"El saber de mis hijos
hará mi grandeza"

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T E S I S

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Contents

Introduction

In this thesis we develop bigraded calculus for differential operators with applications to some problems in Poisson geometry, related to singular foliations.

A Poisson manifold consists in a smooth manifold M and a Poisson bivector field Π, called Poisson structure, which satisfies the integrability condition

$$[\Pi, \Pi] = 0 \tag{1}$$

written in terms of the Schouten - Nijenhuis bracket [16]. Algebraically, this condition means that the Poisson structure induces a Lie bracket $\{,\}$ on the space of smooth functions C^∞_M by

$$\{f, g\} := \Pi(\mathrm{d}f, \mathrm{d}g),$$

called the Poisson bracket. Geometrically, the integrability condition (1) leads to the study of singular foliations. It is well-known that every Poisson structure induces a symplectic foliation, which is singular in general, and it is integrable in the sense of Stefan - Sussman [28, 27]. Thus, one can think of a Poisson manifold as the union of symplectic manifolds (symplectic leaves) of varying dimensions, fitting together in a smooth way. A symplectic leaf is said to be singular if the rank of the Poisson structure is not locally constant at the leaf. Poisson structures near singular symplectic leaves are described in terms of the coupling procedure [36, 33, 34, 37]. This approach is based on the use of a non-linear (Ehresmann) connection, which gives rise to bigraded calculus on Poisson fiber bundles (see [9, 36, 33, 7, 34, 6, 18]).

The goal of this work is to give a unified approach to the Schouten - Frölicher - Nijenhuis - Ehresmann calculus on fibred and foliated manifolds and apply this approach to the study of infinitesimal automorphisms and first cohomology of Poisson manifolds with singular symplectic foliations. Some results on computing Poisson cohomology in the regular case can be found, for example, in [38, 39, 31, 32, 8]. In some special singular cases, Poisson cohomology has been studied in [5, 22, 23, 24, 25].

One of our main results is a geometric splitting for the first cohomology group of coupling Poisson structures in fiber bundles. We give a natural bigrading to the Lichnerowicz - Poisson complex [16], by means of the geometric data of the coupling structure, and split the first Poisson cohomology group using that bigrading. In order to derive this result, we first develop a general scheme to study the first cohomology group of an abstract bigraded cochain complex $(\mathcal{C}, \partial)$, i.e., a cochain complex such that $\mathcal{C}$ is $\mathbb{Z} \times \mathbb{Z}$-graded and the coboundary operator has a bigraded decomposition of the form

$$\partial = \partial_{1,0} + \partial_{0,1} + \partial_{2,-1}. \tag{2}$$

In this context, the coboundary condition $\partial^2 = 0$ implies that $\partial_{1,0}$ and $\partial_{2,-1}$ are also coboundaries. Moreover, we get the following short exact sequence, in which every object is intrinsically defined in terms of the bigraded cochain complex.

Claim 1. *Let $\mathcal{H}^1_\partial$ be the first cohomology group of ∂ and $\mathcal{Z}^{p,q}_{\partial_{0,1}}$ the space of coboundaries of bidegree (p,q) for $\partial_{0,1}$. There is a short exact sequence,*

$$0 \to \mathcal{H}^1_{\overline{\partial}_{1,0}} \hookrightarrow \mathcal{H}^1_\partial \to \frac{\ker\rho}{\mathcal{B}^1_{\partial_{0,1}}} \to 0, \tag{3}$$

where $\overline{\partial}_{1,0} : \mathcal{Z}^{p,0}_{\partial_{0,1}} \longrightarrow \mathcal{Z}^{p+1,0}_{\partial_{0,1}}$ is a coboundary operator given by the restriction of ∂ to $\mathcal{Z}^{\bullet,0}_{\partial_{0,1}}$, and $\rho : \mathcal{A} \longrightarrow \mathcal{H}^2_{\overline{\partial}_{1,0}}$ is a canonical morphism from a subspace $\mathcal{A} \subset \mathcal{Z}^{0,1}_{\partial_{0,1}}$ to the second cohomology space of $\overline{\partial}_{1,0}$.

Up to our knowledge, this special result is not well known in the literature. An important example of a bigraded cochain complex of the type (2) is the de Rham complex of a foliated manifold [32, 30]. Our point is to describe the relationship between the Lichnerowicz - Poisson complex and a bigraded cochain complex of the type (2). To do so, we consider, in a fiber bundle (E, π, B), the bigraded Poisson algebra of *vertical - valued forms in the base* $\mathcal{V}_E := \Omega_B \otimes_{C^\infty_B} \chi_{\mathbb{V}}(E)$, and prove that the Lichnerowicz - Poisson complex of a coupling Poisson structure Π on the fiber bundle (E, π, B) is isomorphic to the bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$, where ∂^γ is a coboundary operator in $\mathcal{V}_E$ induced by the geometric data of Π. Because of this isomorphism, one can apply Claim 1 to the bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$ in order to *split* the first Poisson cohomology group in the sense of (3). In the context of the linearization problems, cochain complexes of the type $(\mathcal{V}_E, \partial^\gamma)$ appeared in [12, 11, 6, 18].

One of the most interesting applications of this result is related to effective ways of describing infinitesimal automorphisms (the first cohomology group) of a Poisson manifold around a symplectic leaf in some "extreme" cases, when the transverse Poisson structure has specific properties.

Now we give a more detailed description of these results, explaining briefly the content of each chapter.

In Chapter 1, some useful results about bigraded cochain complexes of the type (2) are established in full detail. We develop a general scheme to study the first cohomology group of bigraded cochain complexes. More precisely, the main result proved in this chapter states that (3) is a short exact sequence (Theorem 1.2.2). Also, we give the precise definition of all spaces and operators appearing on its formulation. These objects are canonically defined by the bigraded cochain complex (see Section 1.2). In particular, this general scheme is applied to the Lichnerowicz - Poisson complex in Chapter 4.

Chapter 2 is devoted to the description of some algebraic and geometric tools we need along this work. First, we revise the theory of graded derivations and differential operators on vector bundles, in order to define intrinsically the Frölicher and Schouten - Nijenhuis brackets [19, 20]. We also present the concept of generalized connection [21] on bigraded manifolds, which gives a natural generalization of the Ehresmann connections on foliated and fibered manifolds. For a generalized connection γ in the manifold M, with curvature R and co-curvature R', we prove in Theorem 2.5.6 that the exterior differential has the following bigraded decomposition

$$\mathrm{d} = \mathrm{d}_{1,0} + \mathrm{d}_{0,1} + \mathrm{d}_{2,-1} + \mathrm{d}_{-1,2},$$

and the Frölicher - Nijenhuis decompositions of these operators are

$$\mathrm{d}_{1,0} = \mathcal{L}_{\mathrm{Id}_{TM}-\gamma} + 2\mathrm{i}_R - \mathrm{i}_{R'}, \quad \mathrm{d}_{0,1} = \mathcal{L}_\gamma - \mathrm{i}_R + 2\mathrm{i}_{R'}, \quad \mathrm{d}_{2,-1} = -\mathrm{i}_R, \quad \mathrm{d}_{-1,2} = -\mathrm{i}_{R'}.$$

The concept of coupling Poisson structure [36, 33] in a foliated manifold is presented in Chapter 3. Specially, we are interested in the case of coupling Poisson structures Π defined on a fibration $E \xrightarrow{\pi} B$. In this context, the main result presented in this chapter consists in proving that the Lichnerowicz - Poisson complex (χ_E, δ^Π) induced by Π is isomorphic to a bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$ induced by its geometric data (γ, σ, P). This fact can be found without proof in [6, pp. 507-508] and with proof in [18, p. 93]. We present a proof of this result in Theorem 3.4.2. Such isomorphism allows us to apply the theory developed in Chapter 1 to the Lichnerowicz - Poisson complex induced by a coupling Poisson structure.

Our main result is presented in Chapter 4, where we combine the theory developed in Chapters 1 and 3 to obtain a splitting for the first Poisson cohomology group. We first aboard the case of regular Poisson structures, for which we prove that the first Poisson cohomology group splits as

$$\mathcal{H}^1_{LP}(M, \Pi) \simeq H^1_{\mathrm{d}_\mathcal{S}} \oplus \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \mid \mathcal{L}_Y \omega \text{ is } \mathrm{d}_\mathcal{S}\text{-exact}\},$$

where $\mathrm{d}_\mathcal{S}$ denotes the foliated exterior differential of the symplectic foliation $\mathcal{S}$. After that, we apply the theory developed in previous chapters to obtain a similar splitting for the first Poisson cohomology group of a coupling structure in a geometric manner. Starting with a coupling Poisson structure Π in the fiber bundle (E, π, B), we naturally define a triple of geometric data (γ, σ, P) [36, 33], where γ is an Ehresmann connection, σ is an horizontally non-degenerate horizontal 2-form, and P is a vertical Poisson structure. On the other hand, in the fiber bundle (E, π, B), we can define the bigraded Poisson algebra $\mathcal{V}_E$ of *vertical - valued forms in the base*. Furthermore, the geometric data (γ, σ, P) of Π induces a bigraded coboundary operator ∂^γ in $\mathcal{V}_E$ such that the bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$ is isomorphic to the Lichnerowicz

- Poisson complex (χ_E, δ^{Π}) of Π, as proved in Chapter 3. Applying our main result of Chapter 1, we obtain the following splitting-type result for the first cohomology of the coupling Poisson structure Π:

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq \mathcal{H}^1_{\overline{\partial}^{\gamma}_{1,0}} \oplus \frac{\ker\rho}{\mathrm{Ham}(E,P)}.$$

Here, $\overline{\partial}^{\gamma}_{1,0} : \Omega_B \otimes \mathrm{Casim}(E,P) \longrightarrow \Omega_B \otimes \mathrm{Casim}(E,P)$ is a coboundary operator and $\rho : \mathcal{A}^{\gamma} \subset \mathrm{Poiss}_{\mathbb{V}}(E,P) \longrightarrow \mathcal{H}^2_{\overline{\partial}^{\gamma}_{1,0}}$ is a canonical linear morphism from a Lie subalgebra of vertical Poisson vector fields to the second cohomology group of $\overline{\partial}^{\gamma}_{1,0}$. The proof of this (Theorem 4.2.5) is practically a direct consequence of the main results of Chapters 1 and 3. On the other hand, we present some particular cases for which the computation of the first Poisson cohomology group simplifies. For example, a regular Poisson structure around a symplectic leaf B can be modeled by a coupling structure Π in a fiber bundle (E,π,B), with geometric data $P=0$ and σ the pull-back of the symplectic structure in the base B [36]. We show that the first Poisson cohomology group of a coupling Poisson structure Π of such type is isomorphic to the sum of the first leafwise de Rham cohomology of the symplectic foliation and the trivial deformations of the leafwise symplectic structure:

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq H^1_{\mathrm{d}_{\mathcal{S}}} \oplus \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(E,\mathbb{H}) \mid \mathcal{L}_Y\omega \text{ is } \mathrm{d}_{\mathcal{S}}-\text{exact}\}.$$

Another particular case we present is when the vertical Poisson cohomology is trivial,

$$\mathrm{Poiss}_{\mathbb{V}}(E,P) = \mathrm{Ham}(E,P). \tag{4}$$

Under this assumption, the first Poisson cohomology group of Π is isomorphic to $\mathcal{H}^1_{\overline{\partial}^{\gamma}_{1,0}}$. Furthermore, if P satisfies (4), then we show that the first Poisson cohomology groups of every coupling Poisson structure in (E,π,B) having P as vertical part, are isomorphic to each other. Finally, the last family of given examples cannot be directly obtained by the abstract framework developed in Chapter 1. Such examples arise when the Casimir functions of the vertical Poisson structure are precisely the projectable functions in the bundle: $\mathrm{Casim}(E,P) = C^{\infty}_{\mathrm{pr}}(E)$. These examples are important since this condition can be only held in the singular case. In fact, $P=0$ implies $\mathrm{Casim}(E,P) = C^{\infty}_{E} \neq C^{\infty}_{\mathrm{pr}}(E)$. Under this assumption, the first Poisson cohomology group can be computed as

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq H^1_{dR}(B) \oplus \frac{\ker\rho}{\mathrm{Ham}(E,P)},$$

where $H^1_{dR}(B)$ is the first de Rham cohomology group of the base space B.

Finally, we consider a class of coupling Poisson structures on a locally trivial vector bundle whose vertical part P is defined by a 3-dimensional linear Poisson tensor. To apply the above results to this case, we present a classification of all 3-dimensional linear Poisson structures which admit only constant Casimir functions. We show that there are five non-isomorphic families in which this occur, and in most of these cases, the symplectic foliation is an *open book foliation.*

Chapter 1

Bigraded Cochain Complexes. Algebraic Framework

In this chapter, we study the first cohomology group of a cochain complex $(\mathcal{C}, \partial)$, where $\mathcal{C}$ is assumed to be a $\mathbb{Z} \times \mathbb{Z}$-graded $\mathbb{R}$-vector space such that the coboundary operator ∂ has the following *bigraded decomposition*:

$$\partial = \partial_{1,0} + \partial_{0,1} + \partial_{2,-1}. \tag{1.1}$$

We say that a cochain complex of this kind is a *bigraded cochain complex*. In differential geometry, a classical well known example of a coboundary operator with bigraded decomposition as in (1.1) is the exterior differential on fibred or foliated manifolds (see, for example, [30, p. 184], [2, p. 3-6]). Also, bigraded cochain complexes have great importance in the study of the Poisson cohomology groups of regular structures (see [32], [8]).

We study some properties of bigraded cochain complexes in the general (abstract) case. Also, as our main result, we present a short exact sequence for the first cohomology group of bigraded cochain complexes. Such result, Theorem 1.2.2, is, up to our knowledge, not well-known in the literature. To derive it, we develop some properties on bigraded cochain complexes and conclude using some theory on homological algebra.

This chapter is divided in three sections. In Section 1.1 we present some preliminary notions of graded modules and algebras, which will be needed throughout this work. In Section 1.2, we set the problem to be addressed and present the main result. We also define some objects we need in order to state the main result: a coboundary operator, a subspace of cocycles for this new operator, and some morphism from this space to the second cohomology space of such operator. These objects are intrinsic in the bigraded cochain complex, and they acquire geometric meaning when we apply this general scheme to the Lichnerowicz - Poisson complex in the context of Poisson manifolds (see Chapter 4).

The proof of the main result of this chapter (Theorem 1.2.2) is detailed in Section 1.3. We play with the properties of bigraded cochain complexes and, by means of the bigraded components of the operator in (1.1) and the objects presented in the previous section, we first prove the existence of short exact sequences for cocycle and coboundary spaces. Finally, we apply well-known results in homology theory [17] in order to derive de short exact sequence for the first cohomology group of the complex.

In further chapters, some examples of bigraded cochain complexes are studied, and we then apply the main result of this chapter to them. Specially, we apply this abstract result to the Lichnerowicz - Poisson complex in the context of Poisson manifolds with singularities.

1.1 Bigraded modules

Let $\mathcal{R}$ be a commutative ring with identity and $(G,+)$ an Abelian group. An $\mathcal{R}$-module $\mathcal{C}$ is said to be G-**graded** if, for each $g \in G$, there exists an $\mathcal{R}$-module $\mathcal{C}^g$ such that

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}^g.$$

Elements of $\bigsqcup_{g\in G} \mathcal{C}^g$ are called *homogeneous* and, for each nonzero homogeneous element $\eta \in \mathcal{C}$, the **degree** $|\eta|$ is defined by the unique $g \in G$ such that $\eta \in \mathcal{C}^g$.

A G**-graded** $\mathcal{R}$**-algebra** of degree $d \in G$ is a graded G-module $\mathcal{A}$ equipped with an $\mathbb{R}$-bilinear operation

$$\circ : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

such that

$$\mathcal{A}^g \circ \mathcal{A}^h \subseteq \mathcal{A}^{d+g+h} \qquad \forall g, h \in G.$$

If $\mathcal{C}$ and $\mathcal{D}$ are G-graded $\mathcal{R}$-modules, then a **graded morphism** of degree $g \in G$ is an $\mathcal{R}$-linear morphism $L : \mathcal{C} \longrightarrow \mathcal{D}$ such that $L(\mathcal{C}^h) \subset \mathcal{D}^{h+g} \quad \forall h \in G$. The $\mathcal{R}$-module of graded morphisms of degree g is denoted by $\mathrm{Hom}^g_{\mathcal{R}}(\mathcal{C}, \mathcal{D})$. Note that

$$\mathrm{Hom}^g_{\mathcal{R}}(\mathcal{C}, \mathcal{D}) \circ \mathrm{Hom}^h_{\mathcal{R}}(\mathcal{D}, \mathcal{E}) \subset \mathrm{Hom}^{g+h}_{\mathcal{R}}(\mathcal{C}, \mathcal{E}).$$

The $\mathcal{R}$-module of **graded endomorphisms** of degree g in $\mathcal{C}$ is defined by $\mathrm{Hom}^g_{\mathcal{R}}(\mathcal{C}, \mathcal{C})$ and denoted by $\mathrm{End}^g_{\mathcal{R}}\mathcal{C}$. Moreover,

$$\mathrm{End}^\bullet_{\mathcal{R}}\mathcal{C} := \bigoplus_{g \in G} \mathrm{End}^g_{\mathcal{R}}\mathcal{C},$$

is an associative G-graded $\mathcal{R}$-algebra with the composition of graded endomorphisms

$$(E \circ F)(a) := E(F(a)).$$

This work is focused on the following special cases of graded algebras:

- A $\mathbb{Z}$-graded algebra is simply said to be a **graded algebra**.
- A $\mathbb{Z} \times \mathbb{Z}-$graded algebra is said to be a **bigraded algebra**.

It is clear that every bigraded module $\mathcal{C}$ can be endowed with a graded module structure with the following $\mathbb{Z}$-grading

$$\mathcal{C}^n := \bigoplus_{\substack{p,q\in\mathbb{Z}\\ p+q=n}} \mathcal{C}^{p,q}. \tag{1.2}$$

In this case, the $\mathbb{Z}\times\mathbb{Z}$-grading is said to be *compatible* with the $\mathbb{Z}$-grading in $\mathcal{C}$. This means that, for any $\eta \in \mathcal{C}^k$, there exists, for each $p, q \in \mathbb{Z}$ with $p + q = k$, $\eta_{p,q} \in \mathcal{C}^{p,q}$ such that

$$\eta = \sum_{p+q=k} \eta_{p,q}.$$

The right-hand side of the last equation is called the *bigraded decomposition* of η. Moreover, if $(\mathcal{A}, \circ)$ is a bigraded algebra, then $(\mathcal{A}^\bullet, \circ)$ is a graded algebra with the compatible grading. Observe that if $\mathcal{C}^{\bullet,\bullet}$ is a bigraded $\mathcal{R}$-module, then

$$\mathrm{End}^{p,q}_{\mathcal{R}}\mathcal{C}^{\bullet,\bullet} \subset \mathrm{End}^{p+q}_{\mathcal{R}}\mathcal{C}^{\bullet},$$

where the grading of $\mathcal{C}^\bullet$ is defined as in (1.2). Moreover, if the sum in

$$\mathcal{C} = \bigoplus_{j\in\mathbb{Z}} \mathcal{C}^j$$

is finite (i.e., only a finite number of $\mathcal{C}^j$ are non-zero), then the grading of $\mathrm{End}_{\mathcal{R}}\mathcal{C}^\bullet$ is compatible with the bigrading $\mathrm{End}_{\mathcal{R}}\mathcal{C}^{\bullet,\bullet}$, i.e.,

$$\mathrm{End}^k_{\mathcal{R}}\mathcal{C} = \bigoplus_{p+q=k} \mathrm{End}^{p,q}_{\mathcal{R}}\mathcal{C}.$$

Example 1.1.1. *Let M be a differential manifold and E a vector bundle of rank k over M. For each $n \in \mathbb{Z}$, take*

$$\mathcal{A}^n := \Gamma \bigwedge^n E,$$

and $\mathcal{A} := \Gamma \bigwedge E$. It is clear that $(\Gamma \bigwedge E, \wedge)$ is an associative graded C^∞_M-algebra, where $\wedge$ is the usual exterior product. The following are particular cases of such algebras:

- *For $E = T^*M$, denote the C^∞_M-module of* differential n-forms *by $\Omega^n_M := \Gamma \bigwedge^n T^*M$. Moreover,*

$$\Omega_M = \bigoplus_{n\in\mathbb{Z}} \Omega^n_M$$

with its exterior product, is called the ***Cartan's $\mathbb{Z}$-algebra****.*

- *Similarly, for $E = TM$, is denoted $\chi^n_M := \Gamma \bigwedge^n TM$ and its elements are called* n-vector fields*; $\Gamma \bigwedge TM$ is denoted by χ_M and is elements are called* multivector fields*.*

1.2 Main results

In this part we introduce the concept of *bigraded cochain complex*, which is a cochain complex $(\mathcal{C}, \partial)$ with a compatible bigrading for $\mathcal{C}$, in the sense of equation (1.2), and a particular bigraded decomposition of the coboundary operator ∂.

We will give the precise definition of a bigraded cochain complex, set the general problem and state the main result. We derive short exact sequences for cocycles, coboundaries, and cohomology groups of ∂, which are presented in terms of some objects intrinsically defined by the bigraded cochain complex.

Later on, in Chapter 4, we will apply these general-algebraic results to the Lichnerowicz - Poisson complex.

1.2.1 Setting of the problem

Let $\mathcal{C}$ be a graded $\mathbb{R}$-vector space,

$$\mathcal{C}^\bullet = \bigoplus_{n\in\mathbb{Z}} \mathcal{C}^n,$$

and $\partial : \mathcal{C}^\bullet \longrightarrow \mathcal{C}^\bullet$ a *coboundary operator* of degree 1, i.e., $\partial \in \mathrm{End}^1_{\mathbb{R}}\mathcal{C}^\bullet$ such that $\partial^2 = \partial \circ \partial = 0$. Such a pair $(\mathcal{C}^\bullet, \partial)$ is said to be a *cochain complex*.

