \eta = \pi_2^* \eta^0 \tag{4.3.104}$$

for a certain 1-form η^0 on $\pi_2(N) \subset S_2$. Then, according to Lemma 4.3.16, the function J^o in (4.3.101) can be defined as

$$J^o = \mathbf{i}_{V_J^{(2)}} \langle\eta\rangle. \tag{4.3.105}$$

Consider the splitting $TM = \mathcal{H} \oplus TS_2$, where $\mathcal{H}$ is the $\mathbb{S}^1$-invariant horizontal distribution. Then according to this splitting the perturbation vector field $\bar{\mathbb{W}}$ (4.3.100) has the decomposition

$$\bar{\mathbb{W}} = \bar{\mathbb{W}}_H + \bar{\mathbb{W}}_V$$

into the horizontal and vertical parts given by

$$\bar{\mathbb{W}}_H = \Delta_0 \operatorname{hor}_{v_f} + \mathbf{i}_{d\langle F\rangle} \operatorname{hor}(\Pi_1),$$

$$\bar{\mathbb{W}}_V = \frac{1}{2} V_K^{(2)}.$$

It follows that $\bar{\mathbb{W}}_H$ is $\mathbb{S}^1$-invariant and

$$\langle \bar{\mathbb{W}}_V \rangle = \frac{1}{2} V^{(2)}_{\langle K \rangle}.$$

Therefore, the near identity transformation $\mathcal{T}_\varepsilon$ (4.3.46), brings the Hamiltonian vector field V_{H_ε} into an $\mathbb{S}^1$-invariant normal form of first order only in the horizontal direction. Remark that the horizontal lift $\mathrm{hor}(\Pi_1)$ coincides with the horizontal part of $\langle \Pi^{(1)} \rangle$ and is not a Poisson tensor, in general. For example, $\mathrm{hor}(\Pi_1)$ is a Poisson tensor if $\Delta_0 = 0$ and hence $\mathcal{H}$ is integrable. Then, the horizontal normal form $\bar{\mathbb{W}}_H$ is a Hamiltonian vector field relative to the Poisson tensor $\mathrm{hor}(\Pi_1)$ and the function $\langle F \rangle$.

The next question is to remove the non-invariant vertical component by making an additional near identity transformation.

Corollary 4.3.19 *Suppose that there exist functions $G \in C^\infty(N)$ and $c \in C^\infty(\pi_1(N))$ satisfying the homological equation*

$$\mathcal{L}_{\mathbb{V}} G = \frac{1}{2}(K - \langle K \rangle) + c \circ \pi_1. \tag{4.3.106}$$

Then, for small enough $\varepsilon \neq 0$, the near identity transformation

$$\widetilde{\mathcal{T}}_\varepsilon = \mathcal{T}_\varepsilon \circ \mathrm{Fl}^\varepsilon_{V_G^{(2)}} \tag{4.3.107}$$

brings the Hamiltonian vector field V_{H_ε} on (M, σ) into $\mathbb{S}^1$-invariant normal form of first order:

$$\widetilde{\mathcal{T}}_\varepsilon^* V_{H_\varepsilon} = \mathbb{V} + \varepsilon(\bar{\mathbb{W}}_H + \frac{1}{2} V^{(2)}_{\langle K \rangle \rangle}) + O(\varepsilon^2). \tag{4.3.108}$$

Proof. The statement follows from Theorem 4.3.18, the representation

$$\left(\mathrm{Fl}^\varepsilon_{V_K^{(2)}}\right)^* (\mathbb{V} + \varepsilon \bar{\mathbb{W}}) = \mathbb{V} + (-[\mathbb{V}, V_G^{(2)}] + \bar{\mathbb{W}}_H + \frac{1}{2} V_K^{(2)} + O(\varepsilon^2)$$

and the identity $[\mathbb{V}, V_G^{(2)}] = V^{(2)}_{\mathcal{L}_{\mathbb{V}} G}$. ∎

The Adiabatic Invariants. On the phase space $(M = S_1 \times S_2, \sigma = \sigma^{(1)} + \varepsilon \sigma^{(2)})$, we consider a slow-fast Hamiltonian model

$$\dot{\xi} = \varepsilon V_F^{(1)} \quad (\xi \in S_1), \tag{4.3.109}$$

$$\dot{x} = V_F^{(2)} \quad (x \in S_2), \tag{4.3.110}$$

for a certain $F \in C^\infty(M)$. Assume that the regular set $\mathrm{Reg}(V_F^{(2)})$ is dense in M and the flow of $V_F^{(2)}$ is periodic with frequency function $\omega : M \to \mathbb{R}$. Consider the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = \frac{1}{\omega} V_F^{(2)}$. It is clear that the restriction of $V_F^{(2)}$ to each slice $\{m_1\} \times S_2$ gives a Hamiltonian system with periodic flow. Then, applying to this system the period-energy relation argument (see, Proposition 2.5.1), we get the identity

$$d_2 \omega \wedge d_2 F = 0$$

which says that the 1-form $\frac{1}{\omega}d_2F$ is d_2-closed on M. Given an $\mathbb{S}^1$-invariant open domain $N \subset M$ with compact closure, we assume that

$$\frac{1}{\omega}d_2F \text{ is } d_2\text{-}exact \ on \ N. \tag{4.3.111}$$

This implies that $\frac{1}{\omega}d_2F = d_2J$ for a certain smooth function $J : N \to \mathbb{R}$ and hence

$$\frac{1}{\omega}V_F^{(2)} = V_J^{(2)} \ on \ N. \tag{4.3.112}$$

It follows that J is a circle first integral of the unperturbed vector field $V_F^{(2)}$ and hence the initial hypotheses of Theorem 4.3.1 hold.

Theorem 4.3.20 *Under above assumptions, the following assertions are true:*
(a) for small enough ε, the near identity transformation

$$\widetilde{\mathcal{T}}_\varepsilon = \mathcal{T}_\varepsilon \circ \mathrm{Fl}^\varepsilon_{\frac{1}{2}V^{(2)}_{\mathcal{S}(K)}} : N \to M,$$

brings the vector field of slow-fast system (4.3.109), (4.3.110) to $\mathbb{S}^1$-invariant normal form of first order

$$\widetilde{\mathcal{T}}_\varepsilon^*(V_F^{(2)} + \varepsilon V_F^{(1)}) = V_F^{(2)} + \varepsilon(\mathbf{i}_{dF}\mathrm{hor}(\Pi_1) + \frac{1}{2}V^{(2)}_{\langle K_0\rangle}) + O(\varepsilon^2). \tag{4.3.113}$$

where

$$K_0 = -\mathbf{i}_{V_F^{(1)}}\theta^0.$$

(b) If there exists a smooth function $J^0 \in C^\infty(N)$ such that

$$d_2J^0 = \frac{1}{\omega}d_2F, \tag{4.3.114}$$

$$\langle d_1J^0\rangle = 0, \tag{4.3.115}$$

then J^0 is a first integral of the second term in the normal form (4.3.113). In the case when the $\mathbb{S}^1$-action is free on N, the function J_0 is an adiabatic invariant of system of system (4.3.109), (4.3.110).

Proof. It is clear that $\langle F\rangle = F$. By Theorem 4.3.1, in the adiabatic case $f \equiv 0$, the slow-fast Hamiltonian model

$$(M = S_1 \times S_2, \sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)}, H_\varepsilon = \varepsilon F)$$

is transformed under mapping $\mathcal{T}_\varepsilon$ (4.3.46) into the Hamiltonian system

$$\left(N, \langle\sigma\rangle, H_\varepsilon \circ \mathcal{T}_\varepsilon = \varepsilon F + \frac{\varepsilon^2}{2}K + O(\varepsilon^3)\right),$$

where

$$K = \mathcal{L}_{\mathcal{X}}F - \mathbf{i}_{V_F^{(1)}}\theta^0 = \mathcal{L}_{\mathcal{X}}F + K_0.$$

The corresponding Hamiltonian vector field is of the form

$$X_{H_\varepsilon \circ \mathcal{T}_\varepsilon} = V_F^{(2)} + \varepsilon \bar{\mathbb{W}} + O(\varepsilon^2). \tag{4.3.116}$$

Since the flow of $\mathbb{V} = V_F^{(2)}$ is periodic, homological equation (4.3.106) has a solution of the form $G = \frac{1}{2}\mathcal{S}(K)$ and $c = 0$. Moreover, the property $\langle \mathcal{X} \rangle = 0$ implies that $\langle K \rangle = -\langle \mathbf{i}_{V_F^{(1)}} \theta^0 \rangle = \langle K_0 \rangle$, Therefore, this proves item (a). As a consequence, we get the representation

$$\begin{aligned} \langle \bar{\mathbb{W}} \rangle &= \mathbf{i}_{dF} \mathrm{hor}(\Pi_1) + \frac{1}{2} V_{\langle K_0 \rangle}^{(2)} \\ &= (\mathcal{L}_{Y_1^0} F) Y_2^0 - (\mathcal{L}_{Y_2^0} F) Y_1^0 + \frac{1}{2} V_{\langle K_0 \rangle}^{(2)}. \end{aligned}$$

To prove the item (b), we recall that condition (4.3.115) is equivalent to the following

$$\mathcal{L}_{Y_1^0} J^0 = \mathcal{L}_{Y_2^0} J^0 = 0. \tag{4.3.117}$$

Finally, using these properties and condition (4.3.114), we compute

$$\begin{aligned} \mathcal{L}\langle \bar{\mathbb{W}} \rangle J_0 &= (\mathcal{L}_{Y_1^0} F) \mathcal{L}_{Y_2^0} J^0 - (\mathcal{L}_{Y_2^0} F) \mathcal{L}_{Y_1^0} J^0 + \frac{1}{2} \mathcal{L}_{V_{\langle K_0 \rangle}^{(2)}} J^0 \\ &= -\mathcal{L} V_J^{(2)} \langle K_0 \rangle = 0. \end{aligned}$$

In the case when the $\mathbb{S}^1$-action is free, the fact that J^0 is an adiabatic invariant follows from Proposition 3.2.22. ■

Combining Theorem 4.3.20 and Theorem 4.2.2 leads to the following criterion.

Corollary 4.3.21 *The vector field $\tilde{\mathcal{T}}_\varepsilon^*(V_F^{(2)} + \varepsilon V_F^{(1)})$ is in normal form of first order relative to $V_F^{(2)}$ if and only if*

$$\mathrm{hor}(\Pi_1)(dF, d\omega) + \frac{1}{2} \langle \{K_0, \omega\}_2 \rangle = 0.$$

By Theorem 4.3.20, the existence of a function J^0 with properties (4.3.115), (4.3.114) is provided by conditions (4.3.111) which is automatically satisfied in the exact case.