Additionally, assume that $\mathcal{C}$ is a bigraded $\mathbb{R}$-vector space,

$$\mathcal{C}^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{Z}\times\mathbb{Z}} \mathcal{C}^{p,q},$$

such that this bigrading is *compatible* with the original grading in $\mathcal{C}$:

$$\mathcal{C}^n = \bigoplus_{p+q=n} \mathcal{C}^{p,q}, \qquad \forall n \in \mathbb{Z}.$$

Further, assume the following properties:

1. If $p < 0$ or $q < 0$, then $\mathcal{C}^{p,q} = \{0\}$.
2. For each $k \in \mathbb{Z}$, we have the splitting
$$\mathrm{End}^k_{\mathbb{R}}\mathcal{C}^\bullet = \bigoplus_{p+q=k} \mathrm{End}^{p,q}_{\mathbb{R}}\mathcal{C}^{\bullet,\bullet}. \tag{1.3}$$
3. The coboundary operator ∂ splits in the sum of three bigraded operators
$$\partial = \partial_{1,0} + \partial_{0,1} + \partial_{2,-1}, \tag{1.4}$$
where $\partial_{i,j} \in \mathrm{End}^{i,j}_{\mathbb{R}}\mathcal{C}^{\bullet,\bullet}\ \forall (i,j) \in \{(1,0),(0,1),(2,-1)\}$.

Recall that $L \in \mathrm{End}_{\mathbb{R}}^{p,q}\mathcal{C}^{\bullet,\bullet}$ if and only if $L(\mathcal{C}^{r,s}) \subset \mathcal{C}^{r+p,s+q}$ $\forall r, s \in \mathbb{Z}$. The right-hand side of equation (1.4) is the *bigraded decomposition* of the operator ∂.

Definition 1.2.1. *A* ***bigraded cochain complex*** $(\mathcal{C}, \partial)$ *is a cochain complex satisfying all conditions given above.*

Some immediate consequences of the first property are

$$\mathcal{C}^0 = \mathcal{C}^{0,0}, \tag{1.5}$$

$$\mathcal{C}^{-n} = 0 \quad \forall n > 0. \tag{1.6}$$

In particular, equation (1.3) is satisfied if only a finite number of $\mathcal{C}^n$'s are non-zero. It is useful to take into account the following diagrams associated to a bigraded cochain complex $(\mathcal{C}, \partial)$:

$$\begin{tikzcd}
& & \mathcal{C}^{p,q} \arrow[dll, "\partial_{2,-1}"'] \arrow[dl, "\partial_{1,0}"] \arrow[dr, "\partial_{0,1}"] & \\
\mathcal{C}^{p+2,q-1} & \mathcal{C}^{p+1,q} & & \mathcal{C}^{p,q+1}
\end{tikzcd}$$

$$\begin{tikzcd}
& & & & \mathcal{C}^{0,0} \arrow[dl] \arrow[dr] & & & & \\
& & & \mathcal{C}^{1,0} \arrow[dl] \arrow[dr] & & \mathcal{C}^{0,1} \arrow[dl] \arrow[dr] \arrow[dlll] & & & \\
& & \mathcal{C}^{2,0} \arrow[dl] \arrow[dr] & & \mathcal{C}^{1,1} \arrow[dl] \arrow[dr] \arrow[dlll] & & \mathcal{C}^{0,2} \arrow[dl] \arrow[dr] \arrow[dlll] & & \\
& \mathcal{C}^{3,0} \arrow[dl] \arrow[dr] & & \mathcal{C}^{2,1} \arrow[dl] \arrow[dr] \arrow[dlll] & & \mathcal{C}^{1,2} \arrow[dl] \arrow[dr] \arrow[dlll] & & \mathcal{C}^{0,3} \arrow[dl] \arrow[dr] \arrow[dlll] & \\
\mathcal{C}^{4,0} & & \mathcal{C}^{3,1} & & \mathcal{C}^{2,2} & & \mathcal{C}^{1,3} & & \mathcal{C}^{0,4}
\end{tikzcd}$$

Since ∂ is a coboundary operator, representation (1.4) implies that

$$\begin{aligned}
0 = \partial^2 &= (\partial_{1,0} + \partial_{0,1} + \partial_{2,-1})^2 \\
&= \partial_{0,1}^2 + \partial_{2,-1}^2 + (\partial_{1,0}\partial_{0,1} + \partial_{0,1}\partial_{1,0}) \\
&\qquad + (\partial_{1,0}\partial_{2,-1} + \partial_{2,-1}\partial_{1,0}) + (\partial_{1,0}^2 + \partial_{2,-1}\partial_{0,1} + \partial_{0,1}\partial_{2,-1}).
\end{aligned}$$

Note that, in the last sum, operators with the same bidegree are grouped in a parenthesis. Because of (1.3), the condition for ∂ to be a coboundary operator reads

$$\partial_{0,1}^2 = 0, \tag{1.7}$$

$$\partial_{2,-1}^2 = 0, \tag{1.8}$$

$$\partial_{1,0}\partial_{0,1} + \partial_{0,1}\partial_{1,0} = 0, \tag{1.9}$$

$$\partial_{1,0}\partial_{2,-1} + \partial_{2,-1}\partial_{1,0} = 0, \tag{1.10}$$

$$\partial_{1,0}^2 + \partial_{2,-1}\partial_{0,1} + \partial_{0,1}\partial_{2,-1} = 0. \tag{1.11}$$

Equations (1.7) and (1.8) imply that $\partial_{0,1}$ and $\partial_{2,-1}$ are also coboundary operators in $\mathcal{C}$. Therefore, it is convenient to present the following notation: given a cochain complex $(\mathcal{E}^\bullet, \delta)$, we define:

- the space of δ-closed elements in $\mathcal{E}^k$, also named k-cocycles,
$$\mathcal{Z}^k_\delta := \ker(\delta : \mathcal{E}^k \longrightarrow \mathcal{E}^{k+1});$$
- the space of δ-exact elements in $\mathcal{E}^k$, or k-coboundaries,
$$\mathcal{B}^k_\delta := \mathrm{Im}(\delta : \mathcal{E}^{k-1} \longrightarrow \mathcal{E}^k);$$
- the cohomology k-space,
$$\mathcal{H}^k_\delta := \frac{\mathcal{Z}^k_\delta}{\mathcal{B}^k_\delta}.$$

Moreover, if $\mathcal{E}$ has a compatible bigrading, then we will denote
$$\mathcal{Z}^{p,q}_\delta := \mathcal{Z}^{p+q}_\delta \cap \mathcal{E}^{p,q} \qquad \text{and} \qquad \mathcal{B}^{p,q}_\delta := \mathcal{B}^{p+q}_\delta \cap \mathcal{E}^{p,q}.$$

So, our point is to study the first cohomology group of $(\mathcal{C}^\bullet, \partial)$ in terms of the bigraded components of the coboundary operator ∂.

1.2.2 Splitting theorem for the first cohomology

Here, we formulate the main result related to cocycles, coboundaries and cohomology of operator ∂. First of all, taking into account equations (1.7)-(1.11), we observe that $\eta = \eta_{1,0} + \eta_{0,1} \in \mathcal{C}^1$ is a cocycle of ∂, i.e. $\partial\eta = 0$, if and only if
$$\partial_{1,0}\eta_{1,0} + \partial_{2,-1}\eta_{0,1} = 0, \qquad \partial_{0,1}\eta_{1,0} + \partial_{1,0}\eta_{0,1} = 0 \qquad \text{and} \qquad \partial_{0,1}\eta_{0,1} = 0.$$
In the same fashion, $\eta = \eta_{1,0} + \eta_{0,1}$ is a coboundary of ∂, i.e. $\eta = \partial f$ for some $f \in \mathcal{C}^0$, if and only if
$$\partial_{1,0}f = \eta_{1,0}, \qquad \text{and} \qquad \partial_{0,1}f = \eta_{0,1}.$$
Therefore, computation of ∂-cohomology is reduced to the study of above equations.

Our main result is formulated as follows:

Theorem 1.2.2. *Let $(\mathcal{C}, \partial)$ be a bigraded cochain complex. We have the following short exact sequences for cocycles, coboundaries and cohomology groups of $\mathcal{C}$*
$$0 \to \mathcal{Z}^1_{\overline{\partial}_{1,0}} \hookrightarrow \mathcal{Z}^1_\partial \to \ker\rho \to 0, \tag{1.12}$$
$$0 \to \mathcal{B}^1_{\overline{\partial}_{1,0}} \hookrightarrow \mathcal{B}^1_\partial \to \mathcal{B}^1_{\partial_{0,1}} \to 0, \tag{1.13}$$
$$0 \to \mathcal{H}^1_{\overline{\partial}_{1,0}} \hookrightarrow \mathcal{H}^1_\partial \to \frac{\ker\rho}{\mathcal{B}^1_{\partial_{0,1}}} \to 0. \tag{1.14}$$
Here, $\overline{\partial}_{1,0} : \mathcal{Z}^{p,0}_{\partial_{0,1}} \longrightarrow \mathcal{Z}^{p+1,0}_{\partial_{0,1}}$ is a coboundary operator given by the restriction of ∂ to $\mathcal{Z}^{\bullet,0}_{\partial_{0,1}}$, and $\rho : \mathcal{A} \longrightarrow \mathcal{H}^2_{\overline{\partial}_{1,0}}$ is a canonical morphism from a subspace $\mathcal{A} \subset \mathcal{Z}^{0,1}_{\partial_{0,1}}$ to the second cohomology space of $\overline{\partial}_{1,0}$.

We now present some particular cases of the previous theorem.

Corollary 1.2.3. *The following are consequences of equation* (1.14)*.*

1. *Assume that the first cohomology group of $\partial_{0,1}$ is trivial. Then,*

$$\mathcal{H}^1_\partial \simeq \mathcal{H}^1_{\overline{\partial}_{1,0}}.$$

2. *Suppose that the operator $\partial_{0,1}$ is identically zero. Then, $\partial_{1,0}$ is a coboundary and*

$$\mathcal{H}^1_\partial \simeq \mathcal{H}^1_{\partial_{1,0}} \oplus \ker(\rho),$$

where

$$\ker(\rho) = \{Y \in \mathcal{A} \mid \partial_{2,-1}Y \text{ is } \partial_{1,0} - exact\}.$$

Notice that the coboundary operator $\overline{\partial}_{1,0}$, the subspace $\mathcal{A}$ and the morphism ρ are intrinsically defined by the bigrading of the complex. Below, we will describe more precisely these objects.

1.2.3 Coboundary operator $\overline{\partial}_{1,0}$

Lemma 1.2.4. *For any $p, q \in \mathbb{Z}$,*

$$\partial_{1,0}(\mathcal{Z}^{p,q}_{\partial_{0,1}}) \subset \mathcal{Z}^{p+1,q}_{\partial_{0,1}}. \tag{1.15}$$

Moreover, for any $\eta \in \mathcal{Z}^{p,0}_{\partial_{0,1}}$, we have $\partial^2_{1,0}(\eta) = 0$. Therefore, there exists a cochain complex $(\mathcal{Z}^{\bullet,0}_{\partial_{0,1}}, \overline{\partial}_{1,0})$, where

$$\mathcal{Z}^{\bullet,0}_{\partial_{0,1}} := \bigoplus_{p\in\mathbb{Z}} \mathcal{Z}^{p,0}_{\partial_{0,1}}$$

and $\overline{\partial}_{1,0} := \partial_{1,0}|_{\mathcal{Z}^{\bullet,0}_{\partial_{0,1}}}$ is the restriction of $\partial_{1,0}$ to $\mathcal{Z}^{\bullet,0}_{\partial_{0,1}}$.

Proof. If $\eta \in \mathcal{Z}^{p,q}_{\partial_{0,1}}$, then $\partial_{0,1}\eta = 0$. Therefore, $\partial_{0,1}(\partial_{1,0}\eta) = -\partial_{1,0}(\partial_{0,1}\eta) = 0$, in virtue of equation (1.9). So, $\partial_{1,0}\eta \in \mathcal{Z}^{p+1,q}_{\partial_{0,1}}$, proving equation (1.15). For the second part, take $\eta \in \mathcal{Z}^{p,0}_{\partial_{0,1}}$. Then $\partial_{0,1}\eta = 0$, and $\partial_{2,-1}\eta = 0$, since it has bidegree $(p+2,-1)$. From (1.11), it follows that $\partial^2_{1,0}\eta = -\partial_{2,-1}\partial_{0,1}\eta - \partial_{0,1}\partial_{2,-1}\eta = 0$, as desired. ■

By a bigrading argument, it is clear that $\partial|_{\mathcal{Z}^{\bullet,0}_{\partial_{0,1}}} = \partial_{1,0}|_{\mathcal{Z}^{\bullet,0}_{\partial_{0,1}}}$. By (1.15), we conclude that for each $p \in \mathbb{Z}$, we have $\overline{\partial}_{1,0} : \mathcal{Z}^{p,0}_{\partial_{0,1}} \longrightarrow \mathcal{Z}^{p+1,0}_{\partial_{0,1}}$.

1.2.4 Subspace $\mathcal{A}$

Because of our previous development, the operator $\partial_{1,0}$ naturally restricts to the subspaces of $\partial_{0,1}$-cocycles, $\partial_{1,0} : \mathcal{Z}^{0,1}_{\partial_{0,1}} \longrightarrow \mathcal{Z}^{1,1}_{\partial_{0,1}}$. Note that, in $\mathcal{Z}^{1,1}_{\partial_{0,1}}$, there is the subspace of $\partial_{0,1}$-coboundaries, $\mathcal{B}^{1,1}_{\partial_{0,1}} \subset \mathcal{Z}^{1,1}_{\partial_{0,1}}$. We now consider the subspace $\mathcal{A} \subset \mathcal{Z}^{0,1}_{\partial_{0,1}}$ such that the following diagram commutes

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, hook] \arrow[d, "\partial_{1,0}"'] & \mathcal{Z}^{0,1}_{\partial_{0,1}} \arrow[d, "\partial_{1,0}"] \\
\mathcal{B}^{1,1}_{\partial_{0,1}} \arrow[r, hook] & \mathcal{Z}^{1,1}_{\partial_{0,1}}.
\end{tikzcd}
$$

More precisely, define the subspace $\mathcal{A} \subset \mathcal{Z}^{1,0}_{\partial_{0,1}}$ consisting of elements whose image under $\partial_{1,0}$ has trivial $\partial_{0,1}$-cohomology:

$$\mathcal{A} := \{Y \in \mathcal{Z}^{0,1}_{\partial_{0,1}} \mid \partial_{1,0}(Y) \text{ is } \partial_{0,1}\text{-exact}\}. \tag{1.16}$$

The last definition means that for each $Y \in \mathcal{A}$, there exists $\beta \in \mathcal{C}^{1,0}$ such that

$$\partial_{1,0}(Y) = -\partial_{0,1}(\beta). \tag{1.17}$$

Note that $\mathcal{B}^{0,1}_{\partial_{0,1}} \subset \mathcal{A}$. Indeed, a typical element of $\mathcal{B}^{0,1}_{\partial_{0,1}}$ has the form $\partial_{0,1}f$, for some $f \in \mathcal{C}^{0,0}$. Because of equation (1.9), we have $\partial_{1,0}\big(\partial_{0,1}f\big) = -\partial_{0,1}\partial_{1,0}f$. So, $\partial_{0,1}f \in \mathcal{A}$, with $\beta = \partial_{1,0}f$.

Lemma 1.2.5. *For arbitrary $Y \in \mathcal{A}$ and $\beta \in \mathcal{C}^{1,0}$ satisfying equation* (1.17)*, the following equalities are satisfied:*

$$\partial_{0,1}\big(\partial_{1,0}(\beta) + \partial_{2,-1}(Y)\big) = 0, \tag{1.18}$$

$$\partial_{1,0}\big(\partial_{1,0}(\beta) + \partial_{2,-1}(Y)\big) = 0. \tag{1.19}$$

Proof. The first identity follows from equations (1.9), (1.11) and (1.17):

$$
\begin{aligned}
\partial_{0,1}\big(\partial_{1,0}(\beta) + \partial_{2,-1}(Y)\big) &= \partial_{0,1}\partial_{1,0}(\beta) + \partial_{0,1}\partial_{2,-1}(Y) \\
&= -\partial_{1,0}\partial_{0,1}(\beta) - \partial^2_{1,0}(Y) - \partial_{2,-1}\partial_{0,1}(Y) \\
&= -\partial_{1,0}\partial_{0,1}(\beta) + \partial_{1,0}\partial_{0,1}(\beta) - \partial_{2,-1}(0) = 0.
\end{aligned}
$$

On the other hand, using equations (1.10), (1.17), and (1.11), we get

$$
\begin{aligned}
\partial_{1,0}\big(\partial_{1,0}(\beta) + \partial_{2,-1}(Y)\big) &= \partial^2_{1,0}(\beta) + \partial_{1,0}\partial_{2,-1}(Y) \\
&= \partial^2_{1,0}(\beta) - \partial_{2,-1}\partial_{1,0}(Y) \\
&= \partial^2_{1,0}(\beta) + \partial_{2,-1}\partial_{0,1}(\beta) = -\partial_{0,1}\partial_{2,-1}(\beta) = 0.
\end{aligned}
$$

The last equality is clear since $\partial_{2,-1}(\beta)$ has bidegree $(3,-1)$. ∎

1.2.5 Morphism ρ

Observe that equation (1.18) means that $\partial_{1,0}(\beta)+\partial_{2,-1}(Y)\in\mathcal{Z}^{2,0}_{\partial_{0,1}}$, and equation (1.19) implies $\partial_{1,0}(\beta)+\partial_{2,-1}(Y)\in\mathcal{Z}^{2}_{\overline{\partial}_{1,0}}$. Therefore, it makes sense to consider the $\overline{\partial}_{1,0}$-cohomology class of $\partial_{1,0}(\beta)+\partial_{2,-1}(Y)$.

Lemma 1.2.6. *For each $Y\in\mathcal{A}$, the $\overline{\partial}_{1,0}$-cohomology class*

$$[\partial_{1,0}(\beta)+\partial_{2,-1}(Y)]\in\mathcal{H}^2_{\overline{\partial}_{1,0}}$$

does not depend on the choice of β in equation (1.17).

Proof. Fix $Y\in\mathcal{A}$. If $\beta,\beta'\in\mathcal{C}^{1,0}$ satisfy equation (1.17), then $\partial_{1,0}(Y)=-\partial_{0,1}(\beta)$, and $\partial_{1,0}(Y)=-\partial_{0,1}(\beta')$. Note that $[\partial_{1,0}(\beta)+\partial_{2,-1}(Y)]=[\partial_{1,0}(\beta')+\partial_{2,-1}(Y)]$ if and only if $\partial_{1,0}(\beta-\beta')\in\mathcal{B}^2_{\overline{\partial}_{1,0}}$. Since $\overline{\partial}_{1,0}$ is the restriction of $\partial_{1,0}$ to $\mathcal{Z}^1_{\partial_{0,1}}$, it is sufficient to show that $\beta-\beta'\in\mathcal{Z}^1_{\partial_{0,1}}$:

$$\partial_{0,1}(\beta-\beta')=\partial_{0,1}(\beta)-\partial_{0,1}(\beta')=\partial_{2,-1}(Y)-\partial_{2,-1}(Y)=0.$$

■

Corollary 1.2.7. *There exists a canonical morphism $\rho:\mathcal{A}\longrightarrow\mathcal{H}^2_{\overline{\partial}_{1,0}}$ given by*

$$\rho(Y):=[\partial_{1,0}(\beta_Y)+\partial_{2,-1}(Y)],$$

where, for a fixed $Y\in\mathcal{A}$, $\beta_Y\in\mathcal{C}^{1,0}$ is arbitrary satisfying equation (1.17).

Note that $\mathcal{B}^1_{\partial_{0,1}}\subset\ker\rho\subset\mathcal{A}$. Indeed, a typical element in $\mathcal{B}^1_{\partial_{0,1}}$ has the form $\partial_{0,1}f$ for some $f\in\mathcal{C}^{0,0}$. Also, recall that $Y=\partial_{0,1}f$ and $\beta=\partial_{1,0}f$ satisfy equation (1.17). For this choice of β, we have,

$$\rho(\partial_{0,1}f)=[\partial_{1,0}(\beta)+\partial_{2,-1}(Y)]=[\partial^2_{1,0}(f)+\partial_{2,-1}(\partial_{0,1}f)]=[-\partial_{0,1}\partial_{2,-1}f]=[0],$$

proving that $\mathcal{B}^{0,1}_{\partial_{0,1}}\subset\ker\rho$.

1.3 Short exact sequences

Here we present a proof of Theorem 1.2.2, which is divided into few steps.

1.3.1 Cocycles

In the previous section, we have defined a canonical morphism $\rho:\mathcal{A}\longrightarrow\mathcal{H}^2_{\overline{\partial}_{1,0}}$, whose kernel contains the $\partial_{0,1}$-exacts elements. Now, it will be shown the existence of some short exact sequences which allows us to express the first cohomology space of ∂ in terms of its bigraded components.

Proposition 1.3.1. *There is a canonical inclusion* $\mathcal{Z}^1_{\overline{\partial}_{1,0}} \subset \mathcal{Z}^1_{\partial}$. *Furthermore, there exists a short exact sequence*

$$0 \to \mathcal{Z}^1_{\overline{\partial}_{1,0}} \overset{\imath}{\hookrightarrow} \mathcal{Z}^1_{\partial} \xrightarrow{p_{0,1}} \ker\rho \to 0, \tag{1.20}$$

where $p_{0,1} = \mathrm{pr}_{0,1}|_{\mathcal{Z}^1_{\partial}}$ *is the restriction of the projection* $\mathrm{pr}_{0,1} : \mathcal{C}^1 \longrightarrow \mathcal{C}^{0,1}$ *to* $\mathcal{Z}^1_{\partial}$.