Proposition 4.3.22 *Suppose that the flow of $V_F^{(2)}$ is periodic with frequency function $\omega : M \to \mathbb{R}$ and the exactness condition (*4.3.104*) holds. Then, the function*

$$J^0 = \frac{1}{\omega} \mathbf{i}_{V_F^{(2)}} \langle \eta \rangle, \quad \eta = \pi_2^* \eta^0 \tag{4.3.118}$$

*satisfies the conditions (*4.3.115*), (*4.3.114*).*

Proof. Using the invariance of ω and dF with respect to the $\mathbb{S}^1$-action with infinitesimal generator $\Upsilon = \frac{1}{\omega}V_F^{(2)}$ and that $J^0 = \mathbf{i}_\Upsilon\langle\eta\rangle$, we obtain

$$\begin{aligned} d_2J^0 &= d_2(\mathbf{i}_\Upsilon\langle\eta\rangle) = d\circ\mathbf{i}_\Upsilon\langle\eta\rangle - d_1J^0, \\ &= \mathcal{L}_\Upsilon\langle\eta\rangle - \mathbf{i}_\Upsilon\langle d\eta\rangle - d_1J^0, \\ &= -\langle\mathbf{i}_\Upsilon\sigma^{(2)}\rangle - d_1J^0 = \frac{1}{\omega}\langle d_2F\rangle - d_1J^0, \\ &= \frac{1}{\omega}dF - \frac{1}{\omega}\langle d_1F\rangle - d_1J^0. \end{aligned}$$

From here we get the identity

$$d_2J^0 - \frac{1}{\omega}d_2F = \frac{1}{\omega}d_1F - \frac{1}{\omega}\langle d_1F\rangle - d_1J^0$$

which splits into the two relations

$$\begin{aligned} d_2J^0 &= \frac{1}{\omega}d_2F, \\ d_1J^0 &= \frac{1}{\omega}d_1F - \frac{1}{\omega}\langle d_1F\rangle. \end{aligned}$$

This proves (4.3.115), (4.3.114). ∎

Therefore, Proposition 4.3.22 gives us a free action-angle coordinate version of the classical adiabatic theorem [7, 38, 62].

4.3.5 Generalizations

The above results remain true in the general case when we start with a phase space $(M = S_1\times S_2,\ \sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)})$, where S_1 and S_2 are symplectic manifolds of arbitrary dimensions. Below we give some computational formulas for the main objects which appear in the formulations of Theorem 4.3.1. Let $(\xi, x) = (\xi^i, x^\alpha)$ be a (local) coordinate system on M adapted to the symplectic factors, $\xi = (\xi^i)\in S_1$ and $x\in(x^\alpha)\in S_2$. Then, the presymplectic forms $\sigma^{(1)}$ and $\sigma^{(2)}$ have the representations

$$\begin{aligned} \sigma^{(1)} &= \frac{1}{2}\sigma_{ij}^{(1)}(\xi)d\xi^i\wedge d\xi^j, \\ \sigma^{(2)} &= \frac{1}{2}\sigma_{\alpha\beta}^{(2)}(x)dx^\alpha\wedge dx^\beta. \end{aligned}$$

Recall that the summation over repeated indices is understood. Suppose we are given an $\mathbb{S}^1$-action on M with an infinitesimal generator

$$V_J^{(2)} = [\sigma^{(2)}]^{\alpha\beta}\frac{\partial J}{\partial x^\beta}\frac{\partial}{\partial x^\alpha}$$

for certain smooth function $J = J(\xi, x)$ on M. Here $[\sigma^{(2)}]^{\alpha\beta}$ denote the elements of the inverse of the matrix $(\sigma_{\alpha\beta}^{(2)})$. Introduce a 1-form $\theta^0 = \theta_i^o(\xi, x)d\xi^i$ on M whose coefficients are given by

$$\theta_i^0 := \frac{1}{2\pi}\int_0^{2\pi}(t-\pi)(\mathrm{Fl}^t_{V_J^{(2)}})^*\frac{\partial J}{\partial\xi^i}dt$$

and $\langle\theta_i^0\rangle = 0$. Let us associate to θ^0 the following 2-form

$$\mathcal{F} = \frac{1}{2}\mathcal{F}_{ij}(\xi,x)d\xi^i\wedge d\xi^j = \pi_1^*\sigma^{(1)} - \varepsilon(d_1 + \frac{1}{2}\{\theta^0\wedge\theta^0\}_2),$$

$$\mathcal{F}_{ij} := \sigma_{ij}^{(1)} + \varepsilon\left(\frac{\partial\theta_i^0}{\partial\xi_j} - \frac{\partial\theta_j^o}{\partial\xi^i} + [\sigma^{(2)}]^{\alpha\beta}\frac{\partial\theta_i^0}{\partial x^\alpha}\frac{\partial\theta_j^o}{\partial x^\beta}\right),$$

and (local) 1-forms

$$\Gamma^\alpha = dx^\alpha - [\sigma^{(2)}]^{\alpha\beta}\frac{\partial\theta_i^0}{\partial x^\beta}d\xi^i.$$

A multidimensional version of Lemma 4.3.4 and Lemma 4.3.8 is formulated as follows.

Proposition 4.3.23 *The $\mathbb{S}^1$-average $\langle\sigma\rangle$ of the symplectic form σ with respect to the $\mathbb{S}^1$-action $\mathrm{Fl}^t_{V_J^{(2)}}$ has the representations*

$$\langle\sigma\rangle = \sigma - \varepsilon d\theta^0 = \mathcal{F} + \frac{1}{2}\sigma_{\alpha\beta}^{(2)}\Gamma^\alpha\wedge\Gamma^\beta \tag{4.3.119}$$

and is nondegenerate if and only if $\det(\mathcal{F}_{ij}) \neq 0$. The both terms in the last decomposition in (4.3.119) are $\mathbb{S}^1$-invariant. Moreover, there is an $\mathbb{S}^1$invariant splitting

$$TM = \mathcal{H}\oplus TS_2, \tag{4.3.120}$$

where the horizontal distribution $\mathcal{H}$ is generated by the vector fields

$$Y_i^o = \frac{\partial}{\partial\xi^i} + [\sigma^{(2)}]^{\alpha\beta}\frac{\partial\theta_i^0}{\partial x^\beta}\frac{\partial}{\partial x^\alpha}.$$

The $\mathbb{S}^1$-action is Hamiltonian relative to $\langle\sigma\rangle$ if and only if $\langle d_1 J\rangle = 0$ or, equivalently

$$\mathcal{L}_{Y_i^0}J = 0, \tag{4.3.121}$$

for $i = 1, 2, \ldots, \mathrm{dim}S_1$.

The proof of this statement is based on the same arguments as in subsection 4.3.3 (see also [17]).

Remark that the last representation in (4.3.119) says that the averaged form $\langle\sigma\rangle$ is a weak coupling symplectic form which has the following geometrical interpretation [30, 73]. The $\mathbb{S}^1$-invariant splitting (4.3.120) gives the Hannay-Berry connection [55] in the trivial symplectic bundle $\pi_1 : M \to S_1$ associated with the horizontal subbundle $\mathcal{H}$ and the connection form $\Gamma = \Gamma^\alpha\otimes\frac{\partial}{\partial x^\alpha}$. The horizontal 2-form $\mathcal{F}$ controls the curvature of Γ and coincides at $\varepsilon = 0$ with the pull-back $\pi_1^*\sigma_1$ of the symplectic form on the base S_1. For example, in the 2-dimensional case, $\mathcal{F} = (1-\varepsilon\Delta_0)\pi_1^*\sigma_1$ (where Δ_0 is given by 4.3.49) and the zero curvature condition reads $\Delta_0 = 0$. The second term of the last representation in (4.3.119) is a 2-form on M which vanishes along the horizontal distribution $\mathcal{H}$ and coincides with σ_2 on the fiber of π_1. Therefore, the weak coupling symplectic form is the result of coupling of the base of symplectic form σ_1 with the fiberwise symplectic symplectic form σ_2 via the Hannay-Berry connection Γ.

As well as in Theorem 4.3.1, the averaging principle says that in the multidimensional case, a good approximation to the original perturbed system is the model $(\langle\sigma\rangle, \langle H_\varepsilon\rangle)$ which becomes a Hamiltonian system with $\mathbb{S}^1$-symmetry if condition (4.3.121) holds.

4.4 The Quadratic Case

Here we apply the general results to a special type of perturbed Hamiltonian system $(M = S_1 \times S_2, \sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)}, H_\varepsilon = f \circ \pi_1 + \varepsilon F)$ where the fast factor S_2 is a vector space and the perturbation term F is a quadratic function in the fast variables. As we have showed in Section 2, such perturbed models appear in the study of Hamiltonian dynamics near invariant symplectic submanifolds (see, also [39, 74]). In this context, the unperturbed dynamics is defined by the first variation system over an invariant submanifold.

4.4.1 Perturbative setting for linearized dynamics

Let $\mathbb{R}^r = \{x = (x^1, ..., x^r)\}$ be the Euclidean space and S_1 a smooth manifold. Consider the product manifold $M = S_1 \times \mathbb{R}^r$, identifying S_1 with submanifold in M by means of the slice $S_1 \times \{0\}$. Suppose we are given a vector field X on M for which S_1 is an invariant submanifold. Then, we have

$$X = \sum_i X^i(\xi, x)\frac{\partial}{\partial \xi^i} + \sum_\alpha X^\alpha(\xi, x)\frac{\partial}{\partial x^\alpha} \tag{4.4.1}$$

with

$$X^\alpha(\xi, 0) = 0 \ (\alpha = 1, ..., r). \tag{4.4.2}$$

Here, $\xi = (\xi^i)$ is a coordinate system on S_1. The restriction of X to S_1 is a vector field given by

$$v = X \mid_S = \sum_i v^i(\xi)\frac{\partial}{\partial \xi^i},$$

$$v^i(\xi) = X^i(\xi, 0).$$

For every $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$, consider the scaling map $\rho_\varepsilon : M \to M$, $\rho_\varepsilon(\xi, x) = (\xi, \varepsilon x)$. It is clear that ρ_ε is a diffeomorphism if $\varepsilon \neq 0$.