Proof. If $\eta \in \mathcal{Z}^1_{\overline{\partial}_{1,0}}$, then $\eta \in \mathcal{Z}^{1,0}_{\partial_{0,1}}$ and $\overline{\partial}_{1,0}(\eta) = 0$, i.e., $\eta \in \mathcal{C}^{1,0}$, $\partial_{0,1}\eta = 0$ and $\partial_{1,0}\eta = 0$. Therefore, $\partial_{2,-1}\eta = 0$. This proves that

$$\partial\eta = \partial_{0,1}\eta + \partial_{1,0}\eta + \partial_{2,-1}\eta = 0.$$

So, $\mathcal{Z}^1_{\overline{\partial}_{1,0}} \subset \mathcal{Z}^1_{\partial}$. Hence, we have the natural inclusion $\mathcal{Z}^1_{\overline{\partial}_{1,0}} \overset{\imath}{\hookrightarrow} \mathcal{Z}^1_{\partial}$. Now, take $\eta \in \mathcal{Z}^1_{\partial}$, where $\eta = \beta + Y$, $\beta \in \mathcal{C}^{1,0}$ and $Y \in \mathcal{C}^{0,1}$. Condition $\partial\eta = 0$ splits into the following relations:

$$\partial_{1,0}\beta + \partial_{2,-1}Y = 0, \qquad \partial_{1,0}Y + \partial_{0,1}\beta = 0, \qquad \partial_{0,1}Y = 0.$$

Since $\partial_{1,0}Y = -\partial_{0,1}\beta$, we have $Y \in \mathcal{A}$. Moreover, since $\partial_{1,0}\beta + \partial_{2,-1}Y = 0$, it follows that

$$\rho(Y) = [\partial_{1,0}(\beta) + \partial_{2,-1}(Y)] = 0,$$

and $p_{0,1}(\eta) = Y \in \ker\rho$. This proves that $p_{0,1} : \mathcal{Z}^1_{\partial} \longrightarrow \ker\rho$ is well-defined. Moreover, let us show that this map is surjective. Fix $Y \in \ker\rho$. Inclusions $\ker\rho \subset \mathcal{A} \subset \mathcal{Z}^{0,1}_{\partial_{0,1}}$, imply that $Y \in \mathcal{Z}^{0,1}_{\partial_{0,1}}$ and $Y \in \mathcal{A}$. Condition $Y \in \mathcal{Z}^{0,1}_{\partial_{0,1}}$ reads $\partial_{0,1}Y = 0$; since $Y \in \mathcal{A}$, there exists $\beta \in \mathcal{C}^{1,0}$ such that $\partial_{1,0}Y + \partial_{0,1}\beta = 0$. Finally, $Y \in \ker\rho$ implies that there exists $\theta \in \mathcal{Z}^1_{\partial_{0,1}}$ such that

$$\partial_{1,0}(\beta) + \partial_{2,-1}(Y) = \partial_{1,0}\theta.$$

Now, we define $\eta = (\beta - \theta) + Y$. Then, $\mathrm{pr}_{0,1}(\eta) = Y$ and

$$\begin{aligned}\partial\eta &= (\partial_{1,0}(\beta-\theta) + \partial_{2,-1}Y) + (\partial_{0,1}(\beta-\theta) + \partial_{1,0}Y) + \partial_{0,1}Y\\ &= [(\partial_{1,0}\beta + \partial_{2,-1}Y) - \partial_{1,0}\theta] + [(\partial_{0,1}\beta + \partial_{1,0}Y) - \partial_{0,1}\theta] + 0\\ &= 0.\end{aligned}$$

So, $\eta \in \mathcal{Z}^1_{\partial}$ is such that $p_{0,1}(\eta) = Y$, which implies that $p_{0,1} : \mathcal{Z}^1_{\partial} \longrightarrow \ker\rho$ is surjective. It remains to prove that $\mathrm{Im}(\imath) = \ker(p_{0,1})$. It is clear that if $\eta \in \mathcal{Z}^1_{\overline{\partial}_{1,0}} \subset \mathcal{C}^{1,0}$, then $(p_{0,1}\circ\imath)\eta = \mathrm{pr}_{0,1}\eta = 0$, and $\mathrm{Im}(\imath) \subseteq \ker(p_{0,1})$. Now, let $\eta \in \mathcal{Z}^1_{\partial}$ be such that $p_{0,1}\eta = 0$. Then $\eta \in \mathcal{C}^{1,0}$, $\partial_{1,0}\eta = 0$, and $\partial_{0,1}\eta = 0$. This simply means that $\eta \in \mathcal{Z}^1_{\overline{\partial}_{1,0}}$. Hence $\mathrm{Im}(\imath) \supseteq \ker(p_{0,1})$, as desired. ∎

This proves (1.12) in Theorem 1.2.2.

1.3.2 Coboundaries

In a similar fashion, there exists a short exact sequence for coboundary spaces. By definition, it is easy to see that

$$\mathcal{B}^1_{\partial} = \{\partial_{1,0}f + \partial_{0,1}f \mid f \in \mathcal{C}^0\}, \qquad \mathcal{B}^1_{\partial_{0,1}} = \{\partial_{0,1}f \mid f \in \mathcal{C}^0\}, \qquad \mathrm{pr}_{0,1}(\mathcal{B}^1_{\partial}) = \mathcal{B}^1_{\partial_{0,1}}.$$

Moreover, we need the following result.

Lemma 1.3.2. *We have the following equality:*

$$\mathcal{B}^1_{\overline{\partial}_{1,0}} = \mathcal{Z}^1_{\overline{\partial}_{1,0}} \cap \mathcal{B}^1_{\partial} \tag{1.21}$$

Proof. Clearly, $\mathcal{B}^1_{\overline{\partial}_{1,0}} \subset \mathcal{Z}^1_{\overline{\partial}_{1,0}}$. Now, let us prove that $\mathcal{B}^1_{\overline{\partial}_{1,0}} \subset \mathcal{B}^1_{\partial}$. Observe that $\eta \in \mathcal{B}^1_{\overline{\partial}_{1,0}}$ is equivalent to $\eta \in \mathcal{Z}^{1,0}_{\partial_{0,1}}$ and the existence of $f \in \mathcal{Z}^{0,0}_{\partial_{0,1}}$ such that $\eta = \partial_{1,0}f$. Hence,

$$\partial f = \partial_{1,0}f + \partial_{0,1}f = \partial_{1,0}f = \eta,$$

proving that $\eta \in \mathcal{B}^1_{\partial}$, as desired. Conversely, if $\eta \in \mathcal{Z}^1_{\overline{\partial}_{1,0}} \cap \mathcal{B}^1_{\partial}$, then $\eta \in \mathcal{C}^{1,0}$, $\partial_{1,0}\eta = 0$, $\partial_{0,1}\eta = 0$, and $\partial f = \eta$ for some $f \in \mathcal{C}^0$. The last equation means $\eta = \partial_{1,0}f + \partial_{0,1}f$, but condition $\eta \in \mathcal{C}^{1,0}$ implies $\partial_{0,1}f = 0$ and $\eta = \partial_{1,0}f$. Hence, $f \in \mathcal{Z}^{0,0}_{\partial_{0,1}}$ and $\eta = \overline{\partial}_{1,0}f$, proving that $\eta \in \mathcal{B}^1_{\overline{\partial}_{1,0}}$. This completes the proof. ■

Proposition 1.3.3. *There is a short exact sequence*

$$0 \to \mathcal{B}^1_{\overline{\partial}_{1,0}} \xhookrightarrow{\imath} \mathcal{B}^1_{\partial} \xrightarrow{pr_{0,1}} \mathcal{B}^1_{\partial_{0,1}} \to 0$$

where $pr_{0,1} = \mathrm{pr}_{0,1}|_{\mathcal{B}^1_{\partial}}$ *is the restriction of the* $(0,1)$*-projection* $\mathrm{pr}_{0,1} : \mathcal{C}^1 \longrightarrow \mathcal{C}^{0,1}$.

Proof. It is clear by its definition that $pr_{0,1} : \mathcal{B}^1_{\partial} \longrightarrow \mathcal{B}^1_{\partial_{0,1}}$ is well-defined. On the other hand, the existence of the natural inclusion $\imath : \mathcal{B}^1_{\overline{\partial}_{1,0}} \hookrightarrow \mathcal{B}^1_{\partial}$ follows equation (1.21). Moreover, note that $\mathcal{B}^1_{\overline{\partial}_{1,0}} \subset \mathcal{C}^{0,1} = \ker(\mathrm{pr}_{0,1})$ so $\mathrm{Im}(\imath) \subset \ker(pr_{0,1})$. Finally, if $\eta \in \ker(pr_{0,1})$, then $\eta \in \mathcal{B}^1_{\partial}$ and $\mathrm{pr}_{0,1}\eta = 0$. Therefore, there exists $f \in \mathcal{C}^0$ such that

$$\eta = \partial_{1,0}f + \partial_{0,1}f, \qquad \text{and} \qquad \partial_{0,1}f = 0.$$

Hence, $\eta \in \mathcal{Z}^1_{\overline{\partial}_{1,0}} \cap \mathcal{B}^1_{\partial}$, which, in virtue of equation (1.21), means that $\eta \in \mathcal{B}^1_{\overline{\partial}_{1,0}}$, i.e, $\eta \in \mathrm{Im}(\imath)$, as desired. ■

This result proves (1.13) in Theorem 1.2.2.

1.3.3 First cohomology

The following algebraic facts allow us to construct a short exact sequence for the first cohomology of ∂.

Lemma 1.3.4. *Let V, W be $\mathbb{R}$-vector spaces, with $V \subset W$. Consider subspaces $V_0 \subset V$, $W_0 \subset W$. Then, $V_0 \supset V \cap W_0$, if an only if there is a natural inclusion (injective map)*

$$\imath : \frac{W_0}{V_0} \longrightarrow \frac{W}{V}$$

such that the diagram

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & W \arrow[d] \\ \frac{V}{V_0} \arrow[r, dotted, "\imath"] & \frac{W}{W_0} \end{tikzcd}$$

commutes: $\imath[w]_0 = [w]$.

Proof. Define $\imath : \frac{W_0}{V_0} \longrightarrow \frac{W}{V}$ by $\imath[w]_0 := [w]$, where $[w]_0$ denotes the class of w in $\frac{V}{V_0}$ and $[w]$ the class of w in $\frac{W}{W_0}$. Is clear that $\imath$ is well-defined. Indeed, for $w' \in [w]_0$, there exists $v_0 \in V_0$ such that $w = w' + v_0$. Because of $v_0 \in V_0 \subset V$, it follows that $[w] = [w']$. So, $\imath$ is well-defined. On the other hand, this application is $\mathbb{R}$-linear,

$$\imath([w]_0 + \alpha[u]_0) = \imath[w + \alpha u]_0 = [w + \alpha u] = [w] + \alpha[u] = \imath[w]_0 + \alpha\imath[u]_0.$$

Also, $\imath$ it is injective since

$$\begin{aligned} \ker(\imath) &= \left\{[w]_0 \in \tfrac{W_0}{V_0} \mid [w] = 0\right\} = \left\{[w]_0 \in \tfrac{W_0}{V_0} \mid w \in V\right\} \\ &\subset \left\{[w]_0 \in \tfrac{W_0}{V_0} \mid w \in V \cap W_0\right\} = \left\{[w]_0 \in \tfrac{W_0}{V_0} \mid w \in V_0\right\} = \{0\}. \end{aligned}$$

■

Lemma 1.3.5. *Let V, W be $\mathbb{R}$-vector spaces, with $V \subset W$, and $p : W \longrightarrow W_1$ a linear surjective map. Consider a subspace $V_1 \subset W_1$ such that $p(V) \subset V_1$. Under this conditions, there exists a linear surjective map $\widetilde{p} : \frac{W}{V} \longrightarrow \frac{W_1}{V_1}$ such that the following diagram commutes,*

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d] & W_1 \arrow[d] \\ \frac{W}{V} \arrow[r, dotted, "\widetilde{p}"] & \frac{W_1}{V_1}, \end{tikzcd}$$

where the down arrows are the canonical projections, that is, $[p(w)]_1 = \widetilde{p}[w]$.

Proof. Let us show that the quotient map $\widetilde{p}[w] := [p(w)]_1$ is well-defined. Take $w' \in [w]$. Then, there exists $v \in V$ such that $w = w' + v$. Because of $p(V) \subset V_1$, we have $p(v) \in V_1$; therefore, $p(w) = p(w') + p(v)$ implies that $[p(w)]_1 = [p(w')]_1$. This shows that $\widetilde{p}$ is well - defined. $\mathbb{R}$-linearity is clear,

$$\widetilde{p}([w] + \alpha[u]) = \widetilde{p}[w + \alpha u] = [p(w + \alpha u)]_1 = [p(w)]_1 + \alpha[p(u)]_1 = \widetilde{p}[w] + \alpha\widetilde{p}[u].$$

Now, take $w \in W_1$. Since p is surjective, there exists $w' \in W$ such that $p(w') = w$. Note that $\widetilde{p}[w'] = [p(w')]_1 = [w]_1$, proving that $\widetilde{p}$ is surjective. ■

Finally, the following short exact sequence is a consequence of our previous results:

Proposition 1.3.6. *There is a short exact sequence*

$$0 \to \mathcal{H}^1_{\partial_{1,0}} \xhookrightarrow{\imath} \mathcal{H}^1_{\partial} \xrightarrow{\widetilde{p}_{0,1}} \frac{\ker \rho}{\mathcal{B}^1_{\partial_{0,1}}} \to 0.$$

Proof. In the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{B}^1_{\partial_{1,0}} \arrow[r, hook, "\imath"] \arrow[d, hook] & \mathcal{B}^1_{\partial} \arrow[r, "pr_{0,1}"] \arrow[d, hook] & \mathcal{B}^1_{\partial_{0,1}} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{Z}^1_{\partial_{1,0}} \arrow[r, hook, "\imath"] \arrow[d] & \mathcal{Z}^1_{\partial} \arrow[r, "p_{0,1}"] \arrow[d] & \ker \rho \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H}^1_{\partial_{1,0}} \arrow[r, dotted, "\imath"] \arrow[d] & \mathcal{H}^1_{\partial} \arrow[r, dotted, "\widetilde{p}_{0,1}"] \arrow[d] & \frac{\ker \rho}{\mathcal{B}^1_{\partial_{0,1}}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

every column is a short exact sequence by definition, and alse the first two rows, by Propositions 1.3.3 and 1.3.1. Furthermore, this is a commutative 3×3 diagram. Indeed, it is clear that

$$\begin{tikzcd}
\mathcal{B}^1_{\partial_{1,0}} \arrow[r, hook, "\imath"] \arrow[d, hook] & \mathcal{B}^1_{\partial} \arrow[d, hook] \\
\mathcal{Z}^1_{\partial_{1,0}} \arrow[r, hook, "\imath"] & \mathcal{Z}^1_{\partial}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{B}^1_{\partial} \arrow[r, "pr_{0,1}"] \arrow[d, hook] & \mathcal{B}^1_{\partial_{0,1}} \arrow[d, hook] \\
\mathcal{Z}^1_{\partial} \arrow[r, "p_{0,1}"] & \ker \rho
\end{tikzcd}$$

commute since the down arrows are natural inclusions and the right arrows are the restrictions of the same maps: in the left-hand side diagram, the natural inclusion, and in the right-hand side diagram, the map $\mathrm{pr}_{0,1} : \mathcal{C}^1 \longrightarrow \mathcal{C}^{0,1}$. On the other hand, the diagram

$$\begin{tikzcd}
\mathcal{Z}^1_{\partial_{1,0}} \arrow[r, hook, "\imath"] \arrow[d] & \mathcal{Z}^1_{\partial} \arrow[d] \\
\mathcal{H}^1_{\partial_{1,0}} \arrow[r, dotted, "\imath"] & \mathcal{H}^1_{\partial}
\end{tikzcd}$$

commutes if we take $\imath : \mathcal{H}^1_{\partial_{1,0}} \longrightarrow \mathcal{H}^1_{\partial}$ as the quotient map given by Lemma 1.3.4 applied to $W = \mathcal{Z}^1_{\partial}$, $W_0 = \mathcal{B}^1_{\partial}$, $V = \mathcal{Z}^1_{\partial_{1,0}}$, $V_0 = \mathcal{B}^1_{\partial_{1,0}}$, since Lemma 1.3.2 implies

$V_0 = V \cap W_0$ for this case. In a similar fashion,

$$\begin{tikzcd} \mathcal{Z}^1_{\partial} \arrow[r, "p_{0,1}"] \arrow[d] & \ker\rho \arrow[d] \\ \mathcal{H}^1_{\partial} \arrow[r, dotted, "\widetilde{p}_{0,1}"] & \frac{\ker\rho}{\mathcal{B}^1_{\partial_{0,1}}} \end{tikzcd}$$

commutes by taking $\widetilde{p}_{0,1} : \mathcal{H}^1_{\partial} \longrightarrow \frac{\ker\rho}{\mathcal{B}^1_{\partial_{0,1}}}$ as the quotient map in Lemma 1.3.5 applied to $p = p_{0,1}$, $W = \mathcal{Z}^1_{\partial}$, $V = \mathcal{B}^1_{\partial}$, $W_1 = \ker\rho$, $V_1 = \mathcal{B}^1_{\partial_{0,1}}$. This can be done since $p_{0,1}(\mathcal{B}^1_{\partial}) = \mathcal{B}^1_{\partial_{0,1}}$. Therefore, our first diagram satisfies all hypothesis of Lemma 3.4 in [17, p. 366]. So, we can conclude that the last row of this diagram is a short exact sequence, as desired. ■

This completes the proof of Theorem 1.2.2.

Proof of Corollary 1.2.3. Recall that $\mathcal{B}^1_{\partial_{0,1}} \subseteq \ker\rho \subseteq \mathcal{A} \subseteq \mathcal{Z}^1_{\partial_{0,1}}$. If the first cohomology group of $\partial_{0,1}$ is trivial, then $\mathcal{B}^1_{\partial_{0,1}} = \mathcal{Z}^1_{\partial_{0,1}}$ and all the spaces in above coincide. In particular, $\frac{\ker\rho}{\mathcal{B}^1_{\partial_{0,1}}} = \{0\}$, proving the first part of the Corollary. For the second part, if $\partial_{0,1} = 0$, then $\mathcal{Z}^{1,0}_{\partial_{0,1}} = \mathcal{C}^{1,0}$ and $\overline{\partial}_{1,0}$ is precisely $\partial_{0,1}$. Moreover, morphism ρ is just given by $\rho(Y) = [\partial_{2,-1}Y]$. So, $\ker\rho = \{Y \in \mathcal{A} \mid \partial_{2,-1}Y \text{ is } \partial_{1,0} - \text{exact}\}$. ■

Chapter 2

Graded and Bigraded Operators in Manifolds

In this chapter we review the main tools which allow us to apply the results of Chapter 1 in the context of Poisson cohomology:

- Graded derivations of the algebra of sections on vector bundles [13];
- Frölicher - Nijenhuis calculus, which includes Frölicher - Nijenhuis decomposition theorem and Frölicher - Nijenhuis bracket [19, 13];
- differential operators in the Cartan's algebra [14];
- Schouten - Nijenhuis bracket, presented by using the formalism of differential operators [14, 20];
- theory of generalized connections, which gives rise to bigraded calculus in manifolds [13].

We begin with the theory of derivation in graded algebras, specially, we will be focused on the algebra of sections of vector bundles. In particular, for the tangent bundle case, we present the Frölicher - Nijenhuis decomposition theorem, which allows us to define the Frölicher - Nijenhuis bracket of vector-valued differential forms. Then, we review the theory of differential operators in the Cartan's algebra, which we apply in order to present the Schouten - Nijenhuis bracket of multivector fields. Finally, we present the concept of generalized connection in a manifold. This is a generalization of Ehresmann connections in foliated manifolds and fiber bundles. With this tool, we study the bigraded decomposition of the exterior differential, and find the Frölicher - Nijenhuis decomposition of its bigraded components.

2.1 Derivations of graded algebras

Graded and bigraded algebras. Recall that a graded $\mathcal{R}$-algebra of degree $k \in \mathbb{Z}$ is an $\mathcal{R}$-algebra $(\mathcal{A}, \circ)$, where $\mathcal{A}$ is a $\mathbb{Z}$-graded $\mathcal{R}$-module, with

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^n,$$

and $\circ : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ satisfies the following condition for any $m, n \in \mathbb{Z}$:

$$\mathcal{A}^m \circ \mathcal{A}^n \subseteq \mathcal{A}^{m+n+k}.$$

Every non-zero element of $\bigcup_{n\in\mathbb{Z}} \mathcal{A}^n$ is called *homogeneous*, and, for each homogeneous element a, we denote by $|a|$ the only $n \in \mathbb{Z}$ such that $a \in \mathcal{A}^n$.

We now present the concepts of graded exterior, Lie and Poisson algebras, for which we customize the notation of the product.

Definition 2.1.1. *A graded $\mathcal{R}$-algebra of degree zero $(\mathcal{A}, \wedge)$ is said to be a* ***graded exterior algebra*** *if the* product $\wedge$ *satisfies the following properties for any homogeneous elements $a, b, c \in \mathcal{A}$:*

- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ (associativity)*;*
- $a \wedge b = (-1)^{|a||b|} b \wedge a$ (graded symmetry).

Definition 2.1.2. *A graded $\mathcal{R}$-algebra $(\mathcal{A}, [,])$ of degree k is called a* ***Lie algebra*** *if $[,]$ is a* ***Lie bracket****, i.e., it satisfies the following properties for any homogeneous elements $a, b, c \in \mathcal{A}$:*

- $[a, b] = -(-1)^{(|a|+k)(|b|+k)}[b, a]$ (graded skew-symmetry)*;*
- $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|+k)(|b|+k)}[b, [a, c]]$ (graded Jacobi identity).

We will now define the concept of graded Poisson algebra, which is a Lie algebra compatible with an associative operation via the graded Leibniz rule.

Definition 2.1.3. *The triple $(\mathcal{A}, \circ, [,])$ is called a* ***graded Poisson algebra*** *of degree k if the following conditions are satisfied:*

- *$(\mathcal{A}, \circ)$ is an associative graded $\mathcal{R}$-algebra of degree zero,*
- *$(\mathcal{A}, [,])$ is a Lie $\mathcal{R}$-algebra of degree k, and*
- *the* graded Leibniz rule*, $[a, b \circ c] = [a, b] \circ c + (-1)^{(|a|+k)|b|} b \circ [a, c]$, hold for any homogeneous elements $a, b, c \in \mathcal{A}$.*

Example 2.1.4. *For any $\mathcal{R}$-graded algebra $\mathcal{A}$,* $(\mathrm{End}^\bullet_{\mathcal{R}}\mathcal{A}, \circ)$ *is a graded associative $\mathcal{R}$-algebra, where $\circ$ is the composition of graded endomorphisms. Moreover, the* commutator *of graded endomorphisms $[E, F] = E \circ F - (-1)^{|E||F|} F \circ E$ is a Lie bracket and* $(\mathrm{End}^\bullet_{\mathcal{R}}\mathcal{A}, \circ, [,])$ *is a graded Poisson $\mathcal{R}$-algebra of degree zero. Furthermore,* $\mathrm{End}^\bullet_{\mathcal{R}}\mathcal{A}$ *is an $\mathcal{A}$-module with the action $(a, E) \mapsto a \wedge E$ given by $(a \wedge E)(b) := a \wedge E(b)$.*

Also, recall that a bigraded $\mathcal{R}$-algebra $\mathcal{A}$ of degree $(k_1, k_2) \in \mathbb{Z}\times\mathbb{Z}$ is an $\mathcal{R}$-algebra $(\mathcal{A}, \circ)$, where $\mathcal{A}$ is a $\mathbb{Z} \times \mathbb{Z}$-graded $\mathcal{R}$-module

$$\mathcal{A} = \bigoplus_{p,q\in\mathbb{Z}} \mathcal{A}^{p,q}$$

and $\circ : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ satisfies the following condition for any $(m_1, m_2), (n_1, n_2) \in \mathbb{Z}$:

$$\mathcal{A}^{(m_1,m_2)} \circ \mathcal{A}^{(n_1,n_2)} \subseteq \mathcal{A}^{(m_1+n_1+k_1, m_2+n_2+k_2)}.$$

Every non-zero element of $\bigcup_{p,q\in\mathbb{Z}} \mathcal{A}^{p,q}$ is called homogeneous and, for each homogeneous element a, we denote by $|a|$ the only $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ such that $a \in \mathcal{A}^{p,q}$.

Now, consider two graded $\mathcal{R}$-algebras $(\mathcal{A}_1, \circ)$ of degree k, and $(\mathcal{A}_2, \circ)$ of degree l, whose respective gradings are

$$\mathcal{A}_1 = \bigoplus_{p\in\mathbb{Z}} \mathcal{A}_1^p, \qquad \mathcal{A}_2 = \bigoplus_{q\in\mathbb{Z}} \mathcal{A}_2^q.$$

The tensor product $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$ can be naturally turned into a bigraded $\mathcal{R}$-algebra as follows: The bigrading is given by

$$\mathcal{A} = \bigoplus_{p,q\in\mathbb{Z}} \mathcal{A}^{p,q}, \tag{2.1}$$

where $\mathcal{A}^{p,q} := \mathcal{A}_1^p \otimes \mathcal{A}_2^q$ for each $(p,q) \in \mathbb{Z}\times\mathbb{Z}$, and $\circ : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ is defined on decomposable elements $\alpha\otimes A, \beta\otimes B$ by

$$(\alpha\otimes A)\circ(\beta\otimes B) := (-1)^{(|\beta|+k)(|A|+l)}(\alpha\circ\beta)\otimes(A\circ B), \tag{2.2}$$

and then in the whole $\mathcal{A}\times\mathcal{A}$ by linear extension. Moreover, $(\mathcal{A}^\bullet, \circ)$ has a graded algebra structure which is compatible with the bigrading given by

$$\mathcal{A}^n := \bigoplus_{\substack{(p,q)\in\mathbb{Z}\times\mathbb{Z}\\ p+q=n}} \mathcal{A}^{p,q}.$$

Proposition 2.1.5. *Let $(\mathcal{A}_1, \wedge)$ be a graded exterior algebra.*

1. *If $(\mathcal{A}_2, \wedge)$ is a graded exterior algebra, then $(\mathcal{A}^\bullet, \wedge)$, defined by* (2.1) *and* (2.2)*, is a graded exterior algebra.*

2. *If $(\mathcal{A}_2, [,])$ is a graded Lie algebra of degree k, then $(\mathcal{A}^\bullet, [,])$, defined by* (2.1) *and* (2.2)*, is a graded Lie algebra of degree k.*

3. *If $(\mathcal{A}_1, \wedge)$ is an exterior algebra and $(\mathcal{A}_2, \wedge, [,])$ is a graded Poisson algebra of degree k, then $(\mathcal{A}^\bullet, \wedge, [,])$ is a graded Poisson algebra of degree k, where $\mathcal{A}^\bullet$ is defined by* (2.1) *and $\wedge$ and $[,]$ are defined as in* (2.2)*.*

Observe that, for each of the cases of the last Proposition, equation (2.2) reads

$$(\alpha\otimes A)\wedge(\beta\otimes B) := (-1)^{|\beta||A|}(\alpha\wedge\beta)\otimes(A\wedge B),$$
$$[\alpha\otimes A, \beta\otimes B] := (-1)^{|\beta|(|A|+k)}(\alpha\wedge\beta)\otimes[A,B].$$

Graded derivations on the algebra of sections on vector bundles. We first present the concept of graded derivation on graded algebras. Later we develop this notion on sections of tensor product of vector bundles.

Definition 2.1.6. *Let $(\mathcal{A}, \circ)$ be a graded $\mathcal{R}$-algebra of degree k. A* ***graded derivation*** *of degree p on $(\mathcal{A}, \circ)$ is an endomorphism $D \in \mathrm{End}^p_{\mathcal{R}}\mathcal{A}$ such that, for any homogeneous elements $a, b \in \mathcal{A}$,*

$$D(a \circ b) = D(a) \circ b + (-1)^{(|a|+k)p} a \circ D(b).$$

The set of all graded derivations of degree p on $(\mathcal{A}, \circ)$ is denoted by $\mathrm{Der}^p_{\mathcal{R}}(\mathcal{A}, \circ)$, or simply $\mathrm{Der}^p_{\mathcal{R}}\mathcal{A}$. If we define

$$\mathrm{Der}^\bullet_{\mathcal{R}}\mathcal{A} := \bigoplus_{p \in \mathbb{Z}} \mathrm{Der}^p_{\mathcal{R}}\mathcal{A},$$

then it can be shown that $\mathrm{Der}^\bullet_{\mathcal{R}}\mathcal{A}$ is a Lie $\mathcal{R}$-subalgebra of $(\mathrm{End}^\bullet_{\mathcal{R}}\mathcal{A}, [,])$.

Example 2.1.7. *If $(\mathcal{A}, [,])$ is a Lie $\mathcal{R}$-algebra of degree k and $a_0 \in \mathcal{A}$ is an homogeneous element, then its* adjoint operator *$\mathrm{ad}_{a_0} : \mathcal{A} \longrightarrow \mathcal{A}$, defined by $\mathrm{ad}_{a_0}(a) := [a_0, a]$, is a graded derivation, due to the Jacobi identity: $\mathrm{ad}_{a_0} \in \mathrm{Der}^{|a_0|+k}_{\mathcal{R}}(\mathcal{A}, [,])$. Moreover, if $(\mathcal{A}, \circ, [,])$ is a graded Poisson $\mathcal{R}$-algebra of degree k, then $\mathrm{ad}_{a_0} \in \mathrm{Der}^{|a_0|+k}_{\mathcal{R}}(\mathcal{A}, \circ)$, because of the Leibniz rule.*

Example 2.1.8. *If $(\mathcal{A}, \wedge)$ is a graded exterior algebra, then $\mathrm{Der}^\bullet_{\mathcal{R}}(\mathcal{A}, \wedge)$ is a left graded $\mathcal{A}$-module with the action $(a, D) \mapsto a \wedge D$, where $(a \wedge D)(b) := a \wedge D(b)$.*

Now, consider a vector bundle (E, π, M). Recall that

$$\Gamma \bigwedge E = \bigoplus_{n \in \mathbb{Z}} \Gamma \bigwedge^n E$$

is a graded exterior C^∞_M-module, and, in particular, is an $\mathbb{R}$-vector space. We will study the space of its graded $\mathbb{R}$-derivations, $\mathrm{Der}^\bullet_{\mathbb{R}}\Gamma \bigwedge E$. The first important property of these derivations is locality. Roughly speaking, local operators are natural with respect to restrictions to open subsets.

Definition 2.1.9. *A graded endomorphism $D \in \mathrm{End}_{\mathbb{R}}\Gamma \bigwedge E$ is called* ***local*** *if, for each $A \in \Gamma \bigwedge E$ and $U \subset M$ such that $A|_U = 0$, then $DA|_U = 0$. More generally, an $\mathbb{R}$-bilinear map $[,] : \Gamma \bigwedge E \times \Gamma \bigwedge E \longrightarrow \Gamma \bigwedge E$ is called* ***local*** *if, for each $A \in \Gamma \bigwedge E$ and $U \subset M$ such that $A|_U = 0$, then $[A, B]|_U = 0$ for all $B \in \Gamma \bigwedge E$.*

Proposition 2.1.10. *If $D \in \mathrm{Der}^\bullet_{\mathbb{R}}\Gamma \bigwedge E$ is a graded derivation, then D is local.*

This property and the fact that $\Gamma \bigwedge E$ is a locally-free C^∞_M-module imply that graded derivations are determined by its action in C^∞_M and ΓE. Also, $\mathrm{Der}^\bullet_{\mathbb{R}}\Gamma \bigwedge E$ is a locally free graded left $\Gamma \bigwedge E$-module.

Algebraic Derivations.

Definition 2.1.11. *A graded derivation $D \in \mathrm{Der}^\bullet_{\mathbb{R}}\Gamma \bigwedge E$ is called* ***algebraic*** *if $D(f) = 0$ for any $f \in C^\infty_M$.*

Consider the C^∞_M-module

$$\Gamma\left(\bigwedge^{\bullet} E\otimes E^*\right) := \bigoplus_{k\in\mathbb{Z}}\Gamma\left(\bigwedge^{k} E\otimes E^*\right).$$

It is clear that, for each $k\in\mathbb{Z}$, $\Gamma\big(\bigwedge^k E\otimes E^*\big)$ is isomorphic to the C^∞_M-module consisting of all C^∞_M-linear alternating applications

$$K:\Gamma E^*\times\cdots\times\Gamma E^*\longrightarrow \Gamma E^*,$$

where ΓE^* appears k times. In particular, $\Gamma\big(\bigwedge^0 E\otimes E^*\big)\simeq\Gamma E^*$ and $\Gamma\big(\bigwedge^1 E\otimes E^*\big)\simeq \mathrm{End}(\Gamma E^*)$. Also, $\Gamma\big(\bigwedge^\bullet E\otimes E^*\big)$ has a graded left $\Gamma\bigwedge E$-module structure given by $(A,K)\mapsto A\wedge K$, where $A\wedge K\in\Gamma\big(\bigwedge^{p+k} E\otimes E^*\big)$, is defined by

$$(A\wedge K)(\alpha_1,\dots,\alpha_{p+k}) := \sum_{\sigma\in S_{(p,k)}}(-1)^\sigma A(\alpha^{\sigma(1)},\dots,\alpha^{\sigma(p)})K(\alpha^{\sigma(p+1)},\dots,\alpha^{\sigma(p+k)})$$

for any $A\in\Gamma\bigwedge^p E$ and $K\in\Gamma\big(\bigwedge^k E\otimes E^*\big)$.

Remark 2.1.12. *The notation $S_{(p,k)}$ indicates the set of* shuffle permutations

$$S_{(p,k)}=\{\sigma\in S_{p+k}\mid \sigma(1)<\ldots<\sigma(p) \text{ and } \sigma(p+1)<\ldots<\sigma(p+k)\}.$$

As a first example of algebraic derivation in $\Gamma\bigwedge E$, we present the *insertion of sections to the dual bundle*: For each $\alpha\in\Gamma E^*$, define $\mathrm{i}_\alpha\in \mathrm{Der}^{-1}_{\mathbb{R}}\Gamma\bigwedge E$, as $\mathrm{i}_\alpha f:=0$ if $f\in C^\infty_E$, and

$$\mathrm{i}_\alpha A(\alpha^1,\dots,\alpha^k) := A(\alpha,\alpha^1,\dots,\alpha^k), \tag{2.3}$$

if $A\in\Gamma\bigwedge^{k+1}E$, $k\geq 0$. More generally, the insertion of $K\in\Gamma\big(\bigwedge^{k+1} E\otimes E^*\big)$, with $k\geq 0$, defines a graded derivation of degree k. Indeed, define $\mathrm{i}_K\in\mathrm{Der}^k_{\mathbb{R}}\Gamma\bigwedge E$ as follows: For $f\in C^\infty_M$, put $\mathrm{i}_K f:=0$ and, for $A\in\Gamma\bigwedge^a E$, define

$$(\mathrm{i}_K A)(\alpha^1,\dots,\alpha^{k+a}) := \sum_{\sigma\in S_{(k+1,a-1)}}(-1)^\sigma A\big(K(\alpha^{\sigma(1)},\dots,\alpha^{\sigma(k+1)}),\alpha^{\sigma(k+2)},\dots,\alpha^{\sigma(k+a)}\big).$$

It is well-known that i_K defines an algebraic derivation. Furthermore, any algebraic derivation is the insertion of an element of $\bigwedge E\otimes E^*$.

Proposition 2.1.13. *If $D\in\mathrm{Der}^k_{\mathbb{R}}\Gamma\bigwedge E$ is algebraic, then there exists a unique $K\in\Gamma\big(\bigwedge^{k+1}E\otimes E^*\big)$ such that $D=\mathrm{i}_K$.*

A proof of this can be found in [13].

2.2 Frölicher - Nijenhuis calculus

In this section, some of the previous notions will be considered in the case of the algebra of graded derivations of the Cartan's algebra $\Omega_M := \Gamma \bigwedge T^*M$. In other words, we now focus on the case $E = T^*M$. We first present the *exterior differential*, which is an intrinsic graded derivation in Ω_M. Then, we state the *Frölicher - Nijenhuis decomposition theorem*, which says that the algebra of graded derivations splits into the direct sum of two subalgebras: the algebraic derivations and the derivations which commute with the exterior differential.

Derivations in C^∞_M and the exterior differential. Here we give a brief review of vector fields and derivations of C^∞_M.

Recall that C^∞_M has a structure of $\mathbb{R}$-algebra with the usual product of real-valued functions. In this sense, a derivation $D \in \mathrm{Der}_{\mathbb{R}} C^\infty_M$ is an $\mathbb{R}$-linear endomorphism such that

$$D(f \cdot g) = D(f) \cdot g + f \cdot D(g).$$

A way to define derivations of this algebra is using vector fields. Denote by $\mathfrak{X}_M := \Gamma TM$ the $\mathbb{R}$-vector space of vector fields. If $X \in \mathfrak{X}_M$ is a vector field and $p \in M$, then $X_p \in T_pM$ is a tangent vector field, which is a point-wise derivation of C^∞_M:

$$X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g).$$

If we define $X(f) : M \longrightarrow \mathbb{R}$ by $X(f)(p) := X_p(f)$, then $X(f) \in C^\infty_M$, and the map $X : C^\infty_M \longrightarrow C^\infty_M$, $f \mapsto X(f)$ is a derivation of C^∞_M. Conversely, for any $D \in \mathrm{Der}_{\mathbb{R}} C^\infty_M$, there exists a unique vector field X such that $D = X$.

Proposition 2.2.1. *We have a C^∞_M-module isomorphism: $\mathfrak{X}_M \simeq \mathrm{Der}_{\mathbb{R}} C^\infty_M$.*

Since $\mathrm{Der}_{\mathbb{R}} C^\infty_M$ has a Lie $\mathbb{R}$-algebra structure with the commutator $[D, E] = D \circ E - E \circ D$, we can define a **Lie bracket** of vector fields via the above isomorphism by $[X, Y](f) := X(Y(f)) - Y(X(f))$. The Lie bracket of vector fields allows us to define the exterior differential in the whole graded algebra Ω_M.

Definition 2.2.2. *The* ***exterior differential*** $\mathrm{d} : \Omega_M \longrightarrow \Omega_M$ *is the unique graded derivation* $\mathrm{d} \in \mathrm{Der}^1_{\mathbb{R}} \Omega_M$ *satisfying* $\mathrm{d}f(X) := X(f)$ *if* $f \in C^\infty_M, X \in \mathfrak{X}_M$ *and*

$$\mathrm{d}\alpha(X, Y) := X(\alpha(Y)) - Y(\alpha(X)) - \alpha[X, Y] \qquad \forall \alpha \in \Omega^1_M, X, Y \in \mathfrak{X}_M.$$

It follows from its definition that, for any $\alpha \in \Omega^k_M$, we have

$$\begin{aligned} \mathrm{d}\alpha(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i \mathcal{L}_{X_i}(\alpha(X_0, \dots, \widehat{X_i}, \dots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k). \end{aligned}$$

Moreover, because of the Jacobi identity of the Lie bracket of vector fields, the exterior differential is a coboundary operator: $\mathrm{d}^2 = 0$. Thus, the pair (Ω_M, d) is a cochain complex called **de Rham complex**. Also, these properties are also consequence forme the fact that the exterior differential is the differential of a *Lie algebroid* in the tangent bundle.

The Lie derivative and Frölicher - Nijenhuis decomposition theorem. In the previous section we review that every vector - valued differential form induces an algebraic derivation and conversely. Indeed, if $K \in \Omega^k(M;TM)$, then $\mathrm{i}_K \in \mathrm{Der}^{k-1}_{\mathbb{R}}\Omega_M$. Now, we present the notion of the *Lie derivative* along vector - valued differential forms. This extends the well - known Cartan's formula of vector fields.

Definition 2.2.3. *For each $K \in \Omega^k(M;TM)$, the **Lie derivative** along K, $\mathcal{L}_K$, is defined by $\mathcal{L}_K := [\mathrm{i}_K, \mathrm{d}] \in \mathrm{Der}^k_{\mathbb{R}}\Omega_M$, where d is the exterior differential.*

It follows from the definition of the Lie derivative and the Jacobi identity of the graded commutator that the exterior differential commutes with any $\mathcal{L}_K$, $K \in \Omega^k(M;TM)$. Note that for $k = 0$, $\Omega^0(M;TM) = \mathfrak{X}_M$. In this particular case, the identity $\mathcal{L}_X = [\mathrm{i}_X, \mathrm{d}] = \mathrm{i}_K\mathrm{d} + \mathrm{d}\mathrm{i}_K$ is called *the Cartan's formula*. The following are the Lie derivatives of a q-form β along a vector field X and of a 1-form ω along a vector-valued 1-form γ:

$$\mathcal{L}_X\beta(X_1, \ldots, X_q) = \mathcal{L}_X(\beta(X_1, \ldots, X_q)) - \sum_{i=1}^{q} \beta(X_1, \ldots, [X, X_i], \ldots, X_q), \tag{2.4}$$

$$\mathcal{L}_\gamma\omega(X,Y) = \gamma X(\omega(Y)) - \omega[\gamma X, Y] - \gamma Y(\omega(X)) - \omega[X, \gamma Y] + \omega(\gamma[X,Y]).$$

We now review the Frölicher - Nijenhuis decomposition theorem. This result allows us to parameterize the algebra of graded derivations of Ω_M via vector-valued forms on M. The proof of this theorem can be found, for example, in [19].

Theorem 2.2.4 (Frölicher - Nijenhuis Decomposition)**.** *For each $D \in \mathrm{Der}^k_{\mathbb{R}}\Omega_M$, there exists unique $K \in \Omega^k(M;TM)$ and $L \in \Omega^{k+1}(M;TM)$ such that $D = \mathcal{L}_K + \mathrm{i}_L$. This representation for D is called its **Frölicher - Nijenhuis decomposition**.*

From the Frölicher - Nijenhuis decomposition Theorem, one can decide for any graded derivation $D = \mathrm{i}_L + \mathcal{L}_K$ that

- D is algebraic if and only if $K = 0$;
- D commutes with d if and only if $L = 0$.

Frölicher - Nijenhuis bracket. If $K \in \Omega^k(M;TM)$ and $L \in \Omega^l(M;TM)$ are vector - valued forms, then $\mathcal{L}_K \in \mathrm{Der}^k_{\mathbb{R}}\Omega_M$ and $\mathcal{L}_L \in \mathrm{Der}^l_{\mathbb{R}}\Omega_M$ are graded derivations which commute with the exterior differential. It follows from the Jacobi identity that $[\mathcal{L}_K, \mathcal{L}_L]$ is also a graded derivation commuting with the exterior differential. Because of the Frölicher - Nijenhuis decomposition theorem, there exists a unique $[K, L] \in \Omega^{k+l}(M;TM)$ such that $[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_{[K,L]}$.