Proposition 4.4.1 *The pull-back $\mathbf{A}_\varepsilon = \rho_\varepsilon^* X$ is an ε-dependent vector field on M with Taylor expansion at $\varepsilon = 0$*

$$\mathbf{A}_\varepsilon = \mathrm{var}_S(X) + \varepsilon A_1 + \mathcal{O}(\varepsilon^2), \tag{4.4.3}$$

where

$$\mathrm{var}_{S_1}(X) \overset{def}{=} \lim_{\varepsilon \to 0} \rho_\varepsilon^* X = \sum_i v^i(\xi)\frac{\partial}{\partial \xi^i} + \sum_{\alpha,\beta} \frac{\partial X^\alpha(\xi, 0)}{\partial x^\beta} x^\beta \frac{\partial}{\partial x^\alpha},$$

$$\begin{aligned} A_1 &\stackrel{def}{=} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\rho_\varepsilon^* X - \mathrm{var}_{S_1}(X)) \\ &= \sum_{i,\beta}\frac{\partial X^i(\xi,0)}{\partial x^\beta}x^\beta\frac{\partial}{\partial \xi^i} + \frac{1}{2}\sum_{\alpha}\frac{\partial^2 X^\alpha(\xi,0)}{\partial x^\beta \partial x^\gamma}x^\beta x^\gamma\frac{\partial}{\partial x^\alpha}. \end{aligned}$$

Proof. In local coordinates, vector field $\mathbf{A}_\varepsilon$ takes the form

$$\mathbf{A}_\varepsilon(\xi,x) = \rho_\varepsilon^* X = \sum_i X^i(\xi,\varepsilon x)\frac{\partial}{\partial \xi^i} + \sum_\alpha \frac{1}{\varepsilon}X^\alpha(\xi,\varepsilon x)\frac{\partial}{\partial x^\alpha}. \tag{4.4.4}$$

Since the vector field $\mathbf{A}_\varepsilon$ is not defined at $\varepsilon = 0$, the terms of the Taylor expansion of $\mathbf{A}_\varepsilon$ are given by

$$\begin{aligned} A_0 &= \mathrm{var}_{S_1}(X) \stackrel{\mathrm{def}}{=} \lim_{\varepsilon\to 0}\rho_\varepsilon^* X, \\ A_1 &\stackrel{\mathrm{def}}{=} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\rho_\varepsilon^* X - A_0\right). \end{aligned}$$

From (4.4.4), we get

$$\mathrm{var}_{S_1}(X) = \sum_i v^i(\xi)\frac{\partial}{\partial \xi^i} + \sum_{\alpha,\beta}\frac{\partial X^\alpha(\xi,0)}{\partial x^\beta}x^\beta\frac{\partial}{\partial x^\alpha}.$$

Next, we compute the vector field $\frac{1}{\varepsilon}\left(\rho_\varepsilon^* X - A_0\right)$ in local coordinates,

$$\begin{aligned} \frac{1}{\varepsilon}\left(\rho_\varepsilon^* X - A_0\right) &= \sum_i \frac{1}{\varepsilon}\left(X^i(\xi,\varepsilon x) - v^i(\xi)\right)\frac{\partial}{\partial \xi^i} \\ &\quad + \sum_\alpha\left(\frac{1}{\varepsilon^2}X^\alpha(\xi,\varepsilon x) - \frac{1}{\varepsilon}\sum_\beta\frac{\partial X^\alpha(\xi,0)}{\partial x^\beta}x^\beta\right)\frac{\partial}{\partial x^\alpha}. \end{aligned}$$

Taking limit as $\varepsilon \to 0$, we obtain

$$A_1 = \sum_{i,\beta}\frac{\partial X^i(\xi,0)}{\partial x^\beta}x^\beta\frac{\partial}{\partial \xi^i} + \frac{1}{2}\sum_{\alpha,\beta,\gamma}\frac{\partial^2 X^\alpha(\xi,0)}{\partial x^\beta \partial x^\gamma}x^\beta x^\gamma\frac{\partial}{\partial x^\alpha}.$$

∎

Vector field $A_0 = \mathrm{var}_S(X)$, called the *linearized* (or *first variation*) *vector field* of $\mathbf{A}_\varepsilon$ at S, presents the unperturbed dynamics of (4.4.3). The small parameter ε characterized the "radius" of a neighborhood of the invariant submanifold S where we study the original dynamics of X. We have the following properties: the linearized vector field is invariant with respect to scaling, $\rho_\varepsilon^*\,\mathrm{var}_S(X) = \mathrm{var}_S(X)$. And the Lie derivative along $\mathrm{var}_S(X)$ preserves the space of smooth functions on $M = S \times \mathbb{R}^r$ which are linear in $x \in \mathbb{R}^r$.

Consider the following particular case. According to the canonical splitting $TM = TS \oplus \mathbb{R}^r$ the vector field X has the decomposition onto "tangent" and "normal " components relative to S: $X = X^{(1)} + X^{(2)}$. In coordinates, the vector fields $X^{(1)}$ and $X^{(2)}$ correspond to the first and second terms in (4.4.1), respectively. Assume that

$$\rho_{-\lambda}^{*} X^{(1)} = X^{(1)}, \tag{4.4.5}$$

$$\rho_{-\lambda}^{*} X^{(2)} = -\lambda X^{(2)} \quad \forall \lambda \in \mathbb{R}. \tag{4.4.6}$$

Then, one can define the unperturbed vector field as follows

$$\mathbf{A}_{\varepsilon} = \rho_{\sqrt{\varepsilon}}^{*} X.$$

In this case, we have

$$\mathbf{A}_{\varepsilon} = \mathrm{var}_{S}(X) + \varepsilon A_1 + \mathcal{O}(\varepsilon^2)$$

where

$$A_1 = \frac{1}{2} \sum_{i,\beta,\gamma} \frac{\partial^2 X^i(\xi, 0)}{\partial x^\beta \partial x^\gamma} x^\beta x^\gamma \frac{\partial}{\partial \xi^i} + \frac{1}{6} \sum_{i,\beta,\gamma} \frac{\partial^3 X^\alpha(\xi, 0)}{\partial x^\beta \partial x^\gamma \partial x^\sigma} x^\beta x^\gamma x^\sigma \frac{\partial}{\partial x^\sigma}.$$

Finally, let us consider the Hamiltonian case. Suppose that (S_1, σ) is a symplectic manifold with symplectic form

$$\sigma^{(1)} = \frac{1}{2} \sum_{i,j} \sigma_{ij}^{(1)}(\xi) d\xi^i \wedge d\xi^j$$

and the Euclidian space $\mathbb{R}^{2m} = \{x = (x^1, ..., x^{2m})\}$ is equipped with canonical symplectic structure $\sigma^{(1)} = \frac{1}{2} < \mathbf{J} dx \wedge dx >$, where $\mathbf{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Assume that on the phase space $M = S_1 \times \mathbb{R}^{2m}$ with symplectic structure

$$\sigma = \sigma^{(1)} + \frac{1}{2} < \mathbf{J} dx \wedge dx > \tag{4.4.7}$$

we are given a Hamiltonian vector field X_H whose Hamiltonian $H = H(\xi, x)$ satisfies the condition

$$\frac{\partial H}{\partial x}(\xi, 0) = 0 \quad \forall \xi \in S_1.$$

This means that condition (4.4.2) holds and hence the symplectic submanifold $S \approx S \times \{0\}$ is invariant with respect to the flow of X_H. The restriction of X_H to this submanifold is the Hamiltonian vector field v_f on (S_1, σ^1), that is, $\mathbf{i}_{v_f} \omega = -d_1 f$, where $f(\xi) := H(\xi, 0)$. In addition, assuming that $\rho_{-1}^{*} H = H$, we get that X_H satisfies conditions (4.4.5),(4.4.6). Therefore, applying the scaling map by the factor $\sqrt{\varepsilon}$ to symplectic form (4.4.7) and the Hamiltonian vector field X_H gives

$$\sigma_{\varepsilon} = \rho_{\sqrt{\varepsilon}}^{*} \sigma = \sigma^1 + \frac{\varepsilon}{2} < \mathbf{J} dx \wedge dx >,$$

$$\mathbf{A}_{\varepsilon} = \rho_{\sqrt{\varepsilon}}^{*} X_H = \mathrm{var}_{S}(X_H) + \varepsilon A_1 + \mathcal{O}(\varepsilon^2).$$

Here, the unperturbed and perturbation vector fields have the form $\mathrm{var}_{S}(X_H) = v_f + V_{H_1}^{(2)}$, and $A_1 = V_{H_1}^{(1)} + V_{H_2}^{(2)}$, respectively and

$$H_1 = \frac{1}{2} \sum \frac{\partial^2 H}{\theta x^\alpha \partial x^\beta}(\xi, 0) x^\alpha x^\beta.$$

For small $\varepsilon \neq 0$, $\mathbf{A}_{\varepsilon}$ represents a perturbed Hamiltonian system on (M, σ) .

4.4.2 Circle first integrals from Lax's equation

Suppose that we start with a slow-fast phase space ($M = S_1 \times S_2$, $\sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)}$), where S_1 is an arbitrary symplectic manifold and the $S_2 = \mathbb{R}^2 = \{\mathbf{x} = (p_2, q_2)\}$ is the plane equipped with canonical symplectic form. Then,

$$\sigma^{(1)} = \frac{1}{2}\sigma^{(1)}_{ij}(\xi)d\xi^i \wedge d\xi^j,$$

$$\sigma^{(2)} = dp_2 \wedge dq_2 \equiv d(\frac{1}{2}\mathbf{Jx}\cdot\mathbf{dx}).$$

On this phase space, let us consider a perturbed Hamiltonian system of the form

$$H_\varepsilon = f_0(\xi) + \varepsilon\left(f_1(\xi) - \frac{1}{2}\mathbf{JV}(\xi)\mathbf{x}\cdot\mathbf{x}\right) \tag{4.4.8}$$

where $f_0, f_1 \in C^\infty(S_1)$ and $\mathbf{V} : S_1 \to \mathrm{sp}(1;\mathbb{R})$ is a smooth matrix-valued function. Here $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore, we deal with the case when the perturbation term F is a quadratic function in the fast variables p_2, q_2. The corresponding unperturbed vector field is

$$\mathbb{V} = \hat{v}_{f_0} + \mathbf{Vx}\cdot\frac{\partial}{\partial\mathbf{x}}. \tag{4.4.9}$$

Lemma 4.4.2 *Suppose there exists a smooth vector function* $\mathbf{A} : S_1 \to \mathrm{sp}(1;\mathbb{R})$ *satisfying the Lax type equation on* S_1 :

$$\mathcal{L}_{v_{f_0}}\mathbf{A} + [\mathbf{A}, \mathbf{V}] = 0, \tag{4.4.10}$$

and the condition

$$\det\mathbf{A} = \mathbf{1}. \tag{4.4.11}$$

Then,

(a) the function

$$J(\xi, \mathbf{x}) = -\frac{1}{2}\mathbf{JA}(\xi)\mathbf{x}\cdot\mathbf{x} \tag{4.4.12}$$

is a circle first integral of the unperturbed vector field $\mathbb{V}$ *satisfying the adiabatic condition (*4.3.115*).*

(b) the approximate Hamiltonian system with $\mathbb{S}^1$*-symmetry* $(M, \langle\sigma\rangle, \langle H_\varepsilon\rangle)$ *is given by*

$$\langle\sigma\rangle = \sigma^{(1)} + \frac{\varepsilon}{2}d\left(\mathbf{Jx}\cdot(\mathbf{dx} - \frac{1}{2}(\mathbf{A}d_1\mathbf{A})\mathbf{x}\right), \tag{4.4.13}$$

$$\langle H_\varepsilon\rangle = f_0\circ\pi_1 + \varepsilon\left(f_1\circ\pi_1 - \frac{1}{4}\mathbf{J}(\mathbf{V} - \mathbf{A}^{-1}\mathbf{VA})\mathbf{x}\cdot\mathbf{x}\right). \tag{4.4.14}$$

Proof. Consider the function $J = -\frac{1}{2}\mathbf{JA}(\xi)\mathbf{x}\cdot\mathbf{x}$. First, by direct computation we verify that the equality $\mathcal{L}_{\mathbb{V}}J = 0$ is just equivalent to the Lax equation (4.4.10) for

A. Under condition (4.4.11), the flow of $V_J^{(2)} = \mathbf{A}\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{x}}$ is 2π-periodic, because of the representation

$$\mathrm{Fl}^t_{V_J^{(2)}}(\xi,\mathbf{x}) = \exp(t\mathbf{A}(\xi))\mathbf{x} = (\cos t)\mathbf{x} + \sin t\mathbf{A}(\xi)\mathbf{x}.$$

Now, let us justify formulas (4.4.13), (4.4.14). It is clear that the presymplectic form $\sigma^{(2)}$ is exact, $\sigma^{(2)} = d\eta$, where a primitive can be chosen as follows $\eta = \frac{1}{2}\mathbf{J}\mathbf{x}\cdot\mathbf{dx}$. Then, we have the following property of η: the pull-back of η by a diffeomorphism $\Phi : M \to M$ of the form $\Phi(\xi,\mathbf{x}) = (\xi,\mathbf{\Phi}(\xi)\mathbf{x})$, for a certain smooth matrix valued function $\mathbf{\Phi} : S_1 \to \mathrm{Sp}(1;\mathbb{R})$, is given by the formula:

$$\Phi^*\eta = \eta + \frac{1}{2}\mathbf{J}\mathbf{x}\cdot(\mathbf{\Phi}^{-1}d_1\mathbf{\Phi})\mathbf{x}. \tag{4.4.15}$$

Taking $\mathbf{\Phi} = \cos t\mathbf{I} + \sin t\mathbf{A}$, we have $\mathbf{\Phi}^{-1} = \cos t\mathbf{I} - \sin t\mathbf{A}$, $d_1\mathbf{\Phi} = \sin t d_1\mathbf{A}$ and by (4.4.14) we compute

$$(\mathrm{Fl}^t_{V_{J^0}^{(2)}})^*\eta = \eta + \frac{\sin t}{2}\mathbf{J}\mathbf{x}\cdot(\cos t\mathbf{I} - \sin t\mathbf{A})(d_1\mathbf{A})\mathbf{x}.$$

This leads to the following formula for the $\mathbb{S}^1$-average of η:

$$\langle\eta\rangle = \eta - \frac{1}{4}\mathbf{J}\mathbf{x}\cdot\mathbf{A}(d_1\mathbf{A})\mathbf{x}. \tag{4.4.16}$$

Then, representation (4.4.13) follows from the identity $\langle\sigma\rangle = \sigma^{(1)} + d\langle\eta\rangle$ and expression (4.4.16). Moreover, we observe that

$$\mathbf{i}_{V_J^{(2)}}\langle\eta\rangle = \mathbf{i}_{V_J^{(2)}}\eta = -\frac{1}{2}\mathbf{J}\mathbf{A}(\xi)\mathbf{x}\cdot\mathbf{x} = J.$$

Hence $J = J^0$ and by Lemma 4.3.16, J^0 satisfies the adiabatic condition (4.3.115). Finally, we compute the $\mathbb{S}^1$-average of F

$$\begin{aligned}\langle F\rangle(\xi,\mathbf{x}) &= f_1(\xi) - \frac{1}{4\pi}\int_0^{\pi}\mathbf{J}\mathbf{V}(\cos t\mathbf{x} + \sin t\mathbf{A}\mathbf{x})\cdot(\cos t\mathbf{x} + \sin t\mathbf{A}\mathbf{x})dt\\ &= f_1(\xi) - \frac{1}{4}\mathbf{J}(\mathbf{V} - \mathbf{A}^{-1}\mathbf{V}\mathbf{A})\mathbf{x}\cdot\mathbf{x}.\end{aligned}$$

■

By direct computation, we get that the 1-form θ^0 in (4.4.13) is just given by the formula

$$\theta^0 = -\frac{1}{4}\mathbf{x}\cdot\mathbf{J}(\mathbf{A}d_1\mathbf{A})\mathbf{x}.$$

Then, one can show that $\mathcal{C}_{\theta^0} = d_1\theta^0 + \frac{1}{2}\{\theta^0\wedge\theta^0\}_2 = 0$ and hence the zero curvature condition holds. The corresponding horizontal distribution $\mathcal{H}$ is integrable.

Remark 15 *If* $\tilde{\mathbf{A}} : S_1 \to \mathrm{sp}(1;\mathbb{R})$ *is a solution of Lax equation (*4.4.10*) satisfying the condition* $\det\tilde{\mathbf{A}} > 0$, *then* $\det\tilde{\mathbf{A}}$ *is a first integral of* v_{f_0}, $\mathcal{L}_{v_{f_0}}(\det\tilde{\mathbf{A}}) = 0$, *and the matrix valued function* $\mathbf{A} = \left[\det\tilde{\mathbf{A}}\right]^{-\frac{1}{2}}\tilde{\mathbf{A}}$ *is again a solution of (*4.4.10*) with property (*4.4.11*).*

Now, let us consider the adiabatic situation $f_0 \equiv 0$. The corresponding slow-fast Hamiltonian system is of the form

$$\dot{\xi}^i = \varepsilon \left(v_f^i(\xi) - \frac{1}{2}[\sigma^{(1)}(\xi)]^{ij} \mathbf{J} \frac{\partial \mathbf{V}(\xi)}{\partial \xi^{\mathbf{j}}} \mathbf{x} \cdot \mathbf{x} \right), \tag{4.4.17}$$
$$\dot{\mathbf{x}} = \mathbf{V}(\xi)\mathbf{x}.$$

Assuming that $\det \mathbf{V}(\xi) \neq 0 \quad \forall \xi \in S_1$, we get that the matrix valued function $\mathbf{A} = [\det \mathbf{V}]^{-\frac{1}{2}} \mathbf{V}$ satisfies conditions (4.4.10), (4.4.11). In this case, formula (4.4.12) reads

$$J(\xi, \mathbf{x}) = -\frac{1}{2\,[\det \mathbf{V}(\xi)]^{\frac{1}{2}}} \mathbf{J}\mathbf{V}(\xi)\mathbf{x} \cdot \mathbf{x}$$

and by Theorem 4.3.20 J^0 is an adiabatic invariant of system (4.4.17) in the domain where the $\mathbb{S}^1$-action is free.

In the particular case, when

$$H_\varepsilon = \varepsilon \left(\frac{p_1^2 + p_2^2}{2} + \frac{\omega^2(q_1) q_2^2}{2} \right)$$

formula (4.4.12) gives the classical formula [7] for the adiabatic invariant

$$J^0 = \frac{p_2^2 + \omega^2(q_1) q_2^2}{2\omega(q_1)}.$$

4.4.3 Circle first integrals from strong stability

On the 4-dimensional slow-fast space $(\mathbb{R}^4, \sigma = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2)$, we consider again a Hamiltonian system whose perturbation term F is a quadratic form in the fast variables p_2, q_2:

$$H_\varepsilon = f(p_1, q_1) + \frac{\varepsilon}{2}(F_{11} p_2^2 + 2F_{12} p_2 q_2 + F_{22} q_2^2), \tag{4.4.18}$$

where $F_{ij} = F_{ij}(p_1, q_1)$ are some smooth functions on $\mathbb{R}^2$. The corresponding unperturbed vector field is written as

$$\mathbb{V} = \frac{\partial f}{\partial p_1}\frac{\partial}{\partial q_1} - \frac{\partial f}{\partial q_1}\frac{\partial}{\partial p_1} + (F_{11} p_2 + F_{12} q_2)\frac{\partial}{\partial q_2} - (F_{12} p_2 + F_{22} q_2)\frac{\partial}{\partial p_2}. \tag{4.4.19}$$

We assume that the Hamiltonian system $(\mathbb{R}^2, dp_1 \wedge dq_1, f)$ with one degree of freedom admits an invariant open domain $S_1 \subseteq \mathbb{R}^2$ such that the flow $\varphi^t : S_1 \to S_1$ of v_f is periodic with period function $\tau : S_1 \to \mathbb{R}$. Moreover, we suppose that the trajectory of v_f through each point $(p_1, q_1) \in S_1$ is $\tau(p_1, q_1)$-minimally periodic. Therefore, according to general definition (4.2.1), the monodromy of $\mathbb{V}$ at a point $(p_1, q_1) \in S_1$ is a linear symplectomorphism $g_{p_1,q_1} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g_{p_1,q_1} = \mathbf{G}_{p_1,q_1}(t)\mid_{t=\tau(p_1,q_1)}. \tag{4.4.20}$$

Here, $\mathbf{G}_{p_1,q_1}(t) \in \mathrm{Sp}(1; \mathbb{R})$ is the fundamental solution of the linear periodic system:

$$\frac{d}{dt}\mathbf{G}_{p_1,q_1} = \mathbf{V}(\varphi^t(p_1, q_1))\mathbf{G}_{p_1,q_1}, \tag{4.4.21}$$

$$\mathbf{G}_{p_1,q_1}(0) = \mathbf{I}, \tag{4.4.22}$$

where

$$\mathbf{V}(p_1,q_1) := \begin{bmatrix} -\mathcal{F}_{12}(p_1,q_1) & -\mathcal{F}_{22}(p_1,q_1) \\ \mathcal{F}_{11}(p_1,q_1) & \mathcal{F}_{12}(p_1,q_1) \end{bmatrix} \in \mathrm{sp}(1;\mathbb{R}). \tag{4.4.23}$$

Remark that the trace of the monodromy map g_{p_1,q_1} is constant along the trajectories of v_f, $\operatorname{tr} g_{\varphi^t(p_1,q_1)} = \operatorname{tr} g_{p_1,q_1}$, and hence it comes from a function on the orbit space $\mathrm{Orb}(v_f)$. We make also the following assumption:

$$-2 < \operatorname{tr} g_{p_1,q_1} < 2 \quad \forall (p_1,q_1) \in S_1. \tag{4.4.24}$$

As is known [26, 79], this condition implies the strong (parametric) stability of $\tau(p_1,q_1)$-periodic linear Hamiltonian system (4.4.21) for every (p_1,q_1).

We observe that the plane $\{(p_2 = 0, q_2 = 0\}$ is invariant with respect to the flow of the perturbed Hamiltonian system (4.4.18) whose first variation system over the invariant plane is defined by $\mathbb{V}$. First, we observe that condition (4.4.24) means that perturbed Hamiltonian system (4.4.18) is non-integrable.