Definition 2.2.5. *For each $K, L \in \Omega^\bullet(M;TM)$ the unique vector valued form $[K, L]_{FN} \in \Omega^\bullet(M;TM)$ that satisfies the identity*

$$[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_{[K,L]_{FN}}$$

*is called the **Frölicher - Nijenhuis bracket** of K and L.*

As an immediate consequence of definition, the pair $(\Omega^\bullet(M;TM), [,]_{FN})$ is a graded Lie $\mathbb{R}$-algebra. Moreover, if $K, L \in \Omega^0(M;TM) \simeq \mathfrak{X}_M$, then the Frölicher - Nijenhuis bracket $[K, L]_{FN}$ coincides with the Lie bracket $[K, L]$ for vector fields. In particular, for $X, Y \in \mathfrak{X}_M$ and $K, L \in \Omega^1(M;TM)$, we have

$$\begin{aligned}
[K, X]_{FN}(Y) =& [KY, X] + K[X, Y],\\
[K, L]_{FN}(X, Y) =& [KX, LY] - [KY, LX] - L[KX, Y] + L[KY, X]\\
&- K[LX, Y] + K[LY, X] + (LK + KL)[X, Y]. \qquad (2.5)
\end{aligned}$$

Below, we find the Frölicher - Nijenhuis decomposition of the exterior differential.

Example 2.2.6. *Recall that the exterior differential* d *is a graded derivation that commutes with itself (*$[\mathrm{d}, \mathrm{d}] = 0$*). Therefore, there exists $K \in \Omega^1(M;TM)$ such that $\mathcal{L}_K = \mathrm{d}$. Note that the identity* $\mathrm{Id}_{TM} : TM \longrightarrow TM$ *induces a vector - valued 1-form* $\mathrm{Id}_{TM} \in \Omega^1(M;TM)$ *by the identity in $\mathfrak{X}_M$. If $\omega \in \Omega^k(M)$, then*

$$\mathrm{i}_{\mathrm{Id}_{TM}}\omega(X_1, \ldots, X_k) = \sum_{i=1}^{k}(-1)^{i-1}\omega(\mathrm{Id}_{TM}(X_i), X_1, \ldots, X_k) = k\omega(X_1, \ldots, X_k).$$

Therefore, $\mathrm{i}_{\mathrm{Id}_{TM}}\omega = k\omega$*. Furthermore,* $\mathcal{L}_{\mathrm{Id}_{TM}}\omega = [\mathrm{i}_{\mathrm{Id}_{TM}}, \mathrm{d}]\omega = \mathrm{i}_{\mathrm{Id}_{TM}}\mathrm{d}\omega - \mathrm{d}\mathrm{i}_{\mathrm{Id}_{TM}}\mathrm{d}\omega = (k+1)\mathrm{d}\omega - k\mathrm{d}\omega = \mathrm{d}\omega$*, which proves that* $\mathrm{d} = \mathcal{L}_{\mathrm{Id}_{TM}}$*. Since the exterior differential* $\mathrm{d} = \mathcal{L}_{\mathrm{Id}_{TM}}$ *commutes with any graded derivation of the form $\mathcal{L}_K$, we also have* $[\mathrm{Id}_{TM}, K]_{FN} = 0$ *for all $K \in \Omega^\bullet(M, TM)$.*

The Lie algebroid differential. The notion of the exterior differential in the Cartan's algebra can be generalized to vector bundles via Lie algebroid structures. Indeed, some of the most important examples of graded derivations in sections of exterior algebras of vector bundles are coboundary operators. Those are induced by a Lie algebroid in the dual bundle, and the corresponding coboundary operator is called *Lie algebroid differential*.

Definition 2.2.7. *A **Lie algebroid** is a triple $(E, \rho, [\![,]\!])$, where $E \xrightarrow{\pi} M$ is a vector bundle, $\rho : E \longrightarrow TM$ is a vector bundle morphism, called* anchor map*, and $(\Gamma E, [\![,]\!])$ is an $\mathbb{R}$-Lie algebra such that it is satisfied the* Leibniz rule

$$[\![A, fB]\!] = f[\![A, B]\!] + \mathcal{L}_{\rho(A)}f \cdot B, \qquad \forall A, B \in \Gamma E,\ f \in C^\infty_M.$$

It can be shown that the anchor map $\rho : (\Gamma E, [\![,]\!]) \longrightarrow (\mathfrak{X}_M, [,])$ is a Lie algebra morphism, where $[,]$ is the Lie bracket of vector fields: $\rho[\![A, B]\!] = [\rho(A), \rho(B)]$ $\forall A, B \in \Gamma E$.

For each $x \in M$, $\rho_x(E_x) \subset T_xM$ is a subspace of tangent vectors at x. The image of the anchor map $\rho(E) \subset TM$ is a (singular) distribution, which is integrable in the sense of Stefan-Sussmann (see [7]). Furthermore, a Lie algebroid structure on E induces a graded derivation on the algebra of sections to $\bigwedge E^*$.

Definition 2.2.8. *Let $(E, \rho, [\![,]\!])$ be a Lie algebroid over M. The* ***Lie algebroid differential*** $\mathrm{d}_E : \Gamma \bigwedge E^* \longrightarrow \Gamma \bigwedge E^*$ *is defined by*

$$\begin{aligned}\mathrm{d}_E\alpha(A_0, \dots, A_k) = &\sum_{i=0}^{k}(-1)^i \mathcal{L}_{\rho(A_i)}(\alpha(A_0, \dots, \widehat{A_i}, \dots, A_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([\![A_i, A_j]\!], A_0, \dots, \widehat{A_i}, \dots, \widehat{A_j}, \dots, A_k).\end{aligned}$$

As consequence of its definition, $\mathrm{d}_E \in \mathrm{Der}^1_{\mathbb{R}}\Gamma \bigwedge E^*$ and $\mathrm{d}_E^2 = 0$. Alternatively, d_E is the unique graded derivation in $\Gamma \bigwedge E^*$ satisfying

$$\mathrm{d}_E f(A) = \mathcal{L}_{\rho(A)}f, \qquad \mathrm{d}_E\alpha(A, B) = \mathcal{L}_{\rho(A)}(\alpha(B)) - \mathcal{L}_{\rho(B)}(\beta(A)) - \alpha([\![A, B]\!]).$$

It is clear that the exterior differential is the differential of the Lie algebroid $(TM, \mathrm{Id}_{TM}, [,])$, where $\mathrm{Id}_{TM} : TM \longrightarrow TM$ is the identity and $[,] : \mathfrak{X}_M \times \mathfrak{X}_M \longrightarrow \mathfrak{X}_M$ is the Lie bracket of vector fields. Moreover, for any Lie algebroid $(E, \rho, [\![,]\!])$ with differential d_E, the map $\rho^*\alpha(A_1, \dots, A_k) := \alpha(\rho(A_1), \dots, \rho(A_k))$ is a cochain complex morphism $\rho^* : (\Omega_M, \mathrm{d}) \longrightarrow (\Gamma \bigwedge E^*, \mathrm{d}_E)$.

In further sections and chapters, we present examples of Lie algebroid differential. In foliated manifolds, the *leafwise de Rham complex* is induced by the differential of a Lie algebroid defined by the foliation, and the *Lichnerowicz - Poisson operator* is the differential of the Lie algebroid induced by the Poisson structure.

2.3 Differential operators on Ω_M

In the previous section we studied the algebra $\mathrm{Der}^\bullet_{\mathbb{R}}\Omega_M$ of graded derivations on the Cartan's algebra Ω_M. It is clear that $\mathrm{Der}^\bullet_{\mathbb{R}}\Omega_M$ is both a graded Poisson $\mathbb{R}$-subalgebra and an Ω_M-submodule of $\mathrm{End}^\bullet_{\mathbb{R}}\Omega_M$.

In this section, we present the *differential operators* in Ω_M. This subalgebra of endomorphisms is more general than graded derivations. Our goal is to define the Schouten - Nijenhuis bracket of multivector fields via differential operators [14].

Example 2.3.1. *Any $\omega \in \Omega^k_M$ induces a graded endomorphism $\mu_\omega \in \mathrm{End}^k_{\mathbb{R}}\Omega_M$ by $\mu_\omega(\alpha) := \omega \wedge \alpha$. The map $\mu : (\Omega_M, \wedge) \longrightarrow (\mathrm{End}^\bullet_{\mathbb{R}}\Omega_M, \circ)$ given by $\omega \mapsto \mu_\omega$ is an injective graded algebra morphism. So, we can think of Ω_M as a graded $\mathbb{R}$-subalgebra of $(\mathrm{End}^\bullet_{\mathbb{R}}\Omega_M, \circ)$.*

Definition 2.3.2. *If $D \in \mathrm{End}^\bullet_{\mathbb{R}}\Omega_M$ and $r \in \mathbb{N} \cup \{0\}$, D is said to be a graded* ***differential operator*** *of order equal or less than r if, for any $\alpha_0, \dots, \alpha_r \in \Omega_M$,*

$$[\dots[[D, \mu_{\alpha_0}], \mu_{\alpha_1}], \dots, \mu_{\alpha_r}] = 0.$$

The space of graded differential operators of order equal or less than r is denoted by $\mathcal{D}_r(M)$ for $r \geq 0$. Also, define $\mathcal{D}_r(M) = \{0\}$ if $r \leq -1$. Alternatively, $\mathcal{D}_r(M)$ can be recursively defined by $\mathcal{D}_r(M) = \{0\}$ for $r \leq -1$ and, for $r \geq 0$, by

$$\mathcal{D}_r(M) = \{D \in \operatorname{End}^\bullet_{\mathbb{R}}\Omega_M \mid [D, \mu_\alpha] \in \mathcal{D}_{r-1}(M) \quad \forall \alpha \in \Omega_M\}.$$

Clearly, $\{0\} = \mathcal{D}_{-1}(M) \subset \mathcal{D}_0(M) \subset \cdots \subset \mathcal{D}_r(M) \subset \mathcal{D}_{r+1}(M)$. For any $r, k \in \mathbb{Z}$ denote by $\mathcal{D}^k_r(M) := \mathcal{D}_r(M) \cap \operatorname{End}^k_{\mathbb{R}}\Omega_M$ the space of *differential operators of degree k and order equal or less than r*. Finally, we define the space of graded differential operators of the Cartan's algebra Ω_M by

$$\mathcal{D}(M) := \bigoplus_{r\in\mathbb{Z}} \mathcal{D}_r(M).$$

Because of the graded symmetry of the exterior product in Ω_M, it follows that $[\mu_{\alpha_0}, \mu_\alpha] = 0$ for any $\alpha_0, \alpha \in \Omega_M$, implying that $\mu_{\alpha_0} \in \mathcal{D}_0(M)$. Conversely, if $D \in \mathcal{D}_0(M)$, then $0 = [D, \mu_\alpha](\mathbb{1}) = D(\alpha) - D(\mathbb{1}) \wedge \alpha = D(\alpha) - \mu_{D(\mathbb{1})}(\alpha)$ $\forall \alpha \in \Omega_M$, where $\mathbb{1} \in C^\infty_M$ denotes the constant function. This proves that, for each $D \in \mathcal{D}_0(M)$, $D = \mu_{D(\mathbb{1})}$. Therefore, $\mu : \Omega_M \longrightarrow \mathcal{D}_0(M)$ is an isomorphism: $\mathcal{D}_0(M) \simeq \Omega_M$.

Every graded derivation in Ω_M is a differential operator of order equal or less than 1; conversely, if $D \in \mathcal{D}_1(M)$, then $D - D(\mathbb{1})$ is a graded derivation. Hence,

$$\mathcal{D}^\bullet_1(M) = \operatorname{Der}^\bullet_{\mathbb{R}}\Omega_M \oplus \Omega_M.$$

It can be shown that every differential operator in Ω_M is of local type. Moreover, for each nonnegative integer r, $\mathcal{D}_r(M)$ is a locally free Ω_M-module. The following result gives an explicit representation of a differential operator in a local chart.

Theorem 2.3.3. *Let M be a differential manifold of dimension m and (U, φ) a local chart with coordinate functions x^i, $1 \leq i \leq m$. Then, $\mathcal{D}_r(U)$ is a free Ω_U-module, and is generated by*

$$\left\{\mathcal{L}^{\alpha_1}_{\partial_1} \circ \ldots \circ \mathcal{L}^{\alpha_m}_{\partial_m} \circ \mathrm{i}_{\partial_{\beta_{d(\beta)}}} \circ \ldots \circ \mathrm{i}_{\partial_{\beta_1}}\right\}_{|\alpha|+d(\beta)\leq r},$$

where α, β are multi-indexes such that $1 \leq \beta_1 < \ldots < \beta_{d(\beta)} \leq m$. Explicitly, if $D \in \mathcal{D}_r(U)$ and denoting $\overline{\beta} = (\beta_{d(\beta)}, \ldots, \beta_1)$ for $\beta = (\beta_1, \ldots, \beta_{d(\beta)})$, then

$$D = \sum_{|\alpha|+d(\beta)=0}^{r} [[D, x^\alpha], (\mathrm{d}x)_{\overline{\beta}}](\mathbb{1}) \mathcal{L}^{\alpha_1}_{\partial_1} \circ \ldots \circ \mathcal{L}^{\alpha_m}_{\partial_m} \circ \mathrm{i}_{\partial_{\beta_{d(\beta)}}} \circ \ldots \circ \mathrm{i}_{\partial_{\beta_1}}.$$

As consequence of the previous theorem, we have:

Proposition 2.3.4. *Every differential operator $D \in \mathcal{D}_r(M)$ is determined by is action on forms of degree equal or less than r.*

Proof. Consider each coefficient in the local representation of D, $[[D, x^\alpha], (\mathrm{d}x)_{\overline{\beta}}](\mathbb{1})$. Since $d(\beta) \leq r$, if we expand each coefficient, then the only terms appearing are the action of D on local differential forms of degree equal or less than r. ■

Corollary 2.3.5. *If $D \in \mathcal{D}_r^{-k}$, with $k > r$, then $D = 0$.*

Finally, note that $\mathcal{D}(M)$ is a graded Poisson $\mathbb{R}$-subalgebra of $(\mathrm{End}^\bullet_{\mathbb{R}}\Omega_M, \circ, [,])$ with the composition and the graded commutator.

Proposition 2.3.6. *For all $r, s \geq 0$,*

1. $\mathcal{D}_r(M) \circ \mathcal{D}_s(M) \subset \mathcal{D}_{r+s}(M)$,
2. $[\mathcal{D}_r(M), \mathcal{D}_s(M)] \subset \mathcal{D}_{r+s-1}(M)$.

The proof can be done by induction on the sum of the orders r and s, and using the recursive definition of the space of differential operators $\mathcal{D}_r(M)$.

The algebra of multivector fields. The Frölicher - Nijenhuis decomposition theorem allows us to parameterize the graded derivations in Ω_M by vector valued forms (Theorem 2.2.4). We now consider a class of differential operators, which are those of order k and degree $-k$, for each $k \in \mathbb{Z}$. We will show that such operators can be parameterized using the exterior algebra of multivector fields.

The graded exterior C^∞_M-algebra of multivector fields $(\chi_M, \wedge)$ is defined by $\chi_M := \Gamma \bigwedge TM$, where $\wedge$ is the exterior product. It is clear that $\chi^k_M := \Gamma \bigwedge^k TM$ is isomorphic to the C^∞_M-module of all k-linear alternating applications

$$A : \Omega^1_M \times \cdots \times \Omega^1_M \longrightarrow C^\infty_M.$$

In particular, $\chi^0_M \simeq C^\infty_M$ and $\chi^1_M \simeq \mathfrak{X}_M$.

On the other hand, fix a local chart (U, φ). For $\alpha \in \Omega^k_M$ and $A \in \chi^k_M$ consider their local expressions $\alpha|_U = \frac{1}{k!}\alpha_{i_1 \ldots i_k}\mathrm{d}\varphi^{i_1} \wedge \ldots \wedge \mathrm{d}\varphi^{i_k}$ and $A|_U = \frac{1}{k!}A^{j_1 \cdots j_k}\frac{\partial}{\partial \varphi^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial \varphi^{j_k}}$. Now, define locally $\langle \alpha, A\rangle := \frac{1}{k!}\alpha_{i_1 \ldots i_k}A^{i_1 \cdots i_k}$. It is not difficult to check that $\langle , \rangle : \chi^k_M \times \Omega^k_M \longrightarrow C^\infty_M$ is a well - defined C^∞_M-linear operation, called **pairing**. On decomposable elements,

$$\langle \alpha^1 \wedge \ldots \wedge \alpha^k, X_1 \wedge \ldots \wedge X_k\rangle = \det[\alpha^i(X_j)].$$

Moreover, the pairing induces a canonical isomorphism from χ^k_M to $(\Omega^k_M)^*$ given by $A \mapsto \langle \cdot, A\rangle$. Hence, $\chi^k_M \simeq (\Omega^k_M)^*$.

Parametrization of differential operators of order k and degree $-k$. There exists an insertion operator of multivector fields in Ω_M. For a k-vector field $A \in \chi^k_M$, a n-form $\alpha \in \Omega^n_M$ and $p \in M$ we define

$$(\mathrm{i}_A\alpha)(p) := A^{j_1 \cdots j_k}(p)\left(\mathrm{i}_{\frac{\partial}{\partial x^{j_1}}} \circ \ldots \circ \mathrm{i}_{\frac{\partial}{\partial x^{j_k}}}\right)\alpha(p),$$

where $A^{j_1\cdots j_k}\frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_k}}$ is the local representation of A in some chart $(U, x^1, \ldots, x^m)$ around p. By straightforward calculation, it can be shown that $\mathrm{i}_A\alpha$ is a well-defined $(n-k)$-differential form. Furthermore, we have

$$(\mathrm{i}_A\alpha)|_U = A^{j_1\cdots j_k}\mathrm{i}_{\frac{\partial}{\partial\varphi^{j_1}}} \circ \ldots \circ \mathrm{i}_{\frac{\partial}{\partial\varphi^{j_k}}}(\alpha|_U),$$

which shows that i_A is locally the composition of k insertions of vector fields. By Proposition 2.3.6, $\mathrm{i}_A \in \mathcal{D}_k^{-k}(M)$, called **insertion of** $A \in \chi_M$.

It is clear that $A \mapsto \mathrm{i}_A$ is C_M^∞-linear. Also, if $X \in \chi_M^1 = \mathfrak{X}_M$, then i_X coincides with the insertion of vector fields introduced in Section 2.1. Moreover, $\mathrm{i}_{U\wedge V} = \mathrm{i}_U \circ \mathrm{i}_V$. Conversely, if the map $\mathrm{i}: \chi_M \longrightarrow \mathcal{D}(M)$ satisfies the properties:

1. if $X \in \chi_M^1 = \mathfrak{X}_M$, then i_X coincides with the insertion operator defined for vector fields (see Equation (2.3)),
2. for any $U, V \in \chi_M$, $\mathrm{i}_{U\wedge V} = \mathrm{i}_U \circ \mathrm{i}_V$,

then i_A is the insertion of A for each $A \in \chi_M$. Indeed, recall that multivector fields are sum of decomposable elements. By the property 2, the map i is determined on vector fields. Thus, property 1 determines such map. Moreover,

$$\mathrm{i}: (\chi_M, \wedge) \longrightarrow \left(\bigoplus_{l\in\mathbb{Z}} \mathcal{D}_l^{-l}(M), \circ\right)$$

given by $U \mapsto \mathrm{i}_U$ is a graded C_M^∞-algebra morphism.

We now show that the space of differential operators $\mathcal{D}_k^{-k}(M)$ is parameterized by k-vector fields. More precisely, for each $D \in \mathcal{D}_k^{-k}(M)$, there exists $A \in \chi_M^k$ such that $D = \mathrm{i}_A$. Thus, the map $\chi_M^k \ni A \mapsto \mathrm{i}_A \in \mathcal{D}_k^{-k}(M)$ is an isomorphism.

Lemma 2.3.7. *Every differential operator in $\mathcal{D}_k^{-k}(M)$ is determined by its action on the space differential forms Ω_M^k.*

Proof. Fix $D \in \mathcal{D}_k^{-k}(M)$. By Proposition 2.3.4, D is determined by is action on Ω_M^p, with $0 \leq p \leq k$. On the other hand, for any $\alpha \in \Omega_M^p$, with $p < k$, we have $D(\alpha) \in \Omega_M^{p-k} = \{0\}$, since $p-k$ is negative. Therefore, D is zero on forms of degree less than k, proving that D is determined only by its restriction to Ω_M^k, $D|_{\Omega_M^k}: \Omega_M^k \longrightarrow C_M^\infty$. ■

Proposition 2.3.8. *The map*

$$\mathrm{i}: (\chi_M, \wedge) \longrightarrow \left(\bigoplus_{k\in\mathbb{Z}} \mathcal{D}_k^{-k}(M), \circ\right)$$

is a graded C_M^∞-algebra isomorphism.