Proposition 4.4.3 *If condition (*4.4.24*) holds, then perturbed Hamiltonian system (*4.4.18*) does not admit a first integral G defined in a neighborhood of the invariant submanifold $S_1 \times \{0\}$ and such that $d_2G \neq 0$ at $S_1 \times \{0\}$.*

Proof. As we have mentioned above the vector field $\mathbb{V}$ represents the first variation equations at the invariant domain foliated by periodic trajectories of Hamiltonian system (4.4.18). If Hamiltonian system (4.4.18) admits an additional first integral independent with H_ε, then by the well-known criterion [26, 77], the monodromy of $\mathbb{V}$ must satisfy the condition $\operatorname{tr} g_{p_1,q_1} = 2$. ∎

Now, we will show that under above hypothesis the unperturbed system admits a circle first integral. We need the following interpretation of condition (4.4.24) (see, for example [26]). Let us associate to the unperturbedvector field $\mathbb{V}$ the Riccati equation on S_1

$$\mathcal{L}_{v_f} D + \mathcal{F}_{11} D^2 + 2\mathcal{F}_{12} D + \mathcal{F}_{22} = 0. \tag{4.4.25}$$

Proposition 4.4.4 *Stability condition (*4.4.24*) is equivalent to the following: the Riccati equation (*4.4.25*) admits a unique smooth, complex-valued solution*

$$S_1 \ni (p_1,q_1) \mapsto D(p_1,q_1) = D_1(p_1,q_1) + iD_2(p_1,q_1)$$

satisfying the condition

$$D_2(p_1,q_1) > 0 \quad \forall (p_1,q_1) \in S_1 \tag{4.4.26}$$

This criterion leads to the following fact.

Proposition 4.4.5 *Under above hypotheses, the unperturbed vector field $\mathbb{V}$ (*4.4.19*) admits a circle first integral on $S_1 \times \mathbb{R}^2$ of the form*

$$J = \frac{1}{2D_2}[(p_2 - D_1 q_2)^2 + (D_2 q_2)^2] \tag{4.4.27}$$

where $D = D_1 + i\ D_2$ is the solution of (4.4.25), (4.4.26). Moreover, this integral satisfies the the adiabatic condition

$$\langle d_1 J\rangle = 0. \tag{4.4.28}$$

Here, the average $\langle\ \cdot\ \rangle$ is taken with respect to the the $\mathbb{S}^1$-action with infinitesimal generator $V_J^{(2)}$.

Proof. By direct computation and by using Riccati equation (4.4.25), we verify that formula (4.4.61) gives a first integral of $\mathbb{V}$. The 2π-periodicity of the flow of $V_J^{(2)}$ follows from the following argument. Consider the transformation $\phi : S_1 \times \mathbb{R}^2 \to S_1 \times \mathbb{R}^2$ given by

$$\phi(p_1, q_1, p_2, q_2) = \left(p_1, q_1, \sqrt{D_2}p_2 + \frac{D_1}{\sqrt{D_2}}q_2, \frac{q_2}{\sqrt{D_2}}\right) \tag{4.4.29}$$

it is clear that this is a diffeomorphism with inverse

$$\phi^{-1}(p_1, q_1, p_2, q_2) = \left(p_1, q_1, \frac{p_2 - D_1 q_2}{\sqrt{D_2}}, \sqrt{D_2}q_2\right) \tag{4.4.30}$$

Moreover, we have

$$\phi^*(\frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}) = \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2} \tag{4.4.31}$$

and

$$J \circ \phi = \frac{1}{2}\left({p_2}^2 + {q_2}^2\right) \tag{4.4.32}$$

It follows that the transformation ϕ takes the vector field $V_J^{(2)}$ into the form

$$\phi^* V_J^{(2)} = V_{J\circ\phi}^{(2)} = -q_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial q_2}$$

and hence

$$\mathrm{Fl}^t_{V_{J\circ\phi}^{(2)}}(p_1, q_1, p_2, q_2) = (p_1, q_1, p_2 \cos t - q_2 \sin t, p_2 \sin t + q_2 \cos t)\,, \tag{4.4.33}$$

$$\mathrm{Fl}^t_{V_J^{(2)}} = \phi^{-1} \circ \mathrm{Fl}^t_{V_{J\circ\phi}^{(2)}} \circ \phi.$$

Next, to prove (4.4.28) we observe that

$$\phi^*\langle d_1 J\rangle = \phi^* d\langle J\rangle - \phi^*\langle d_2 J\rangle = d\langle\phi^* J\rangle - \langle\phi^* d_2 J\rangle.$$

Taking into account the equalities

$$\phi^* p_2 = \sqrt{D_2}p_2 + \frac{D_1}{\sqrt{D_2}}q_2, \quad \phi^* q_2 = \frac{q_2}{\sqrt{D_2}},$$

$$d_2 J = \frac{p_2 - D_1 q_2}{D_2} dp_2 + \frac{(-D_1 p_2 + (D_1^2 + D_2^2) q_2)}{D_2} dq_2,$$

we get

$$\phi^* d_2 J = \frac{p_2}{\sqrt{D_2}} d\left(\frac{D_2 p_2 + D_1 q_2}{\sqrt{D_2}}\right) + \frac{1}{\sqrt{D_2}}\left(-D_1 p_2 + D_2 q_2\right) d\left(\frac{q_2}{\sqrt{D_2}}\right). \quad (4.4.34)$$

The averaging with respect to the trivial $\mathbb{S}^1$-action (4.4.33) gives

$$\langle p_2^2\rangle = \langle q_2^2\rangle = \frac{1}{2}(p_2^2+q_2^2), \quad \langle p_2 q_2\rangle = 0. \quad (4.4.35)$$

Using these relations and equalities (4.4.32), (4.4.34), we compute

$$\begin{aligned}\langle \phi^* d_2 J\rangle &= \frac{1}{\sqrt{D_2}}\langle p_2 d\left(\frac{D_1 q_2}{\sqrt{D_2}}\right)\rangle - \frac{D_1}{\sqrt{D_2}}\langle p_2 d\left(\frac{q_2}{\sqrt{D_2}}\right)\rangle \\ &+ \frac{1}{\sqrt{D_2}}\langle p_2 d\left(\sqrt{D_2} q_2\right)\rangle - \sqrt{D_2}\langle q_2 d\left(\frac{q_2}{\sqrt{D_2}}\right)\rangle \\ &= d\langle \frac{1}{2}(p_2^2+q_2^2)\rangle = d\langle \phi^* J\rangle.\end{aligned}$$

It follows that $\phi^*\langle d_1 J\rangle = 0$. ∎

Now, let us consider the $\mathbb{S}^1$-action defined by the infinitesimal generator $V_J^{(2)}$ associated to the circle first integral (4.4.27). The next proposition gives us effective formulas for computation of the $\mathbb{S}^1$-averages of the symplectic form and the perturbation term F of H_ε in terms of the solution D of the Riccati equation.

Lemma 4.4.6 *The $\mathbb{S}^1$-average of the symplectic form $\sigma = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2$ has representation $\langle\sigma\rangle = \sigma - \varepsilon d\theta^0$, where the 1-form $\theta^0 = \theta_1^0 dp_1 + \theta_2^0 dq_1$ with zero average is given by*

$$\phi^*\theta^0 = -\frac{1}{4D_2}\left((q_2^2 - p_2^2) d_1 D_1 + 2p_2 q_2 d_1 D_2\right). \quad (4.4.36)$$

Moreover,

$$\phi^*\langle F\rangle = \frac{1}{4D_2}\left(F_{11}(D_1^2+D_2^2) + 2F_{12}D_1 + F_{22}\right)\left(p_2^2+q_2^2\right). \quad (4.4.37)$$

Proof. First, we remark that

$$\phi^*(dp_2 \wedge dq_2) = dp_2 \wedge dq_2 + d\beta, \quad (4.4.38)$$

where the 1-form β is given by

$$\beta = -\frac{1}{2}\left(\frac{q_2^2}{D_2} d_1 D_1 + \frac{p_2 q_2}{D_2} d_1 D_2\right). \quad (4.4.39)$$

Indeed, under mapping (4.4.29) the 1-form $p_2 dq_2$ is transformed as follows

$$\begin{aligned}\phi^*(p_2 dq_2) &= \left(\sqrt{D_2} p_2 + \frac{D_1}{\sqrt{D_2}} q_2\right) d\left(\frac{q_2}{\sqrt{D_2}}\right) \\ &= \left(p_2 + \frac{D_1}{D_2} q_2\right) dq_2 - \frac{1}{2}\left(p_2 + \frac{D_1}{D_2} q_2\right) q_2 \frac{dD_2}{D_2},\end{aligned}$$

and hence

$$\phi^*(dp_2 \wedge dq_2) = d(\phi^*(p_2 dq_2)) \tag{4.4.40}$$
$$= dp_2 \wedge dq_2 - d\left(\frac{q_2^2}{2} d(\frac{D_1}{D_2})\right) - d\left(\frac{1}{2D_2}(p_2 + \frac{D_1}{D_2} q_2) q_2 dD_2\right).$$

Using relations (4.4.35) , we deduce the following formula for the average of 1-form β (4.4.39) relative to $\mathbb{S}^1$-action (4.4.33) :

$$\langle \beta \rangle = -\frac{1}{4D_2}(p_2^2 + q_2^2) d_1 D_1. \tag{4.4.41}$$

Next, by definition of the 1-form θ^0, we have $\langle dp_2 \wedge dq_2 \rangle = dp_2 \wedge dq_2 - d\theta^0$. It follows from here and identity (4.4.38) that

$$\phi^* \langle dp_2 \wedge dq_2 \rangle = \phi^* \langle dp_2 \wedge dq_2 \rangle - d(\phi^* \theta^0) \tag{4.4.42}$$
$$= dp_2 \wedge dq_2 + d\beta - d(\phi^* \theta^0).$$

On the other hand, using the property that the trivial $\mathbb{S}^1$-action (4.4.33) preserves the 2-form $dp_2 \wedge dq_2$, we get

$$\phi^* \langle dp_2 \wedge dq_2 \rangle = \langle \phi^*(dp_2 \wedge dq_2) \rangle = \langle dp_2 \wedge dq_2 + d\beta \rangle = dp_2 \wedge dq_2 + d\langle \beta \rangle.$$

Comparing this equality with (4.4.42) gives

$$\beta - \phi^* \theta^0 = \langle \beta \rangle + \gamma, \tag{4.4.43}$$

where γ is a closed 1-form which has the representation $\gamma = \gamma_1 dp_1 + \gamma_2 dq_1$. By closedness of γ, the coefficients γ_1 and γ_2 are independent of the variables p_2, q_2 and hence γ is the the pull-back under π_1 of a 1-closed on S_1. This means that γ is invariant with respect to the $\mathbb{S}^1$-action (4.4.33), $\langle \gamma \rangle = \gamma$. Moreover, by the property $\langle \theta^0 \rangle = 0$ and equality (4.4.43) we conclude that $\langle \gamma \rangle = 0$. Therefore, $\gamma = 0$ and the identity

$$\phi^* \theta^0 = \beta - \langle \beta \rangle \tag{4.4.44}$$

together with identities (4.4.39) and (4.4.41) implies (4.4.36). Finally, from

$$\phi^* F = \frac{1}{2D_2}\left(F_{11} D_2^2 p_2^2 + 2(F_{11} D_1 D_2 + F_{12} D_2) p_2 q_2 + (F_{22} + F_{11} D_1^2 + 2F_{12} D_1) q_2^2\right),$$

relations (4.4.35) and the identity $\phi^* \langle F \rangle = \langle \phi^* F \rangle$ we deduce representation (4.4.37). ∎

Remark 16 *Formula (4.4.37) can be also derived from identity*

$$\phi^* \langle F \rangle = \phi^* F - \mathbf{i}_{\hat{v}_f} \phi^* \theta^0$$

which is a consequence of (4.3.78).