Proof. We already proved that the above map is a graded C^∞_M-algebra morphism. So, it suffices to prove that each $D \in \mathcal{D}^{-k}_k(M)$ is the insertion of a multivector field. Note that, if $f \in C^\infty_M$, then $[D, f] \in \mathcal{D}^{-k}_{k-1}(M) = \{0\}$. Thus, for any $\alpha \in \Omega^k_M$, $0 = [D, f](\alpha) = D(f\alpha) - fD(\alpha)$, proving that $D|_{\Omega^k_M} : \Omega^k_M \longrightarrow C^\infty_M$ is C^∞_M-linear. So, $D|_{\Omega^k_M} \in (\Omega^k_M)^*$. Therefore, there exists a unique $B \in \chi^k_M$ such that $D(\alpha) = \langle \alpha, B\rangle$ $\forall \alpha \in \Omega^k_M$. Taking $A = (-1)^{\frac{k(k-1)}{2}} B$, we get $D(\alpha) = \langle \alpha, B\rangle = \mathrm{i}_A\alpha$, $\forall \alpha \in \Omega^k_M$. This shows that D and i_A are differential operators of degree $-k$ and order k coinciding in Ω^k_M. By Lemma 2.3.7, they coincide on Ω_M, as desired. ■

Remark 2.3.9. *Along this work the insertion of multivector fields is defined to have an algebra isomorphism:* $\mathrm{i}_{X_1\wedge\ldots\wedge X_k} = \mathrm{i}_{X_1} \circ \ldots \circ \mathrm{i}_{X_k}$. *However, it is pretty common to find the following definition (see, for example, [7, p. 30] [16, p. 254] o [32, p. 9]):* $i_{X_1\wedge\ldots\wedge X_k}\omega = \omega(X_1, \ldots, X_k)$ $\forall \omega \in \Omega^k_M$ *(note the difference of the symbols* i *and* i*). Both operators are related by the formula*

$$i_A = (-1)^{\frac{|A|(|A|-1)}{2}} \mathrm{i}_A.$$

The definition presented in this chapter coincides with the given in [14, p. 266]. We prefer to work with it because of its algebraic properties and its relation to the Schouten - Nijenhuis bracket.

2.4 The Schouten - Nijenhuis bracket

In this part, the notion of Lie derivative along vector fields is extended to multivector fields, by means of the Cartan's formula. This notion, and Proposition 2.3.8, will be used to define the Schouten - Nijenhuis bracket.

Let M be a differential manifold. Recall that the Lie derivative of a tensor field $\tau \in \mathcal{T}^r_s(M)$ along a vector field X is defined in terms of its flow Fl^t_X by

$$\mathcal{L}_X\tau := \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Fl}^t_X)^*\tau.$$

It turns out that $\mathcal{L}_X\tau \in \mathcal{T}^r_s(M)$. Moreover, for $X_i \in \mathfrak{X}_M$ and $\alpha_j \in \Omega^1_M$,

$$\begin{aligned} \mathcal{L}_X\tau(X_1, \ldots, X_s; \alpha_1, \ldots, \alpha_r) = \mathcal{L}_X(\tau(X_1, \ldots, X_s; \alpha_1, \ldots, \alpha_r)) \\ - \sum \tau(X_1, \ldots, \mathcal{L}_X X_i, \ldots, X_s; \alpha_1, \ldots, \alpha_r) \\ - \sum \tau(X_1, \ldots, X_s; \alpha_1, \ldots, \mathcal{L}_X\alpha_j, \ldots, \alpha_r). \end{aligned}$$

In case of functions, vector fields and 1-forms, this is

$$\mathcal{L}_X f = X(f), \qquad \mathcal{L}_X Y = [X, Y], \qquad (\mathcal{L}_X\alpha)(Y) = X(\alpha(Y)) - \alpha[X, Y].$$

For differential forms, we get the formula (2.4). So, $\mathcal{L}_X \in \mathrm{Der}^0_{\mathbb{R}}\Omega_M$. Also, for multivector fields, we have $\mathcal{L}_X \in \mathrm{Der}^0_{\mathbb{R}}\chi_M$, since

$$\mathcal{L}_X A(\alpha^1, \ldots, \alpha^k) = \mathcal{L}_X(A(\alpha^1, \ldots, \alpha^k)) - \sum_{i=1}^{k} A(\alpha^1, \ldots, \mathcal{L}_X\alpha^i, \ldots, \alpha^k).$$

Now, it is presented the notion of Lie derivative along multivector fields as differential operators in Ω_M, generalizing the Cartan's formula for vector fields.

Definition 2.4.1. *Let $A \in \chi^k_M$ be a multivector field. The* ***Lie derivative*** *along A, $\mathcal{L}_A \in \mathcal{D}^{-(k-1)}_k(M)$, is defined by*

$$\mathcal{L}_A := [\mathrm{i}_A, \mathrm{d}] = \mathrm{i}_A\mathrm{d} - (-1)^k \mathrm{d}\mathrm{i}_A.$$

It is easy to see that $[\mathcal{L}_A, \mathrm{d}]$, due to the Jacobi identity. Now, note that if $A \in \chi^k_M$ and $B \in \chi^l_M$, then $\mathcal{L}_A \in \mathcal{D}^{-k+1}_k(M)$, $\mathrm{i}_B \in \mathcal{D}^{-l}_l(M)$, and $[\mathcal{L}_A, \mathrm{i}_B] \in \mathcal{D}^{-(k+l-1)}_{k+l-1}(M)$. By Proposition 2.3.8, there exists a unique $(k+l-1)$-vector field whose insertion coincides with $[\mathcal{L}_A, \mathrm{i}_B]$.

Definition 2.4.2. *Let $A, B \in \chi_M$ be multivector fields. The unique multivector field $[A, B] \in \chi_M$ such that*

$$\mathrm{i}_{[A,B]} = [\mathcal{L}_A, \mathrm{i}_B] \tag{2.6}$$

is called the ***Schouten - Nijenhuis bracket*** *of A and B.*

Because of the Jacobi identity for the graded commutator of differential operators, the Schouten - Nijenhuis bracket satisfies the following relation, given in terms of the Lie derivative along multivector fields:

$$\mathcal{L}_{[A,B]} = [\mathcal{L}_A, \mathcal{L}_B]. \tag{2.7}$$

Alternatively, equation (2.7) could have been used as definition of the Schouten - Nijenhuis bracket, excepting for some low-degree cases.

As an immediate consequence of these definitions, the triple $(\chi_M, \wedge, [,])$ is a graded Poisson algebra of degree -1 with the exterior product $\wedge$ and the Schouten - Nijenhuis bracket $[,]$ for multivector fields. Explicitly, the Schouten - Nijenhuis bracket satisfies the following properties:

- $[A, B] = -(-1)^{(|A|-1)(|B|-1)}[B, A]$ *(graded skew-symmetry)*;
- $[A, [B, C]] = [[A, B], C] + (-1)^{(|A|-1)(|B|-1)}[B, [A, C]]$ *(graded Jacobi identity)*;
- $[A, B \wedge C] = [A, B] \wedge C + (-1)^{(|A|-1)|B|} B \wedge [A, C]$ *(graded Leibniz rule)*.

Moreover, because of the Cartan's formula, the Lie bracket of vector fields satisfies (2.6), showing that the Schouten - Nijenhuis and the Lie brackets coincide on vector fields. Therefore, we denote by $[,]$ both the Schouten - Nijenhuis and Lie brackets without ambiguity. In other words, $(\chi_M, \wedge, [,])$ is the natural algebraic extension of $(\mathfrak{X}_M, [,])$ to multivector fields. Finally, the Schouten - Nijenhuis bracket is of local type since differential operators are of local type.

The following are important properties of the Schouten - Nijenhuis bracket.

Proposition 2.4.3. *Let $X, X_1, \dots, X_k, Y_1, \dots, Y_l \in \mathfrak{X}_M$, $B \in \chi_M$, and $f, g \in C^\infty_M$. The Schouten - Nijenhuis bracket satisfies the following properties:*

1. $[f, B] = -\mathrm{i}_{\mathrm{d}f} B$,
2. $[X, B] = \mathcal{L}_X B$,
3. $[X_1 \wedge \dots \wedge X_k, B] = \sum_{i=1}^k (-1)^{(k-i)(|B|-1)} X_1 \wedge \dots \wedge \mathcal{L}_{X_i} B \wedge \dots \wedge X_k$.
4. $[X_1 \wedge \dots \wedge X_k, Y_1 \wedge \dots \wedge Y_l] = \sum_{i=1}^k \sum_{j=1}^l (-1)^{k-i+j-1} X_1 \wedge \dots \widehat{X_i} \dots \wedge X_k \wedge [X_i, Y_j] \wedge Y_1 \wedge \dots \widehat{Y_j} \dots \wedge Y_l$.
5. $[X_1 \wedge \dots \wedge X_k, Y_1 \wedge \dots \wedge Y_l] = \sum_{i=1}^k \sum_{j=1}^l (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \dots \widehat{X_i} \dots \wedge X_k \wedge Y_1 \wedge \dots \widehat{Y_j} \dots \wedge Y_l$.
6. $[f, X_1 \wedge \dots \wedge X_k] = \sum_{i=1}^k (-1)^i \mathcal{L}_{X_i} f X_1 \wedge \dots \widehat{X_i} \dots \wedge X_k$.

Proof. In 1 and 2 suppose that $|B| = k$.

1. If $\omega \in \Omega^{k-1}_M$, then

$$\begin{aligned}
\mathrm{i}_{[f,B]}\omega &= [\mathcal{L}_f, \mathrm{i}_B]\omega = [[\mathrm{i}_f, \mathrm{d}], \mathrm{i}_B]\omega = [[\mu_f, \mathrm{d}], \mathrm{i}_B]\omega = -[\mu_{\mathrm{d}f}, \mathrm{i}_B]\omega \\
&= -\mu_{\mathrm{d}f}\mathrm{i}_B\omega + (-1)^k \mathrm{i}_B \mu_{\mathrm{d}f}\omega = (-1)^k \mathrm{i}_B(\mathrm{d}f \wedge \omega) \\
&= (-1)^k \mathrm{i}_{\mathrm{d}f\wedge\omega} B = -\mathrm{i}_\omega(\mathrm{i}_{\mathrm{d}f} B) = -\mathrm{i}_{\mathrm{i}_{\mathrm{d}f} B}\omega.
\end{aligned}$$

2. Without lost of generality, assume that $B = X_1 \wedge \dots \wedge X_k$. By the Leibniz rule of the Schouten - Nijenhuis bracket and the exterior product in χ_M,

$$[X, B] = \sum_{i=1}^k X_1 \wedge \dots \wedge [X, X_i] \wedge \dots \wedge X_k = \sum_{i=1}^k X_1 \wedge \dots \wedge \mathcal{L}_X X_i \wedge \dots \wedge X_k = \mathcal{L}_X B.$$

3. It follows from the Leibniz rule and incise 2.

4. Denote $B = Y_1 \wedge \dots \wedge Y_l$. Applying incise 3,

$$[X_1 \wedge \dots \wedge X_k, Y_1 \wedge \dots \wedge Y_l] = \sum_{i=1}^k (-1)^{k-i} X_1 \wedge \dots \widehat{X_i} \dots \wedge X_k \wedge \mathcal{L}_{X_i} B.$$

   Since $\mathcal{L}_{X_i} \in \mathrm{Der}^0_{\mathbb{R}} \chi_M$, $\mathcal{L}_{X_i} B = \sum_{j=1}^m (-1)^{j-1} [X_i, Y_j] \wedge Y_1 \wedge \dots \widehat{Y_j} \dots \wedge Y_l$. Substituting this equation in the previous one, we get the desired result.

5. By the graded symmetry of the exterior product, it is equivalent to incise 4.

6. It follows from the Leibniz rule and the incise 1.

■

Recall that the Lie bracket of vector fields is natural respect to pullbacks and, in particular, to restrictions. Therefore, it follows from incise 4 of Proposition that the Schouten - Nijenhuis bracket is also natural.

Remark 2.4.4. *There are various ways to introduce the Schouten - Nijenhuis bracket. Basically, there exist two different Schouten - Nijenhuis brackets of multivector fields in the literature. The one presented here coincides with the used in [6], [21], [20] and [18] but it differs to the used in [16] or [32]. The Schouten - Nijenhuis bracket presented in these books satisfies the graded symmetry of degree zero, Leibniz rule and a third one, similar to the graded Jacobi identity:*

$$\begin{aligned}
&[P,Q]_{SN} = (-1)^{|P||Q|}[Q,P]_{SN},\\
&[P,Q\wedge R]_{SN} = [P,Q]_{SN}\wedge R + (-1)^{(|P|-1)|Q|}Q\wedge [P,R]_{SN},\\
&\sum_{(P,Q,R)} (-1)^{|P|(|R|-1)}[P,[Q,R]_{SN}]_{SN} = 0.
\end{aligned}$$

Also, is characterized by the following property

$$[X_1\wedge\ldots\wedge X_p, B]_{SN} = \sum_{i=1}^{p}(-1)^{i+1}X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_p\wedge\mathcal{L}_{X_i}B.$$

Comparing this with Proposition 2.4 incise 3, we deduce the following relation between this Schouten - Nijenhuis bracket and the one presented in this work:

$$[P,Q]_{SN} = (-1)^{|P|-1}[P,Q].$$

2.5 Generalized connections in manifolds

In this section, following [13], we present the concept of generalized connection, i.e., a tangent bundle endomorphism over the identity which is a projection of constant rank. Also, we introduce the concept of *bigraded manifold*, which is a manifold equipped with generalized connection. A generalized connection is equivalent to a splitting of the tangent bundle by two distributions. The notions of *curvature* and *co-curvature* are presented, which are a measure of the integrability of the splitting distributions.

Generalized connections and bigrading. Let M be a differential manifold. A **generalized connection** in M is a vector valued 1-form $\gamma\in\Omega^1(M;TM)$ such that the corresponding vector bundle endomorphism $\gamma : TM \longrightarrow TM$ satisfies $\gamma^2=\gamma$. If γ has constant rank, then $\mathbb{V} := \mathrm{Im}(\gamma)$ is a regular distribution, called **vertical distribution**. Also, its kernel $\mathbb{H} := \ker(\gamma)$ is a regular distribution satisfying

$$TM = \mathbb{H}\oplus\mathbb{V}, \tag{2.8}$$

called **horizontal distribution**. Equivalently, each splitting of the form (2.8) induces a connection γ in M, defined as the projection on the vertical subbundle: $\gamma := \mathrm{pr}_{\mathbb{V}}$.

Remark 2.5.1. *For the rest of this work, the term* generalized connection *will only refer to a generalized connection of constant rank.*

A vector field X *tangent* to $\mathbb{H}$, $X \in \Gamma\mathbb{H}$, is called **horizontal vector field**. A vector field Y *tangent* to $\mathbb{V}$, $X \in \Gamma\mathbb{V}$, is called **vertical vector field**. Note that a connection in M induces the splitting $\mathfrak{X}_M = \Gamma\mathbb{H} \oplus \Gamma\mathbb{V}$ for vector fields. We also have a splitting in the cotangent bundle $T^*M = \mathbb{V}^0 \oplus \mathbb{H}^0$, where $\mathbb{V}^0$ is the annihilator of $\mathbb{V}$ and $\mathbb{H}^0$ is the annihilator of $\mathbb{H}$. Moreover, we have the following splittings,

$$\bigwedge^k T^*M = \bigoplus_{p+q=k} \left(\bigwedge^p \mathbb{V}^0 \wedge \bigwedge^q \mathbb{H}^0 \right), \qquad \Omega_M^k = \bigoplus_{p+q=k} \Omega_M^{p,q},$$

where $\Omega_M^{p,q} = \Gamma\left(\bigwedge^p \mathbb{V}^0 \wedge \bigwedge^q \mathbb{H}^0\right)$ is identified with the C_M^∞-module of $(p+q)$-linear applications $\alpha : \mathfrak{X}_M \times \cdots \times \mathfrak{X}_M \longrightarrow C_M^\infty$ such that, $\omega(X_1, \ldots, X_r, Y_1, \ldots, Y_s) = 0$ for all $r, s \in \mathbb{Z}$ with $r \neq p$, $X_i \in \Gamma\mathbb{H}$, $Y_j \in \Gamma\mathbb{V}$. In the same fashion, we have

$$\bigwedge^k TM = \bigoplus_{p+q=k} \left(\bigwedge^p \mathbb{H} \wedge \bigwedge^q \mathbb{V} \right), \qquad \chi_M^k = \bigoplus_{p+q=k} \chi_M^{p,q},$$

with $\chi_M^{p,q} = \Gamma\left(\bigwedge^p \mathbb{H} \wedge \bigwedge^q \mathbb{V}\right)$. Therefore, a connection γ in M induces the following bigrading of the tensor algebras Ω_M and χ_M:

$$\Omega_M = \bigoplus_{p,q\in\mathbb{Z}} \Omega_M^{p,q}, \qquad \chi_M = \bigoplus_{p,q\in\mathbb{Z}} \chi_M^{p,q}. \tag{2.9}$$

Those are bigraded algebras with their respective exterior products, and the homogeneous elements in $\Omega_M^{p,q}$ and $\chi_M^{p,q}$ are called differential forms and multivector fields of bidegree (p,q). Each $\alpha \in \Omega_M^k$ has a decomposition of the form

$$\alpha = \sum_{p+q=k} \alpha_{p,q}, \tag{2.10}$$

where $\alpha_{p,q} \in \Omega_M^{p,q}$. The right-hand side in equation (2.10) is called **bigraded decomposition** of α. Similarly, multivector fields have a bigraded decomposition. Furthermore, Since M is finite dimensional, the sums in (2.9) are finite. So, there is a bigrading for their graded endomorphisms, derivations, and differential operators

$$\mathrm{End}_{\mathbb{R}}^k \chi_M = \bigoplus_{p+q\in\mathbb{Z}} \mathrm{End}_{\mathbb{R}}^{p,q} \chi_M, \qquad \mathrm{End}_{\mathbb{R}}^k \Omega_M = \bigoplus_{p+q\in\mathbb{Z}} \mathrm{End}_{\mathbb{R}}^{p,q} \Omega_M,$$
$$\mathrm{Der}_{\mathbb{R}}^k \chi_M = \bigoplus_{p+q\in\mathbb{Z}} \mathrm{Der}_{\mathbb{R}}^{p,q} \chi_M, \qquad \mathrm{Der}_{\mathbb{R}}^k \Omega_M = \bigoplus_{p+q\in\mathbb{Z}} \mathrm{Der}_{\mathbb{R}}^{p,q} \Omega_M,$$
$$\mathcal{D}_r^k(M) = \bigoplus_{p+q\in\mathbb{Z}} \mathcal{D}_r^{p,q}(M).$$

This motivates the following definition.

Definition 2.5.2. *A* ***bigraded manifold*** *is a pair (M, γ), where M is a differential manifold and $\gamma : TM \longrightarrow TM$ is a generalized connection in M of constant rank.*

Example 2.5.3. *Let (M,γ) be a bigraded manifold. Recall that every vector-valued form induces an algebraic derivation given by its insertion. Fix $\alpha \in \Omega_M^{p,q}$, $X_1,\dots,X_p \in \Gamma\mathbb{H}$ and $Y_1,\dots Y_q \in \Gamma\mathbb{V}$. Note that*

$$\begin{aligned}
&\mathrm{i}_\gamma\alpha(X_1,\dots,X_p,Y_1,\dots Y_q) = \\
&\sum_{i=1}^{p}\alpha(X_1,\dots,\gamma X_i,\dots,X_p,Y_1,\dots Y_q) + \sum_{i=1}^{q}\alpha(X_1,\dots,X_p,Y_1,\dots,\gamma Y_i,\dots Y_q) = \\
&\sum_{i=1}^{q}\alpha(X_1,\dots,X_p,Y_1,\dots,Y_i,\dots Y_q) = q\alpha(X_1,\dots,X_p,Y_1,\dots Y_q),
\end{aligned}$$

so $\mathrm{i}_\gamma\alpha = q\alpha$, where q is the vertical bidegree *of α. Similarly, $\mathrm{i}_{\mathrm{Id}_{TM}-\gamma}\alpha = p\alpha$, where p is the* horizontal bidegree *of α.*

Define the following vector valued 2-forms $R, R' \in \Omega^2(M;TM)$ by

$$R(X,Y) = \gamma[(\mathrm{Id}_{TM}-\gamma)X, (\mathrm{Id}_{TM}-\gamma)Y], \tag{2.11}$$
$$R'(X,Y) = (\mathrm{Id}_{TM}-\gamma)[\gamma(X),\gamma(Y)], \tag{2.12}$$

where R is called the **curvature** and R' the **co-curvature** of γ. Note that $\mathbb{V}$ is involutive if and only if $R' = 0$; also, $\mathbb{H}$ is involutive if and only if $R = 0$. A connection γ in M, is related to its curvature and its co-curvature by $[\gamma,\gamma]_{FN} = 2R + 2R'$, where $[,]_{FN}$ denotes the *Frölicher - Nijenhuis bracket* for vector valued forms (see formula (2.5)). Also, the *Bianchi identities*

$$[\gamma, R+R']_{FN} = 0, \qquad [R,\gamma]_{FN} = \mathrm{i}_R R' + \mathrm{i}_{R'} R \tag{2.13}$$

are satisfied (see also equation (3.22)).

Remark 2.5.4. *A generalized connection γ in M induces a splitting on the space of tensor fields in bigraded components. In particular, it is easy to verify that $R \in \Omega^{2,0}(M,\mathbb{V})$ and $R' \in \Omega^{0,2}(M,\mathbb{H})$. Hence, the bigraded decomposition of $[\gamma,\gamma]_{FN}$ is*

$$[\gamma,\gamma]_{FN} = 2R + 2R'.$$

A generalized connection γ is called *flat* if $[\gamma,\gamma]_{FN} = 0$. By a bidegree argument, this is equivalent to $R = 0$ and $R' = 0$. In this case, the horizontal and vertical distributions are both integrable, due to Frobenius Theorem. The resulting foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are called a *co-foliation*, which is $T\mathcal{F}_1 \oplus T\mathcal{F}_2 = TM$.

Remark 2.5.5. *Note that if γ is a connection in M, then $\mathrm{Id}_{TM} - \gamma$ is also a connection in M. Furthermore, $\mathrm{Im}(\mathrm{Id}_{TM}-\gamma) = \ker(\gamma)$, $\mathrm{Im}(\gamma) = \ker(\mathrm{Id}_{TM}-\gamma)$. So, the vertical distribution of γ and the horizontal distribution of $(\mathrm{Id}_{TM}-\gamma)$ coincide, an conversely. Thus, $R_\gamma = R'_{\mathrm{Id}_{TM}-\gamma}$ and $R'_\gamma = R_{\mathrm{Id}_{TM}-\gamma}$.*

The exterior differential in bigraded manifolds. Let γ be a connection in the differential manifold M. This induces a bigrading in Ω_M by

$$\Omega_M = \bigoplus_{p,q\in\mathbb{Z}} \Omega_M^{p,q},$$

where $\Omega_M^{p,q} = \Gamma\left(\bigwedge^p \mathbb{V}^0 \wedge \bigwedge^q \mathbb{H}^0\right)$. With this bigrading, Ω_M becomes a bigraded algebra with the exterior product, i.e, $\Omega_M^{p,q} \wedge \Omega_M^{r,s} \subset \Omega_M^{p+r,q+s}$. On the other hand, every graded operator has a bigraded decomposition. We now show that the exterior differential $\mathrm{d} \in \mathrm{Der}^1_{\mathbb{R}}\Omega_M$, has a bigraded decomposition of the form

$$\mathrm{d} = \mathrm{d}_{1,0} + \mathrm{d}_{0,1} + \mathrm{d}_{2,-1} + \mathrm{d}_{-1,2}, \tag{2.14}$$

regardless of the connection γ. Moreover, the bigraded components in (2.14) are graded derivations. Their Frölicher - Nijenhuis decompositions depend on γ, its curvature $R \in \Omega^2(M;TM)$ and its co-curvature $R' \in \Omega^2(M;TM)$.

Theorem 2.5.6. *Let (M,γ) be a bigraded manifold, R the curvature and R' the co-curvature of γ. The bigraded components of the exterior differential are graded derivations whose Frölicher - Nijenhuis decompositions are given by*

$$\mathrm{d}_{1,0} = \mathcal{L}_{\mathrm{Id}_{TM}-\gamma} + 2\mathrm{i}_R - \mathrm{i}_{R'}, \quad \mathrm{d}_{0,1} = \mathcal{L}_\gamma - \mathrm{i}_R + 2\mathrm{i}_{R'}, \quad \mathrm{d}_{2,-1} = -\mathrm{i}_R, \quad \mathrm{d}_{-1,2} = -\mathrm{i}_{R'}.$$

Proof. Let $\mathrm{d} = \sum_{i+j=1} \mathrm{d}_{i,j}$ be the bigraded decomposition of the exterior differential. First, note that if $j \leq -2$, then $\mathrm{d}_{i,j} = 0$. Indeed, if $\alpha \in \Omega^k_M$, with $k \leq 1$, then $\mathrm{d}_{i,j}\alpha = 0$ because of its negative vertical bidegree. Since $\mathrm{d}_{i,j}$ is a graded derivation, this implies that $\mathrm{d}_{i,j} = 0$. Similarly, if $i \leq -2$, then $\mathrm{d}_{i,j} = 0$. Therefore, the only non-zero bigraded components of d are $\mathrm{d}_{1,0}$, $\mathrm{d}_{0,1}$, $\mathrm{d}_{2,-1}$, $\mathrm{d}_{-1,2}$. To find their Frölicher - Nijenhuis decompositions, we begin by proving $\mathrm{d}_{2,-1} = -\mathrm{i}_R$. Observe that, if $f \in C^\infty_M$, then $\mathrm{d}_{2,-1}f = 0$, since its vertical bidegree is negative. Hence, $\mathrm{d}_{2,-1}$ is an algebraic derivation. Now, fix $\alpha \in \Omega^{1,0}_M$ and $\mu \in \Omega^{0,1}_M$. Note that $\mathrm{d}_{2,-1}\alpha = 0$ because of its bidegree, and $-\mathrm{i}_R\alpha = 0$ since R values on $\Gamma\mathbb{V}$. On the other hand, $\mathrm{d}_{2,-1}\mu \in \Omega^{2,0}$. Evaluating in horizontal vector fields $X, Y \in \Gamma\mathbb{H}$, we get

$$\begin{aligned} \mathrm{d}_{2,-1}\mu(X,Y) &= X(\mu(Y)) - Y(\mu(X)) - \mu[X,Y] = -\mu[X,Y], \\ -\mathrm{i}_R\mu(X,Y) &= -\mu(R(X,Y)) = -\mu(\gamma[X,Y]) = -\mu[X,Y]. \end{aligned}$$

So, $-\mathrm{i}_R$ and $\mathrm{d}_{2,-1}$ coincide on 1-forms, Thus, $\mathrm{d}_{2,-1} = -\mathrm{i}_R$. The proof of $\mathrm{d}_{-1,2} = -\mathrm{i}_{R'}$ is completely analogous. Now, it follows from Example 2.5.3 that, for any $\alpha \in \Omega^{p,q}_M$,

$$\begin{aligned} \mathcal{L}_\gamma\alpha =& \mathrm{i}_\gamma\mathrm{d}\alpha - \mathrm{d}\mathrm{i}_\gamma\alpha = \mathrm{i}_\gamma(\mathrm{d}_{1,0}\alpha + \mathrm{d}_{0,1}\alpha + \mathrm{d}_{2,-1}\alpha + \mathrm{d}_{-1,2}\alpha) - q\mathrm{d}\alpha \\ =& (q\mathrm{d}_{1,0}\alpha + (q+1)\mathrm{d}_{0,1}\alpha + (q-1)\mathrm{d}_{2,-1}\alpha + (q+2)\mathrm{d}_{-1,2}\alpha) \\ &- q(\mathrm{d}_{1,0}\alpha + \mathrm{d}_{0,1}\alpha + \mathrm{d}_{2,-1}\alpha + \mathrm{d}_{-1,2}\alpha) = \mathrm{d}_{0,1}\alpha - \mathrm{d}_{2,-1}\alpha + 2\mathrm{d}_{-1,2}\alpha \\ =& \mathrm{d}_{0,1}\alpha + \mathrm{i}_R\alpha - 2\mathrm{i}_{R'}\alpha. \end{aligned}$$

This proves that $\mathrm{d}_{0,1} = \mathcal{L}_\gamma - \mathrm{i}_R + 2\mathrm{i}_{R'}$. Finally, it follows from Example 2.2.6 that

$$\mathrm{d}_{1,0} = \mathrm{d} - \mathrm{d}_{0,1} - \mathrm{d}_{2,-1} - \mathrm{d}_{-1,2} = \mathcal{L}_{\mathrm{Id}_{TM}-\gamma} + 2\mathrm{i}_R - \mathrm{i}_{R'},$$

completing the proof. ∎

Corollary 2.5.7. *Let (M,γ) be a bigraded manifold. If the vertical distribution $\mathbb{V}$ is involutive, then the bigraded components of the exterior differential satisfy*

$$\mathrm{d}_{-1,2}=0, \quad \mathrm{d}_{1,0}=\mathcal{L}_{\mathrm{Id}_{TM}-\gamma}+2\mathrm{i}_R, \quad \mathrm{d}_{0,1}=\mathcal{L}_\gamma-\mathrm{i}_R, \quad \mathrm{d}_{2,-1}=-\mathrm{i}_R, \quad \mathrm{d}_{1,0}^2=\mathcal{L}_R.$$

In particular, $\mathbb{H}$ is also an involutive distribution if and only if $\mathrm{d}_{1,0}$ is a coboundary.

Proof. Recall that $\mathbb{V}$ is involutive if and only if $R'=0$. By Theorem 2.5.6, it remains to prove that $\mathrm{d}_{1,0}^2=\mathcal{L}_R$. Since $\mathrm{d}=\mathrm{d}_{1,0}+\mathrm{d}_{0,1}+\mathrm{d}_{2,-1}$ and $\mathrm{d}^2=0$, it follows, from the bigraded equation (1.11), that

$$\mathrm{d}_{1,0}^2=-(\mathrm{d}_{0,1}\mathrm{d}_{2,-1}+\mathrm{d}_{0,1}\mathrm{d}_{2,-1})=-[\mathrm{d}_{0,1},\mathrm{d}_{2,-1}]=-[\mathrm{d},\mathrm{d}_{2,-1}]=[\mathrm{d},\mathrm{i}_R]=\mathcal{L}_R.$$

Thus, $\mathrm{d}_{1,0}^2=0$ if and only if $R=0$, which is equivalent to the involutivity of $\mathbb{H}$. ■

Some of the results of Corollary 2.5.7 can be found, for example, in [30].

2.6 Ehresmann connections in fiber bundles

A **fiber bundle** is a triple (E,π,B), where $\pi:E\longrightarrow B$ is a surjective submersion between differential manifolds E and B. The manifold E is called the *total space*, B is the *base space* and π is called *projection.* It is well-known that every fibred manifold has an intrinsic involutive distribution. Indeed, note that the diagram

$$\begin{tikzcd} TE \arrow[r, "\pi_*"] \arrow[d, "\tau_E"'] & TB \arrow[d, "\tau_B"] \\ E \arrow[r, "\pi"'] & B \end{tikzcd}$$

commutes. So, $\pi_*:TE\longrightarrow TB$ is a vector bundle morphism over the surjective submersion π. Thus, the *kernel distribution* $\mathbb{V}$, $\mathbb{V}_p:=\ker(\pi_{*_p})$, is a subbundle of TE of codimension $\dim B$, which is called **vertical subbundle** or **vertical distribution**.

On the other hand, each fiber $E_b:=\pi^{-1}(b)$ is a regular submanifold of E, of codimension $\dim B$. The fibers define a foliation $\mathcal{F}$ in E, by $\mathcal{F}=\{E_{\pi(b)}\}_{b\in B}$, which is called **characteristic foliation** of E. It is clear that $\mathbb{V}=T\mathcal{F}$, where $T_p\mathcal{F}=T_pE_{\pi(p)}$, i.e., the vertical distribution at each point identifies with the tangent subspace to the fiber passing at the point. Thus, $\mathbb{V}$ is involutive.

The involutivity of the vertical subbundle implies that the space of *vertical vector fields* $\mathfrak{X}_{\mathbb{V}}(E):=\Gamma\mathbb{V}$ is a Lie subalgebra of vector fields. Moreover, the exterior algebra of *vertical multivector fields* $\chi_{\mathbb{V}}(E):=\Gamma\bigwedge\mathbb{V}$ is a Poisson $\mathbb{R}$-subalgebra of multivector fields $(\chi_M,\wedge,[,])$ with the Schouten - Nijenhuis bracket. Also, the algebra of *horizontal differential forms*, $\Omega_{\mathbb{V}^0}(E):=\Gamma\bigwedge\mathbb{V}^0$, where $\mathbb{V}^0\subset T^*M$

denotes de annihilator of $\mathbb{V}$, is an exterior subalgebra of the Cartan's algebra $(\Omega_M, \wedge)$, but it fails to be a sub-complex of the de Rham complex (Ω_M, d).

Projectability properties. Involutive distributions and the fiber bundle case. It is useful to review the concept of *projectable functions*, *projectable vector fields* and *projectable differential forms* respect to a regular involutive distribution.

Definition 2.6.1. *Let M be a differential manifold, $D \subset TM$ a regular and involutive distribution in M and $D^0 \subset T^*M$ the annihilator subbundle.*

- *A smooth function $f \in C^\infty_M$ is called **projectable** if $\mathrm{d}f \in \Gamma D^0$.*
- *A vector field $X \in \mathfrak{X}_M$ is said to be **projectable** if $[X, \Gamma D] \subset \Gamma D$.*

The $\mathbb{R}$-vector spaces of projectable functions and vector fields are respectively denoted by $C^\infty_{\mathrm{pr}}(M, D)$ and $\mathfrak{X}_{\mathrm{pr}}(M, D)$.

Equivalently, $f \in C^\infty_{\mathrm{pr}}(M, D)$ if and only if $\mathcal{L}_Y f = 0$, $\forall Y \in \Gamma D$. Furthermore, projectable functions and vector fields satisfy the following properties:

1. $X \in \mathfrak{X}_{\mathrm{pr}}(M, D)$ if and only if $[X, \chi_D(M)] \subset \chi_D(M)$,
2. $f \in C^\infty_{\mathrm{pr}}(M, D)$ if and only if $[f, \chi_D(M)] = 0$,
3. $C^\infty_{\mathrm{pr}}(M, D)$ is a sub-ring of C^∞_M,
4. $\mathfrak{X}_{\mathrm{pr}}(M, D)$ is a $C^\infty_{\mathrm{pr}}(M, D)$-submodule of $\mathfrak{X}_M$,
5. $\mathfrak{X}_{\mathrm{pr}}(M, D)$ is a Lie $\mathbb{R}$-subalgebra of $\big(\mathfrak{X}_M, [,]\big)$,
6. $(\Gamma D, [,])$ is a Lie $C^\infty_{\mathrm{pr}}(M, D)$-algebra with the usual Lie bracket for vector fields,
7. $(\chi_D(M), \wedge, [,])$ is a Poisson $C^\infty_{\mathrm{pr}}(M, D)$-algebra of degree -1 with the Schouten - Nijenhuis bracket.

The proof of 1, 2 and 4 follows from the Leibniz rule of the Schouten - Nijenhuis bracket of multivector fields; the proof of 3 follows from $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$; properties 5 and 6 follow from the Jacobi identity of the Lie bracket; and property 7 follow from the Leibniz rule and Jacobi identity of the Schouten - Nijenhuis bracket. Also, vector fields are locally generated by projectable vector fields.

Definition 2.6.2. *A differential k-form $\alpha \in \Omega^k_M$ is **projectable** if*

$$\mathrm{i}_Y \alpha = 0, \tag{2.15}$$

$$\mathrm{i}_Y \mathrm{d}\alpha = 0, \tag{2.16}$$

for any vector field $Y \in \Gamma D$. The space of projectable differential k-forms is denoted by $\Omega^k_{\mathrm{pr}}(M, D)$. The graded vector space of projectable differential forms is defined by

$$\Omega^\bullet_{\mathrm{pr}}(M, D) := \bigoplus_{k \in \mathbb{Z}} \Omega^k_{\mathrm{pr}}(M, D).$$

It is clear by equation (2.15) that $\Omega^k_{\mathrm{pr}}(M, D) \subset \Omega^k_{D^0}(M)$. Moreover, (2.15), (2.16) are equivalent to $\mathrm{i}_Y\alpha = 0$ and $\mathcal{L}_Y\alpha = 0$. Projectable differential forms satisfy:

1. $\Omega^\bullet_{\mathrm{pr}}(M, D)$ is a graded exterior subalgebra of $(\Omega_{D^0}(M), \wedge)$.
2. $\Omega^\bullet_{\mathrm{pr}}(M, D)$ is a cochain sub-complex of (Ω_M, d).
3. $\alpha \in \Omega^k_{D^0}(M)$ is projectable if and only if, for any $X_i \in \mathfrak{X}_{\mathrm{pr}}(M, D)$,
$$\alpha(X_1, \dots, X_k) \in C^\infty_{\mathrm{pr}}(M, D).$$
4. $\Omega^0_{\mathrm{pr}}(M, D) = C^\infty_{\mathrm{pr}}(M, D)$.
5. $\Omega^\bullet_{\mathrm{pr}}(M, D)$ is a $C^\infty_{\mathrm{pr}}(M, D)$-submodule of Ω_M.

First property follows from the fact that, for any $Y \in \Gamma \mathrm{D}$, i_Y and $\mathcal{L}_Y$ are graded derivations of the exterior product. Second property is immediate. Third property follows from equation (2.4). The last two properties follows from the previous ones. Moreover, projectable differential forms locally generate horizontal differential forms.

Remark 2.6.3. *When the choice of the involutive distribution D in M is clear, we simplify the notation referring to projectable structures. We respectively denote $C^\infty_{\mathrm{pr}}(M)$, $\mathfrak{X}_{\mathrm{pr}}(M)$ and $\Omega_{\mathrm{pr}}(M)$ the sets of projectable functions, vector fields and projectable differential forms.*

Now, we review more properties of projectable structures when those are defined on a fiber bundle, and the involutive distribution is the vertical distribution.

Let (E, π, B) be a fiber bundle, and $\mathbb{V}$ its vertical subbundle. It can be shown, for instance, that a smooth function $f \in C^\infty_E$ is projectable if and only if there exists $g \in C^\infty_B$ such that $f = g \circ \pi$, i.e., $C^\infty_{\mathrm{pr}}(E) = \pi^* C^\infty_B$. In this case, we denote by $\pi_* f$ the unique $g \in C^\infty_B$ satisfying $f = g \circ \pi$. It is clear that $\pi_* : C^\infty_{\mathrm{pr}}(E) \longrightarrow C^\infty_B$ is a ring isomorphism. Also, $X \in \mathfrak{X}_E$ is projectable if and only if it is π-related to a vector field $u \in \mathfrak{X}_B$, i.e., $\pi_* \circ X = u \circ \pi$.

We will see that projectable differential forms $\Omega_{\mathrm{pr}}(E)$ actually *project* to differential forms in the base space. Before that, we need to review the concepts of Ehresmann connection and horizontal lift.

Ehresmann connections. An Ehresmann connection in a fiber bundle is a generalized connection such that its vertical distribution is the vertical subbundle $\mathbb{V} = \ker(\pi_*)$. Equivalently, it is a choice of a complementary distribution to the vertical subbundle.

Definition 2.6.4. *An **Ehresmann connection** in the fiber bundle (E, π, B) is a generalized connection γ in E such that its vertical distribution coincides with the vertical subbundle of (E, π, B). In other words, is a vector bundle morphism $\gamma : TM \longrightarrow TM$ satisfying $\gamma^2 = \gamma$ and $\mathrm{Im}(\gamma) = \mathbb{V}$, where $\mathbb{V} = \ker(\pi_*)$. Equivalently, an Ehresmann connection is a distribution $\mathbb{H}$ complementary to $\mathbb{V}$: $TE = \mathbb{H} \oplus \mathbb{V}$.*

By means of an Ehresmann connection one can define the *horizontal lift* of vector fields in the base space to the total space. In particular, this allows us to project horizontal projectable differential forms. Since $\mathbb{V} = \ker(\pi_*)$, the restriction $\pi_*|_{\mathbb{H}} : \mathbb{H} \longrightarrow TB$ is a vector bundle isomorphism over π. This induces an injective morphism of modules on their sections, $\mathrm{hor}^\gamma : (\mathfrak{X}_B, C^\infty_B) \longrightarrow (\Gamma\mathbb{H}, C^\infty_{\mathrm{pr}}(E))$, called the **horizontal lift** of vector fields. For each $u \in \mathfrak{X}_B$ the vector field $\mathrm{hor}^\gamma(u) \in \Gamma\mathbb{H} \subset \mathfrak{X}_E$ is said to be the *horizontal lift* of $u \in \mathfrak{X}_B$. Point-wisely, it is given by

$$\mathrm{hor}^\gamma(u)_p := (\pi_*|_{\mathbb{H}})^{-1}(u_{\pi(p)}).$$

Note that $\mathrm{hor}^\gamma(u)$ and u are π-related. Moreover, $\mathrm{hor}^\gamma(u)$ is the only horizontal vector field π-related to u. Therefore, any horizontal projectable vector field is the horizontal lift of some vector field in the base space: $\mathrm{hor}^\gamma(\mathfrak{X}_B) = \mathfrak{X}_{\mathrm{pr}}(E) \cap \Gamma\mathbb{H}$. Since $\mathrm{hor}^\gamma : \mathfrak{X}_B \longrightarrow \Gamma\mathbb{H}$ is a morphism of modules, we have

$$\mathrm{hor}^\gamma(gu) = (g \circ \pi)\mathrm{hor}^\gamma(u). \tag{2.17}$$

Remark 2.6.5. *Let D and $\tilde{D}$ be distributions in M, such that D is involutive and $TM = D \oplus \tilde{D}$. The $\mathbb{R}$-vector space of projectable vector fields tangent to $\tilde{D}$ is*

$$\mathfrak{X}^{\tilde{D}}(M, D) := \{X \in \Gamma\tilde{D} \mid [X, \Gamma D] \subset \Gamma D\} = \mathfrak{X}(M, D) \cap \Gamma\tilde{D}.$$

When the choice of the distribution D is clear, we simply denote it by $\mathfrak{X}^{\tilde{D}}(M)$.