Remark 17 *By the same arguments as in the proof of Lemma* 4.3.16*, one can show that the circle first integral* J *in (*4.4.27*) satisfies the relation* $J = \mathbf{i}_{V_J^{(2)}}\eta$*, where* $\eta = \frac{1}{2}(p_2 dq_2 - q_2 dp_2)$*.*

Resuming, we arrive at the following result.

Proposition 4.4.7 *Under above assumptions, for small enough* ε*, there exists a near identity transformation* $\mathcal{T}_\varepsilon$ *such that the pull back of the original perturbed Hamiltonian system (*4.4.18*) by the mapping* $\mathcal{T}_\varepsilon \circ \phi$ *is* ε^2*-close to the completely integrable Hamiltonian system* $(\tilde{\sigma}, \tilde{H}_\varepsilon)$ *given by*

$$\tilde{\sigma} = d\left(p_1 dq_1 + \varepsilon p_2 dq_2 - \frac{\varepsilon}{4D_2}(p_2^2 + q_2^2) d_1 D_1\right), \tag{4.4.45}$$

$$\tilde{H}_\varepsilon = f(p_1, q_1) + \frac{\varepsilon}{4D_2}\left(\digamma_{11}(D_1^2 + D_2^2) + 2\digamma_{12} D_1 + \digamma_{22}\right)\left(p_2^2 + q_2^2\right). \tag{4.4.46}$$

The corresponding additional first integral is just $J \circ \phi = \frac{1}{2}[{p_2}^2 + q_2^2]$*.*

Proof. Let $\mathcal{T}_\varepsilon$ be a near identity transformation defined in Theorem 4.3.1. Then, $\mathcal{T}_\varepsilon^* \sigma = \langle\sigma\rangle$, $H_\varepsilon \circ \mathcal{T}_\varepsilon = f \circ \pi_1 + \varepsilon\langle F\rangle + O(\varepsilon^2)$. Applying transformation ϕ (4.4.29) by formulas (4.4.38), (4.4.44), we get

$$\begin{aligned}
\tilde{\sigma} := \phi^* \mathcal{T}_\varepsilon^* \sigma = \phi^* \ \langle\sigma\rangle &= dp_1 \wedge dq_1 + \varepsilon\phi^*(dp_2 \wedge dq_2) - \varepsilon d(\phi^*\theta^0), \\
&= dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2 + \varepsilon d(\beta - \phi^*\theta^0), \\
&= dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2 + \varepsilon d\langle\beta\rangle.
\end{aligned}$$

This together with representation (4.4.41) leads to (4.4.45). Moreover,

$$H_\varepsilon \circ \mathcal{T}_\varepsilon \circ \phi = f \circ \pi_1 + \varepsilon\langle F\rangle \circ \phi + O(\varepsilon^2)$$

and from (4.4.37) we derive representation (4.4.46) for $\tilde{H}_\varepsilon = f \circ \pi_1 + \varepsilon\langle F\rangle \circ \phi$. ■

It follows from this theorem and the Liouville-Arnold theorem that for small enough $\varepsilon \neq 0$, the motion along the trajectories of the model Hamiltonian system (4.4.45), (4.4.46) is quasiperiodic and the corresponding Liouville tori in the phase space $(S_1 \times \mathbb{R}^2, \tilde{\sigma})$ are given as

$$\mathbb{T}^2_{c_1,c_2}(\varepsilon) = \left\{ f + \frac{\varepsilon}{2D_2}\left(\digamma_{11}(D_1^2 + D_2^2) + 2\digamma_{12}D_1 + \digamma_{22}\right) c_2 = c_1, \quad \frac{1}{2}(p_2^2 + {q_2}^2) = c_2 \right\}.$$

By (4.3.82) the Hamiltonian vector field of (4.4.46) is ε-close to the vector field $\phi^*\mathbb{V}$ which has two first integrals $f \circ \pi_1$ and $J \circ \phi$. The level sets of these functions give quasiperiodic 2-tori

$$\mathbb{T}^2_{c_1,c_2}(0) = \{f(p_1, q_1) = c_1\} \times \{\frac{1}{2}({p_2}^2 + {q_2}^2) = c_2\},$$

that is, the trajectory through a point $(p_1, q_1, p_2, q_2) \in \mathbb{T}^2_{c_1,c_2}(0)$ is quasiperiodic with frequencies $\omega_1 = \varpi(p_1, q_1) = \frac{2\pi}{\tau(p_1,q_1)}$ and $\omega_2 = \omega_2(p_1, q_1)$, where

$$2\cos\left(2\pi\frac{\omega_2}{\omega_1}\right) = \operatorname{tr} g_{p_1,q_1}. \tag{4.4.47}$$

This formula follows from the Floquet theory for linear periodic systems [26]. Moreover, there are two commuting vector fields $\tilde{\Upsilon}^{(1)}$ and $\tilde{\Upsilon}^{(2)}$ on $S_1 \times \mathbb{R}^2$ whose flows are 2π-periodic and such that

$$\phi^*\mathbb{V} = \omega_1\tilde{\Upsilon}^{(1)} + \omega_2\tilde{\Upsilon}^{(2)}. \tag{4.4.48}$$

Here, $\tilde{\Upsilon}^{(2)} = V^{(2)}_{J\circ\phi} = p_2\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial p_2}$ is just the infinitesimal generator of the trivial $\mathbb{S}^1$-action (4.4.33). Remark also that the"normal" frequency ω_2 is just the Floquet exponent of linear periodic system (4.4.21) and can be expressed in terms of the solution of the Riccati equation as follows [26]

$$\omega_2 = \langle \digamma_{11} D_2\rangle_1.$$

Here $\langle\cdot\rangle_1$ denotes the average with respect to the $\mathbb{S}^1$-action on S_1 associated to the periodic flow of v_f.

Therefore, combining Theorem 4.4.7 and the KAM-theory [6, 14] one can try to establish the persistence of the quasiperiodic tori $\mathbb{T}^2_{c_1,c_2}(0)$ for the perturbed Hamiltonian system $(\tilde{\sigma}, H_\varepsilon \circ \mathcal{T}_\varepsilon \circ \phi)$.

Now, let us consider the resonance case. Taking into account equalities (4.4.47), (4.4.48), we get the following criterion.

Proposition 4.4.8 *(The Resonance Case) The flow of the unperturbed vector field $\mathbb{V}$ is periodic if the parameter $\varkappa$ satisfies the condition*

$$\operatorname{tr} g_{p_1,q_1} = 2\cos\left(2\pi\frac{m}{k}\right) \tag{4.4.49}$$

for arbitrary coprime integers $m, k \in \mathbb{Z}$ such that and $0 < m < \frac{k}{2}$. The corresponding period function $T : S_1 \times \mathbb{R}^2 \to \mathbb{R}$ is given as

$$T = k\tau(p_1, q_1). \tag{4.4.50}$$

*and presents an adiabatic invariant of system (*4.4.18*).*

Condition (4.4.49) means that $\frac{\omega_2}{\omega_1} = \frac{m}{k}$ and hence we have

$$\phi^*\mathbb{V} = \frac{\omega_1}{k}\left(k\tilde{\Upsilon}^{(1)} + m\tilde{\Upsilon}^{(2)}\right).$$

The $\mathbb{S}^1$-action associated with periodic flow of $\phi^*\mathbb{V}$ is the product of two $\mathbb{S}^1$-actions with infinitesimal generators $k\tilde{\Upsilon}^{(1)}$ and $m\tilde{\Upsilon}^{(2)}$. Therefore, under condition (4.4.49), Theorem 4.3.18 and Averaging Theorem 3.2.15 to the perturbed vector field $\phi^*\mathbb{V} + \varepsilon\phi^*\tilde{\mathbb{W}} + O(\varepsilon^2)$ coming from the non-integrable Hamiltonian system, we get

$$(\tilde{\sigma}, H_\varepsilon \circ \mathcal{T}_\varepsilon \circ \phi = f \circ \pi_1 + \varepsilon\langle F\rangle \circ \phi + \frac{\varepsilon^2}{2}K \circ \phi + O(\varepsilon^3)).$$

4.4.4 Hamiltonian systems of Yang-Mills type

As an application of the above results, we consider a family of Hamiltonian systems of the Yang -Mills type [41] on the slow-fast space ($\mathbb{R}^4$, $\sigma = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2$):

$$H_\varepsilon = \frac{1}{2}p_1^2 + \frac{1}{4}q_1^4 + \frac{\varepsilon}{2}\left[p_2^2 + \frac{\varkappa(\varkappa+1)}{2}q_1^2q_2^2\right], \tag{4.4.51}$$

where $\varepsilon \ll 1$ is a perturbation parameter and $\varkappa \in \mathbb{R}$ is a constant. Therefore, the perturbed Hamiltonian is of the form (4.3.3) with

$$f = \frac{1}{2}p_1^2 + \frac{1}{4}q_1^4, \quad \text{and} \quad F = \frac{1}{2}\left[p_2^2 + \frac{\varkappa(\varkappa+1)}{2}q_1^2q_2^2\right] \tag{4.4.52}$$

The corresponding Hamiltonian equations of motions are

$$\frac{dp_1}{dt} = -q_1^3 - \varepsilon\frac{\varkappa(\varkappa+1)}{2}q_1q_2^2, \qquad \frac{dq_1}{dt} = p_1, \tag{4.4.53}$$

$$\frac{dp_2}{dt} = -\frac{\varkappa(\varkappa+1)}{2}q_1^2q_2 - \varepsilon q_2^3, \qquad \frac{dq_2}{dt} = p_2. \tag{4.4.54}$$

The unperturbed and perturbation vector fields are written as follows

$$\mathbb{V} = p_1\frac{\partial}{\partial q_1} - q_1^3\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial q_2} - \frac{\varkappa(\varkappa+1)}{2}q_1^2q_2\frac{\partial}{\partial p_2}, \tag{4.4.55}$$

and

$$\mathbb{W} = -\frac{\varkappa(\varkappa+1)}{2}q_1q_2^2\frac{\partial}{\partial p_1} - q_2^3\frac{\partial}{\partial p_2}. \tag{4.4.56}$$

The plane $\{p_2 = 0, q_2 = 0\} \subset \mathbb{R}^4$ is invariant under the flow of system (4.4.53), (4.4.54) and the unperturbed vector field $\mathbb{V}$ just represents the first variation system at the invariant plane. Under natural projection $\pi_1 : \mathbb{R}^4 \to \mathbb{R}^2$, the vector field $\mathbb{V}$ descends to the Hamiltonian vector field

$$v_f = -q_1^3\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_1}$$

of the Hamiltonian system $(\mathbb{R}^2, dp_1 \wedge dq_1, f)$ with one degree of freedom. The trajectory of v_f through each point (p_1, q_1) in the open domain $S_1 = \mathbb{R}^2 \setminus \{0\}$ is periodic with minimal period

$$\tau(p_1, q_1) = \frac{\tau_0}{(4f)^{\frac{1}{4}}} = \frac{\tau_0}{(2p_1^2 + q_1^4)^{\frac{1}{4}}}$$

where $\tau_0 = 4\sqrt{2}\int_0^1 \frac{dz}{\sqrt{1-z^4}}$. Therefore, the flow φ^t of v_f is periodic with frequenct function $\varpi : \mathbb{R}_0^2 \to \mathbb{R}$ given by

$$\varpi(p_1, q_1) = \frac{2\pi}{\tau_0}(2p_1^2 + q_1^4)^{\frac{1}{4}}.$$

The monodromy of $\mathbb{V}$ at a point $(p_1, q_1) \in \mathbb{R}_0^2$ is a linear symplectomorphism $g_{p_1,q_1} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$g_{p_1,q_1} = \mathbf{G}_{p_1,q_1}(t)\mid_{t=\tau(p_1,q_1)}.$$

Here $\mathbf{G}_{p_1,q_1}(t) \in \mathrm{Sp}(1;\mathbb{R})$ is the fundamental solution of the linear periodic system (4.4.21), (4.4.22) with

$$\mathbf{V}(p_1,q_1) := \begin{pmatrix} 0 & -\frac{\varkappa(\varkappa+1)}{2}q_1^2 \\ 1 & 0 \end{pmatrix}.$$

Notice that the trajectory of v_f passing through a point $(p_1^0, q_1^0) \in \mathbb{R}_0^2$ intersects the semiline $\{p_1 = 0, q_1 > 0\}$ exactly at the point $(0, (4f(p_1^0, q_1^0))^{\frac{1}{4}})$. This says that the dependence of the mondromy mapping in varibles p_1, q_1 is determinated up to conjugacy by symplectomorphisms g_{0,q_1}. In terms of the Jacobi elliptic functions [4], the flow of v_f is represented as

$$\varphi^t(0,,q_1) = ((4f)^{\frac{1}{4}}\,\mathrm{cn}((4f)^{\frac{1}{4}}t, \frac{1}{\sqrt{2}}), -2f^{\frac{1}{2}}\,\mathrm{sn}(4f)^{\frac{1}{4}}t, \frac{1}{\sqrt{2}})\,\mathrm{dn}(4f)^{\frac{1}{4}}t, \frac{1}{\sqrt{2}})).$$

For the function $\mathbf{G}_{0,1}(t)$, system (4.4.53) takes the form

$$\frac{d}{dt}\mathbf{G}_{0,1} = \begin{pmatrix} 0 & -\frac{\varkappa(\varkappa+1)}{2}\,\mathrm{cn}^2(t, \frac{1}{\sqrt{2}}) \\ 1 & 0 \end{pmatrix}\mathbf{G}_{0,1}. \tag{4.4.57}$$

The coefficients of this system are periodic in t with minimal period $\frac{1}{2}\tau(0,1) = \frac{\tau_0}{2}$. By the Yoshida formula [77] for the monodromy of time periodic system (4.4.57), we have

$$\mathrm{tr}\,\mathbf{G}_{0,1}(\frac{\tau_0}{2}) = 2\sqrt{2}\cos\left(\frac{\pi}{4}(1+2\varkappa)\right).$$

It is clear that

$$g_{0,1} = \mathbf{G}_{0,1}(\tau_0) = \mathbf{G}_{0,1}(\frac{\tau_0}{2}) \cdot \mathbf{G}_{0,1}(\frac{\tau_0}{2})$$

Moreover,

$$\mathbf{G}_{0,q_1}(t) = \begin{pmatrix} \frac{1}{(4f)^{\frac{1}{4}}} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{G}_{0,1}((4f)^{\frac{1}{4}}t)\begin{pmatrix} (4f)^{\frac{1}{4}} & 0 \\ 0 & 1 \end{pmatrix}$$

and hence

$$g_{0,q_1} = \begin{pmatrix} \frac{1}{(4f)^{\frac{1}{4}}} & 0 \\ 0 & 1 \end{pmatrix} g_{0,q_1} \begin{pmatrix} (4f)^{\frac{1}{4}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking into account the identity $\mathrm{tr}(\mathbf{G}^2) = (\mathrm{tr}\,\mathbf{G})^2 - 2$, for any symplectic matrix $\mathbf{G} \in \mathrm{Sp}(1;\mathbb{R})$, we arrive at the following fact.

Proposition 4.4.9 *For every $(p_1,q_1) \in \mathbb{R}_0^2$, the trace of the monodromy map of $\mathbf{V}$ is given by the formula*

$$\mathrm{tr}\,g_{p_1,q_1} = 8\cos^2\left(\frac{\pi}{4}(1+2\varkappa)\right) - 2. \tag{4.4.58}$$

By Proposition 4.4.3, a necessary condition for the existence of an additional first integral of system (4.4.53),(4.4.54) is $\mathrm{tr}\,g_{p_1,q_1} = 2$. By formula (4.4.58), this condition is written as $\cos\left(\frac{\pi}{4}(1+2\varkappa)\right) = \pm\frac{1}{2}$.

Corollary 4.4.10 *If the parameter $\varkappa$ is not integer,*

$$\varkappa \notin \mathbb{Z} \tag{4.4.59}$$

then original perturbed Hamiltonian system (4.4.53),(4.4.54) is not integrable in the sense that does not admit an additional first integral in a neighborhood of $\mathbb{R}^2_0 \times \{0\}$ which is functionally independent with H_ε.

In the non-integrable case (4.4.59), using above results we will approximate the perturbed system (4.4.53),(4.4.54) by a completely integrable Hamiltonian system.

Proposition 4.4.11 *Suppose that the parameter $\varkappa$ takes values in the open set*

$$\varkappa \in \bigcup_{s\in\mathbb{Z}} (2s, 2s+1). \tag{4.4.60}$$

Then, the unperturbed vector field $\mathbb{V}$ (4.4.55) admits a circle first integral on $\mathbb{R}^2_0 \times \mathbb{R}^2$ of the form

$$J = \frac{1}{2D_2}[(p_2 - D_1 q_2)^2 + (D_2 q_2)^2] \tag{4.4.61}$$

where $D_1 = D_1(p_1, q_1) + i\ D_2(p_1, q_1)$ is a smooth complex valued solution on $\mathbb{R}^2_0$ of the Riccati equation

$$L_{v_f} D + D^2 + \frac{\varkappa(\varkappa+1)}{2} q_1^2 = 0 \tag{4.4.62}$$

with $D_2 > 0$. Moreover, the circle first integral satisfies the adiabatic condition (4.4.28).

Proof. It follows from (4.4.58) that condition (4.4.24) holds for the monodromy g_{p_1,q_1} if and only if the parameter $\varkappa$ satisfies (4.4.60). Then, the statement follows from Proposition 4.4.5. ■

It follows from this proposition and Theorem 4.4.7 that for every parameter $\varkappa$ satisfying (4.4.60) and $\varepsilon \ll 1$, Hamiltonian systems of the Yang -Mills type (4.4.53), (4.4.54) is approximated by the completely integrable Hamiltonian system relative to the symplectic form $\tilde{\sigma}$(4.4.45) and Hamiltonian

$$\tilde{H}_\varepsilon = \frac{1}{2}p_1^2 + \frac{1}{4}q_1^4 + \frac{\varepsilon}{4D_2}\left(D_1^2 + D_2^2 + \frac{\varkappa(\varkappa+1)}{2}q_1^2\right)\left(p_2^2 + q_2^2\right).$$

By Proposition 4.4.8 and formula (4.4.58), we derive the resonance criterion.

Proposition 4.4.12 *The flow of vector field $\mathbb{V}$ (4.4.55) is periodic if the parameter $\varkappa$ satisfies the condition*

$$\sqrt{2}\cos\left(\frac{\pi}{4}(1+2\varkappa)\right) = \cos\left(\pi\frac{m}{k}\right) \tag{4.4.63}$$

for arbitrary coprime integers $m, k \in \mathbb{Z}$ such that and $0 < m < k$. The corresponding period function $T : \mathbb{R}^2_0 \times \mathbb{R}^2 \to \mathbb{R}$,

$$T = k\tau(p_1, q_1)$$

is an adiabatic invariant of system (4.4.53), (4.4.54).

Remark that the parameter $\varkappa$ satisfying condition (4.4.63) runs over an dense numerable subset in the open set $\bigcup_{s\in\mathbb{Z}}(2s, 2s+1)$. Therefore, by Theorem 4.2.2, under condition (4.4.63), perturbed system (4.4.53), (4.4.54) admits a first order normalization relative to $\mathbb{V}$ and the function $f = \frac{1}{2}p_1^2 + \frac{1}{4}q_1^4$ is an adiabatic invariant of this system.

4.5 Particle Dynamics with Spin in a Magnetic Field

Let us consider a slow-fast phase space ($M = S_1 \times S_2$, $\sigma = \sigma^{(1)} + \varepsilon\sigma^{(2)}$) in the case when S_1 is an arbitrary symplectic manifold and $S_2 = \mathbb{S}^2 \subset \mathbb{R}^3 = \{\mathbf{x} = (x^1, x^2, x^3)\}$ is the unit sphere equipped with standard area form. Therefore,

$$\sigma^{(1)} = \frac{1}{2}\sigma^{(1)}_{ij}(\xi)d\xi^i \wedge d\xi^j,$$

$$\sigma^{(2)} = \frac{1}{2}\epsilon_{ijk}x^i dx^j \wedge dx^k.$$

Suppose that we are given a smooth mapping $\mathbf{n} : S_1 \to \mathbb{S}^2$ and define the function $J \in C^\infty(M)$ by

$$J(\xi, \mathbf{x}) = \mathbf{n}(\xi) \cdot \mathbf{x} \tag{4.5.1}$$

for $\xi \in S_1$ and $\mathbf{x} \in \mathbb{S}^2$. Then, we have the $\mathbb{S}^1$-action on M associated to the infinitesimal generator

$$V_J^{(2)} = -\mathbf{n} \times \mathbf{x} \cdot \frac{\partial}{\partial \mathbf{x}}$$

which is given by the rotations $\exp(-t\Lambda \circ \mathbf{n})$ in $\mathbb{R}^3_{\mathbf{x}}$ about the axis $-\mathbf{n}$. Here $\Lambda \circ \mathbf{n}$ denotes the matrix of the cross product in $\mathbb{R}^3_{\mathbf{x}}$ with $\mathbf{n}$. We have the following explicit formula for the $\mathbb{S}^1$-action

$$\mathrm{Fl}^t_{V_J^{(2)}}(\xi, \mathbf{x}) = \cos t\mathbf{x} + (1 - \cos t)(\mathbf{n}(\xi) \cdot \mathbf{x})\mathbf{n}(\xi) - \sin t(\mathbf{n}(\xi)\times\mathbf{x}). \tag{4.5.2}$$

Lemma 4.5.1 *The $\mathbb{S}^1$-average of the symplectic form has the representation $\langle\sigma\rangle = \sigma - \varepsilon d\theta^0$, where*

$$\theta^0 = (\mathbf{n} \times \mathbf{x}) \cdot d_1\mathbf{n}. \tag{4.5.3}$$

Moreover,

$$d_1\theta^0 = \mathbf{x} \cdot d_1\mathbf{n} \wedge d_1\mathbf{n}, \tag{4.5.4}$$

$$\{\theta^0 \wedge \theta^0\}_2 = 0, \tag{4.5.5}$$

and, function J in (4.5.1) satisfies the adiabatic condition $\langle d_1 J\rangle = 0$. Hence, the $\mathbb{S}^1$-action is Hamiltonian relative to $\langle\sigma\rangle$ with momentum map εJ.

Proof. First, we have that $d_1 J = \mathbf{x} \cdot d_1\mathbf{n}$. Hence, $(\mathrm{Fl}^t_{V_J^{(2)}})^* d_1 J = \mathrm{Fl}^t_{V_J^{(2)}}(\xi, x) \cdot d_1\mathbf{n}$. By (4.5.2) and direct computations, we have $\langle d_1 J\rangle = (\mathbf{n} \cdot \mathbf{x})(\mathbf{n} \cdot d_1\mathbf{n})$. Since $\mathbf{n}$ has constant norm, the adiabatic condition holds. Analogously, we get that $\theta^0 = \mathcal{S}(d_1 J) = (\mathbf{n} \times \mathbf{x}) \cdot d_1\mathbf{n}$. Therefore, formula (4.5.4) tell us the the curvature of the Hannay-Berry connection associated to 1-form (4.5.3) is nonzero. ∎

Now, suppose that on the slow fast space we are given a perturbed Hamiltonian system of the form

$$H_\varepsilon(\xi, \mathbf{x}) = f(\xi) + \varepsilon \mathbf{B}(\xi) \cdot \mathbf{x}, \tag{4.5.6}$$

for some $f \in C^\infty(S_1)$ and a smooth vector function $\mathbf{B} : S_1 \to \mathbb{R}^3$ which will play the role of a magnetic field. The corresponding unperturbed and perturbation vector fields are given by

$$\mathbb{V} = \hat{v}_f - \mathbf{B} \times \mathbf{x} \cdot \frac{\partial}{\partial \mathbf{x}}, \tag{4.5.7}$$

$$\mathbb{W} = -[\sigma^{(1)}]^{ij} (x_k \frac{\partial B_k}{\partial \xi^i}) \frac{\partial}{\partial \xi^j},$$

where $\hat{v}_f = [\sigma^{(1)}]^{ij} \frac{\partial f}{\partial \xi^i} \frac{\partial}{\partial \xi^j}$.

Lemma 4.5.2 *Function J in (4.5.1) is a circle first integral of unperturbed vector field (4.5.7) if and only if the vector functions $\mathbf{n}$ and $\mathbf{B}$ are related by the condition*

$$\mathcal{L}_{\hat{v}_f} \mathbf{n} + \mathbf{B} \times \mathbf{n} = 0, \tag{4.5.8}$$

where $\hat{v}_f = [\sigma^{(1)}]^{ij} \frac{\partial f}{\partial \xi^i} \frac{\partial}{\partial \xi^j}$.

Proof. Let $\hat{v}_f = (\hat{v}_f)_j \frac{\partial}{\partial \xi^j}$. By straight forward computation, we get $\frac{\partial f}{\partial \xi^i} = -\frac{1}{2} (\hat{v}_f)_j \sigma^{(1)}_{ji}$. Since σ^1 in nondegenerate, $\hat{v}_f = [\sigma^{(1)}]^{ij} \frac{\partial f}{\partial \xi^i} \frac{\partial}{\partial \xi^j}$. Now, we compute the Lie derivative of J along $\mathbb{V}$.

$$\mathcal{L}_{\mathbb{V}} J = (\mathcal{L}_{\hat{v}_f} \mathbf{n}) \cdot \mathbf{x} - \mathbf{B} \times \mathbf{x} \cdot \mathbf{n} = (\mathcal{L}_{\hat{v}_f} \mathbf{n} + \mathbf{B} \times \mathbf{n}) \cdot \mathbf{x} \cdot$$

Hence, J is first integral of $\mathbb{V}$ if and only if condition (4.5.8) holds. ■

Now, let us apply this result to the equations describing the motion of a non relativistic particle with spin in a slow varying magnetic field [12, 46, 76]. In this case, the slow phase space $S_1 = \mathbb{R}^3_{\mathbf{p}} \times \mathbb{R}^3_{\mathbf{q}}$ is equipped with symplectic form

$$\sigma^{(1)} = \frac{1}{2} d\mathbf{p} \wedge d\mathbf{q} + \frac{1}{2} (\mathbf{B}(\mathbf{q}) \times d\mathbf{q}) \wedge d\mathbf{q},$$

where $\mathbf{B} = \mathbf{B}(\mathbf{q})$ is a divergence free field on $\mathbb{R}^3_{\mathbf{q}}$, $\operatorname{div} \mathbf{B} = 0$. Therefore, $\sigma^{(1)}$ equals to the canonical symplectic form on $\mathbb{R}^3_{\mathbf{p}} \times \mathbb{R}^3_{\mathbf{q}}$ pulse the " magnetic" term. Putting $f = \frac{\mathbf{p}^2}{2}$ into (4.5.6), we get he following perturbed Hamiltonian dynamical system on $M = (\mathbb{R}^3_{\mathbf{p}} \times \mathbb{R}^3_{\mathbf{q}}) \times \mathbb{S}^2$:

$$\frac{d\mathbf{q}}{dt} = \mathbf{p},$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{p} \times \mathbf{B} - \varepsilon (\frac{\partial \mathbf{B}}{\partial \mathbf{q}})^T \mathbf{x},$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{x} \times \mathbf{B}.$$

This system describes the motion of a non relativistic particle with spin in a slow varying magnetic field, where the mass and charge of the particle $m = 1$ and $e = 1$, the gyromagnetic ratio $g = 2$. The unperturbed system is of the form

$$\frac{d\mathbf{q}}{dt} = \mathbf{p}, \tag{4.5.9}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{p} \times \mathbf{B}, \tag{4.5.10}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{x} \times \mathbf{B}. \tag{4.5.11}$$

System (4.5.9), (4.5.10) corresponds to the Lorentz force equations and equation (4.5.11) describes the spin precession.

Lemma 4.5.3 *Condition (4.5.8) holds for the following choice of the vector function* $\mathbf{n}$:

$$\mathbf{n} = \frac{1}{\| \mathbf{p} \|}\mathbf{p}.$$

Proof. The vector field $\mathbb{V}$ generating the system (4.5.9), (4.5.10) and (4.5.11) has the form

$$\mathbb{V} = \mathbf{p} \cdot \frac{\partial}{\partial \mathbf{q}} + \mathbf{p} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{P}} + \mathbf{x} \times \mathbf{B} \cdot \frac{\partial}{\partial \mathbf{x}}$$

By direct computation, we have

$$\begin{aligned} \mathcal{L}_{\mathbb{V}} J &= \mathbf{p} \times \mathbf{B} \cdot \left(\frac{\mathbf{x}}{\|\mathbf{p}\|} - \frac{\mathbf{p}\cdot\mathbf{x}}{\|\mathbf{p}\|^3}\mathbf{p}\right) + \mathbf{x} \times \mathbf{B} \cdot \frac{\mathbf{p}}{\|\mathbf{p}\|}, \\ &= \frac{1}{\|\mathbf{p}\|}\mathbf{p} \times \mathbf{B} \cdot \mathbf{x} - \left(\frac{\mathbf{p}\cdot\mathbf{x}}{\|\mathbf{p}\|^3}\right)\mathbf{p} \times \mathbf{B} \cdot \mathbf{p} - \frac{1}{\|\mathbf{p}\|}\mathbf{p} \times \mathbf{B} \cdot \mathbf{x} = 0. \end{aligned}$$

∎

It follows that the infinitesimal generator of the Hamiltonian $\mathbb{S}^1$-action and the momentum map are of the form

$$V_J = \frac{1}{\| \mathbf{p} \|}\mathbf{x} \times \mathbf{p} \quad \text{and} \quad J = \frac{1}{\| \mathbf{p} \|}\mathbf{p} \cdot \mathbf{x}. \tag{4.5.12}$$

Moreover,

$$\theta^0 = -\frac{1}{\mathbf{p}^2}(\mathbf{x} \times \mathbf{p}) \cdot d\mathbf{p}.$$

Finally, we get that the approximate Hamiltonian system with $\mathbb{S}^1$-symmetry is

$$\langle \sigma \rangle = \frac{1}{2} d\mathbf{p} \wedge d\mathbf{q} + \frac{1}{2}(\mathbf{B}(\mathbf{q}) \times d\mathbf{q}) \wedge d\mathbf{q} + \frac{\varepsilon}{2}(\mathbf{x} \times d\mathbf{x}) \wedge d\mathbf{x} - \varepsilon \mathbf{x} \cdot d_1\left(\frac{\mathbf{p}}{\|\mathbf{p}\|}\right) \wedge d_1\left(\frac{\mathbf{p}}{\|\mathbf{p}\|}\right),$$

and

$$\langle H_\varepsilon \rangle = \frac{\mathbf{p}^2}{2} + \frac{\varepsilon}{\mathbf{p}^2}(\mathbf{p} \cdot \mathbf{x})(\mathbf{p} \cdot \mathbf{B}).$$

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