Recall that $\Omega^\bullet_{\mathrm{pr}}(E)$ and Ω_B are modules over $C^\infty_{\mathrm{pr}}(E)$ and C^∞_B, respectively, and $\pi_* : C^\infty_{\mathrm{pr}}(E) \longrightarrow C^\infty_B$ is a ring morphism. For an Ehresmann connection γ, define $\pi^\gamma_* : \Omega^\bullet_{\mathrm{pr}}(E) \longrightarrow \Omega^\bullet_B$ by

$$(\pi^\gamma_*\alpha)(u_1, \ldots, u_k) := \pi_*(\alpha(\mathrm{hor}^\gamma u_1, \ldots, \mathrm{hor}^\gamma u_1)).$$

First, observe that $\pi^\gamma_*\alpha$ is well-defined, in virtue of Property 3 of projectable differential forms in page 40. On the other hand, since α is skew-symmetric, $\pi^\gamma_*\alpha$ is also skew-symmetric. Finally, the C^∞_B-linearity of $\pi^\gamma_*\alpha$ follows from equation (2.17), the $C^\infty_{\mathrm{pr}}(E)$-linearity of α, and the fact that $\pi_* : C^\infty_{\mathrm{pr}}(E) \longrightarrow C^\infty_B$ is a ring morphism. Furthermore, the projection $\pi^\gamma_* : \Omega^\bullet_{\mathrm{pr}}(E) \longrightarrow \Omega^\bullet_B$ is a canonical exterior algebra isomorphism, i.e., it is bijective and it does not depend on the choice of the Ehresmann connection γ used to define it. This isomorphism can be naturally extended by tensor product to

$$\pi^\gamma_* : \Omega^\bullet_{\mathrm{pr}}(E) \otimes_{C^\infty_{\mathrm{pr}}(E)} \mathcal{A} \longrightarrow \Omega^\bullet_B \otimes_{C^\infty_B} \mathcal{A},$$

where $\mathcal{A}$ is any C^∞_B-module. In particular, it can be extended to *horizontal differential forms* $\Omega_{\mathbb{V}^0}(E) := \Gamma \bigwedge \mathbb{V}^0 \simeq \Omega_{\mathrm{pr}}(E) \otimes C^\infty_E$ by

$$\pi^\gamma_* : \Gamma \bigwedge \mathbb{V}^0 \longrightarrow \Omega^\bullet_B \otimes_{C^\infty_B} C^\infty_E,$$

and to *vertical-valued horizontal forms* $K \in \Omega^{k,0}(E; \mathbb{V})$ by

$$\pi^\gamma_* K(u_1, \ldots, u_k) = K(\mathrm{hor}^\gamma u_1, \ldots, \mathrm{hor}^\gamma u_k) \in \Gamma\mathbb{V}.$$

Using the above discussion, we are in position to present the standard interpretation of curvature in the context an Ehresmann connections on fiber bundles. Recall that the curvature of a generalized connection is defined as the vertical component of the Lie bracket of the horizontal components of vector fields (see (2.11)). If γ is an Ehresmann connection in the fiber bundle (E, π, B), then its curvature R defines a vertical-valued horizontal 2-form, $R \in \Omega^2(E; \mathbb{V})$, which can be considered as an element of $\Omega^2_{\mathrm{pr}}(E) \otimes_{C^\infty_{\mathrm{pr}}(E)} \Gamma\mathbb{V}$. Now, fix $u, v \in \mathfrak{X}_B$. Since $\mathrm{hor}^\gamma u$ is π-related to u, and $\mathrm{hor}^\gamma v$ is π-related to v, their Lie brackets are also π-related: $[\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v]$ is π-related to $[u, v]$. On the other hand, $\mathrm{hor}^\gamma[u, v]$ is π-related to $[u, v]$. Therefore, $[\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v] - \mathrm{hor}^\gamma[u, v] \in \Gamma \ker(\pi_*) = \Gamma\mathbb{V}$, and

$$[\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v] - \mathrm{hor}^\gamma[u, v] = R(\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v).$$

Definition 2.6.6. *The **curvature of an Ehresmann connection** γ, $\mathrm{Curv}^\gamma \in \Omega^2_B \otimes_{C^\infty_B} \Gamma\mathbb{V}$, is defined by*

$$\mathrm{Curv}^\gamma(u, v) := R(\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v) = [\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v] - \mathrm{hor}^\gamma[u, v].$$

Remark 2.6.7. *One can derive the transition rules under varying the Ehresmann connection of the horizontal lift and the curvature. Let γ and $\tilde{\gamma}$ be two Ehresmann connections in (E, π, B) and define $\Theta \in \Omega^1(E; \mathbb{V})$ by $\Theta := \gamma - \tilde{\gamma}$. It is clear that $\mathrm{Im}(\Theta) \subset \mathbb{V} \subset \ker(\Theta)$. So, $\Theta^2 = 0$. Moreover, it can be shown that*

$$\begin{aligned} \mathrm{hor}^\gamma(u) - \mathrm{hor}^{\tilde{\gamma}}(u) = -\Theta(\mathrm{hor}^\gamma(u)) = -\Theta(\mathrm{hor}^{\tilde{\gamma}}(u)), \\ R = \tilde{R} + [\Theta, \gamma_2]_{FN} + \tfrac{1}{2}[\Theta, \Theta]_{FN}. \end{aligned}$$

This implies that $\pi^\gamma_*(\Theta) = \mathrm{hor}^{\tilde{\gamma}} - \mathrm{hor}^\gamma$.

Chapter 3

Bigrading of the Lichnerowicz - Poisson Operator

In this chapter we present the concept of coupling Poisson structure in foliated manifolds. The main goal is, given a coupling Poisson structure in a fiber bundle, to define a cochain complex isomorphism between its Lichnerowicz - Poisson complex and a bigraded cochain complex defined by its associated geometric data.

We begin with a review of some basics in Poisson manifolds, specially the notions of Hamiltonian vector field, infinitesimal Poisson automorphism, Casimir function, Lichnerowicz - Poisson complex and low - dimensional cohomology groups. Then, we present the concept of coupling Poisson structure, its relation to geometric data, and the *factorization* of the Jacobi identity into four integrability equations. Finally, we show that integrable geometric data in fiber bundles define a bigraded cochain complex. Furthermore, if the geometric data correspond to a coupling Poisson structure, then the induced bigraded cochain complex is isomorphic to the Lichnerowicz Poisson complex. This main result is presented and proved in Theorem 3.4.2. For various presentations of this result, see also [12, 11, 6, 18].

3.1 Preliminary on Poisson manifolds

A Poisson manifold is a pair $(M, \{,\})$, where M is a differential manifold and $\{,\} : C^\infty_M \times C^\infty_M \longrightarrow C^\infty_M$ is an $\mathbb{R}$-bilinear skew-symmetric operation, called **Poisson bracket**, satisfying the Jacobi identity

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\} \tag{3.1}$$

and the Leibniz rule

$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\}. \tag{3.2}$$

Leibniz rule means that the Poisson bracket is a bi-derivation of C^∞_M. So, there exists a bivector field $\Pi \in \chi^2_M$ defined by $\Pi(\mathrm{d}f, \mathrm{d}g) := \{f, g\}$. Note that

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\} - \frac{1}{2}[\Pi, \Pi](\mathrm{d}f, \mathrm{d}g, \mathrm{d}h),$$

where $[,] : \chi_M \times \chi_M \longrightarrow \chi_M$ is the Schouten - Nijenhuis bracket (see Definition 2.4.2). Therefore, a Poisson bracket $\{,\}$ in M is equivalent to have a bivector field Π satisfying

$$[\Pi, \Pi] = 0. \tag{3.3}$$

Such a bivector field Π is said to be a **Poisson structure** or a **Poisson bivector field** in M. The Poisson manifold is also denoted by (M, Π).

Symplectic foliations on Poisson manifolds. For any bivector field $\Pi \in \chi^2_M$, there is an induced vector bundle morphism $\Pi^\sharp : T^*M \longrightarrow TM$ defined by

$$\langle \beta, \Pi^\sharp_p(\alpha) \rangle := \Pi_p(\alpha, \beta) \qquad \forall\ \alpha, \beta \in T^*_p M, p \in M. \tag{3.4}$$

Furthermore, the map $\sharp_\Pi : \Gamma \bigwedge T^*M \longrightarrow \Gamma \bigwedge TM$ given by $(\sharp_\Pi \omega)(\alpha^1, \dots, \alpha^k) := \omega(\Pi^\sharp \alpha^1, \dots, \Pi^\sharp \alpha^k)$ is an exterior algebra morphism. Note that $\Pi^\sharp(T^*M)$ is a distribution in M, whose dimension at each point $\Pi^\sharp_p(T^*_p M)$, $p \in M$, may vary. In the case in which Π is a Poisson structure, one can associate to each $f \in C^\infty_M$ a vector field

$$X_f := \Pi^\sharp(\mathrm{d}f) = \mathrm{i}_{\mathrm{d}f}\Pi = -[f, \Pi] = -[\Pi, f]$$

called the **Hamiltonian vector field** of f. The $\mathbb{R}$-vector space of Hamiltonian vector fields is denoted by $\mathrm{Ham}(M, \Pi)$, and it turns out to be a Lie $\mathbb{R}$-subalgebra of $\mathfrak{X}_M$ since $[X_f, X_g] = X_{\{f,g\}}$.

Note that Hamiltonian vector fields generate the distribution $D^\Pi := \Pi^\sharp(T^*M)$, which is called **characteristic distribution**. It is well-known that D^Π is, in general, integrable in the sense of Stefan - Sussmann [28, 27]. Furthermore, it can be shown that, for an arbitrary bivector field Π, the distribution $\Pi^\sharp(T^*M)$ is integrable if and only if Π is a Poisson structure.

Therefore, a Poisson structure Π in M induces a (singular) foliation $\mathcal{S}$ given by $T\mathcal{S} := D^\Pi$. Moreover, the Poisson structure can be restricted at each leaf S, determining a non-degenerate Poisson structure at S. In consequence, each leaf S is endowed with a symplectic structure ω_S given by $\omega_S(X_f|_S, X_g|_S) = \Pi(\mathrm{d}f, \mathrm{d}g)|_S$, defining a leafwise symplectic structure ω. The pair $(\mathcal{S}, \omega)$ is called **symplectic foliation**. The leaves of $\mathcal{S}$ are called **symplectic leaves**, which are generated by flows of Hamiltonian vector fields.

A point $x \in M$ in the Poisson manifold (M, Π) is called **regular** if $\mathrm{rank}\Pi$ is locally constant at x; in other case, x is said to be **singular**. A symplectic leaf $S \in \mathcal{S}$ is called **regular** if every $x \in S$ is a regular point. Otherwise, S is called **singular**. A Poisson manifold (M, Π) is called **regular** if every $x \in M$ is regular; otherwise, (M, Π) is called **singular**.

Let $(M, \{,\})$ be an m-dimensional Poisson manifold. It is well-known [35] that, for each $x \in M$ of rank $2s$ and a $(m-2s)$-dimensional submanifold N of M, with $x \in N$, there exists a local system of coordinates $(p_1, \dots, p_s, q_1, \dots, q_s, z_1, \dots, z_{m-2s})$ around x such that the Poisson bracket has the *normal form*

$$\{f, g\} = \sum_{i,j=1}^{m-2s} \{z_i, z_j\} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} + \sum_{i=1}^{s} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i},$$

and the following conditions are satisfied:

(a) $p_i(N_x) = q_i(N_x) = 0$, where N_x is a small neighborhood of x in N,

(b) $\{z_i, z_j\}(x) = 0$.

Observe that $\{f, g\}_N := \sum_{i=1}^{s} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$ defines a Poisson structures in N_x, called *transverse Poisson structure* [4]. In other words, (M, Π) is locally isomorphic to the direct product of a symplectic manifold $(S, \sum_{i=1}^{s} \mathrm{d}p_i \wedge \mathrm{d}q_i)$ with a Poisson manifold $(N_x, \{,\}_N)$, where $\{,\}_N$ vanishes at x.

For a Poisson manifold (M, Π), the set of regular points $N^{reg} = \{p \in M \mid p \text{ is regular}\}$ is an open dense subset of M. Thus, one can restrict the Poisson structure to N^{reg} getting the regular Poisson manifold $(N^{reg}, \Pi_{N^{reg}})$. Moreover, $\mathrm{rank}\Pi$ is constant on each connected component of N^{reg}.

Poisson cohomology. Note that, if a vector field Z is Hamiltonian, then

$$\mathcal{L}_Z \Pi = 0. \tag{3.5}$$

In general, a vector field Z satisfying (3.5) is called **infinitesimal Poisson automorphism** or **Poisson vector field**. The $\mathbb{R}$-vector space of Poisson vector fields is denoted by $\mathrm{Poiss}(M, \Pi)$. A vector field Z is an infinitesimal Poisson automorphism if and only if its flow $\mathrm{Fl}_Z^t : M \longrightarrow M$ satisfies $(\mathrm{Fl}_Z^t)^*\Pi = \Pi$. Equivalently, a vector field Z is an infinitesimal automorphism if and only if $\mathcal{L}_Z \in \mathrm{Der}_{\mathbb{R}}(C^\infty_M, \{,\})$.

A function $K \in C^\infty_M$ is called **Casimir** for Π if its Hamiltonian vector field is zero, $X_K = 0$. The $\mathbb{R}$-algebra of Casimir functions is denoted by $\mathrm{Casim}(M, \Pi)$. It is easy to check that Poisson vector fields preserve the subspace of Casimir functions.

If (M, Π) is a Poisson manifold, then there is an operator $\delta^\Pi : \chi_M \longrightarrow \chi_M$ defined as the adjoint of the Poisson bivector Π respect to the Schouten - Nijenhuis bracket: $\delta^\Pi(A) := [\Pi, A]$. This is called **Lichnerowicz - Poisson operator**. In virtue of the Leibniz rule and the Jacobi identity of the Schouten - Nijenhuis bracket, the Lichnerowicz - Poisson operator is a graded derivation for the exterior product and the Schouten - Nijenhuis bracket of multivector fields,

$$\delta^\Pi \in \mathrm{Der}^1_{\mathbb{R}}(\chi_M, \wedge, [,]). \tag{3.6}$$

Moreover, δ^Π is a coboundary operator [16], because of (3.3) and the Jacobi identity for the Schouten - Nijenhuis bracket. Thus, (χ_M, δ^Π) is a cochain complex called the **Lichnerowicz-Poisson complex**. The cocycles, coboundaries and cohomology spaces are denoted by $\mathcal{Z}^k_{LP}(M, \Pi)$, $\mathcal{B}^k_{LP}(M, \Pi)$ and $\mathcal{H}^k_{LP}(M, \Pi)$, respectively.

Because of (3.6), it turns out that cocycles

$$\mathcal{Z}^\bullet_{LP}(M, \Pi) := \bigoplus_{k \in \mathbb{Z}} \mathcal{Z}^k_{LP}(M, \Pi)$$

form a graded Poisson $\mathbb{R}$-subalgebra of $(\chi_M, \wedge, [,])$. Furthermore, coboundaries

$$\mathcal{B}^\bullet_{LP}(M,\Pi) := \bigoplus_{k\in\mathbb{Z}} \mathcal{B}^k_{LP}(M,\Pi)$$

form an ideal of $\mathcal{Z}^\bullet_{LP}(M,\Pi)$, for both operations $\wedge$ and $[,]$. Hence, there is a well-defined product an bracket in

$$\mathcal{H}^\bullet_{LP}(M,\Pi) := \bigoplus_{k\in\mathbb{Z}} \mathcal{H}^k_{LP}(M,\Pi)$$

given by $[A]\wedge[B] := [A\wedge B]$ and $\big[[A],[B]\big] := [[A,B]]$, which endow $\mathcal{H}^\bullet_{LP}(M,\Pi)$ with a Poisson $\mathbb{R}$-algebra structure of degree -1.

It is also well - known that a Poisson structure Π in the manifold M induces a Lie algebroid structure on the cotangent bundle T^*M. Indeed, for $\alpha,\beta\in\Omega^1_M$, define $\{\alpha,\beta\}_\Pi := \mathcal{L}_{\Pi^\sharp\alpha}\beta - \mathcal{L}_{\Pi^\sharp\beta}\alpha - \mathrm{d}\Pi(\alpha,\beta) = \mathrm{i}_{\Pi^\sharp\alpha}\mathrm{d}\beta - \mathrm{i}_{\Pi^\sharp\beta}\mathrm{d}\alpha + \mathrm{d}\Pi(\alpha,\beta)$. It can be shown that the triple $(T^*M, \Pi^\sharp, \{,\}_\Pi)$ is a Lie algebroid. Furthermore, its Lie algebroid differential is precisely the Lichnerowicz - Poisson complex. In other words, for any $A\in\chi^k_M$ and $\alpha_0,\dots,\alpha_k\in\Omega^1_M$, we have

$$\begin{aligned}(\delta^\Pi A)(\alpha_0,\dots,\alpha_k) = &\sum_{i=0}^{k}(-1)^i\mathcal{L}_{\Pi^\sharp(\alpha_i)}(A(\alpha_0,\dots,\widehat{\alpha_i},\dots,\alpha_k))\\ &+\sum_{i<j}(-1)^{i+j}A(\{\alpha_i,\alpha_j\}_\Pi,\alpha_0,\dots,\widehat{\alpha_i},\dots,\widehat{\alpha_j},\dots,\alpha_k).\end{aligned}$$

In consequence, the map $\sharp_\Pi : (\Omega_M, \mathrm{d}) \longrightarrow (\chi_M, \delta^\Pi)$ is a cochain complex morphism: $\sharp_\Pi\mathrm{d}\alpha = \delta^\Pi\sharp_\Pi\alpha$. Hence, this induces a cohomology morphism $\sharp^*_\Pi : H^k_{dR}(M) \longrightarrow \mathcal{H}^k_{LP}(M,\Pi)$ by $\sharp^*_\Pi[\alpha] := [\sharp_\Pi\alpha]$. If Π is a regular Poisson structure, then the map $\sharp^*_\Pi$ is a morphism to the *tangential Poisson cohomology*, which we present in Chapter 4.

On a Poisson manifold (M,Π), there is an homology operator on the algebra of differential forms. This is called the **Koszul - Brylinski operator** and is given by

$$\mathcal{L}_\Pi : \Omega_M \longrightarrow \Omega_M, \qquad \mathcal{L}_\Pi\alpha := [\mathrm{i}_\Pi, \mathrm{d}](\alpha) = \mathrm{i}_\Pi\mathrm{d}\alpha + \mathrm{d}\mathrm{i}_\Pi\alpha.$$

It is clear by its definition that $\mathcal{L}_\Pi \in \mathcal{D}^{-1}_2(M)$. Furthermore, the Koszul - Brylinski operator is an homology operator. Indeed, since Π is a Poisson structure, $\mathcal{L}^2_\Pi = \frac{1}{2}[\mathcal{L}_\Pi,\mathcal{L}_\Pi] = \frac{1}{2}\mathcal{L}_{[\Pi,\Pi]} = 0$. Also, the Koszul - Brylinski operator is a *generator* of the bracket $\{,\}_\Pi$ for 1-forms. Indeed, using the language of differential operators, for any $\alpha,\beta\in\Omega^1_M$ we have

$$[[\mu_\alpha,\mathcal{L}_\Pi],\mu_\beta](\mathbb{1}) = \mathcal{L}_\Pi\beta\cdot\alpha + \mathcal{L}_\Pi(\alpha\wedge\beta) - \mathcal{L}_\Pi\alpha\cdot\beta = \mathrm{i}_{\Pi^\sharp\alpha}\mathrm{d}\beta - \mathrm{i}_{\Pi^\sharp\beta}\mathrm{d}\alpha + \mathrm{d}\Pi(\alpha,\beta) = \{\alpha,\beta\}_\Pi.$$

Furthermore, for any $\alpha,\beta\in\Omega_M$, define $\{\!\!\{,\}\!\!\}_\Pi : \Omega_M\times\Omega_M \longrightarrow \Omega_M$ by

$$\{\!\!\{\alpha,\beta\}\!\!\}_\Pi := [[\mu_\alpha,\mathcal{L}_\Pi],\mu_\beta](\mathbb{1}) = -(-1)^{|\alpha|}\mathcal{L}_\Pi(\alpha\wedge\beta) + (-1)^{|\alpha|}\mathcal{L}_\Pi\alpha\wedge\beta + \alpha\wedge\mathcal{L}_\Pi\beta.$$

It is clear that $\{\!\{,\}\!\}_\Pi$ extends the bracket of 1-forms $\{,\}_\Pi$. Moreover, $\{\!\{,\}\!\}_\Pi$ is a graded Lie bracket in Ω_M and satisfies the Leibniz rule with the exterior product. In other words, the triple $(\Omega_M, \wedge, \{\!\{,\}\!\}_\Pi)$ is the graded Poisson algebra of degree -1, extending the bracket of 1-forms.

Finally, it is clear that $\mathrm{d} \circ \mathcal{L}_\Pi + \mathcal{L}_\Pi \circ \mathrm{d} = [\mathrm{d}, \mathcal{L}_\Pi] = 0$. Hence, $(\Omega_M, \mathrm{d}, \mathcal{L}_\Pi)$ is a bi-complex. Furthermore, in some special cases, there exists duality on the Poisson homology and cohomology groups.

Below, we describe the low-dimensional Poisson cohomology groups.

- If $k = 0$, $\mathcal{H}^0(M, \Pi) = \mathrm{Casim}(M, \Pi)$.
- If $k = 1$, $\mathcal{H}^1(M, \Pi) = \frac{\mathrm{Poiss}(M,\Pi)}{\mathrm{Ham}(M,\Pi)}$.
- If $\Lambda \in \mathcal{Z}^2_{LP}(M, \Pi)$, then $[\Pi, \Lambda] = 0$. So, $\Pi + \epsilon\Lambda$ is a Poisson structure up to terms of degree ϵ^2:

$$[\Pi + \epsilon\Lambda, \Pi + \epsilon\Lambda] = \epsilon^2[\Lambda, \Lambda] = 0 \mod \epsilon^2.$$

  Therefore, $\Pi + \epsilon\Lambda$ is an *infinitesimal deformation* of the Poisson structure Π. If $\Lambda \in \mathcal{B}^2_{LP}(M, \Pi)$, there exists $X \in \mathfrak{X}_M$ such that $\Lambda = [\Pi, X]$. So,

$$\Pi + \epsilon\Lambda = \Pi + \epsilon[\Pi, X] = (\mathrm{Fl}^\epsilon_X)_*\Pi \mod \epsilon^2.$$

  Hence, the second Poisson cohomology $\mathcal{H}^2_{LP}(M, \Pi)$ is the quotient of all possible infinitesimal deformations of Π by the space of trivial deformations.

In general, the computation of the Poisson cohomology groups is a hard problem [38, 39, 32, 8], especially in the case of singular Poisson structures [5, 22, 23, 24, 25].

3.2 Poisson structures on foliated manifolds

In this part, we present the concept of *coupling Poisson structure* on foliated manifolds, and describe the Jacobi identity in terms of its associated geometric data. This concept naturally arises in the study of a Poisson structure around closed symplectic leaves [37] and in the context of Poisson vector bundles [36], but it can also be defined for any regular foliated manifold [33]. We show that, in a foliated manifold $(M, \mathcal{F})$, coupling Poisson structures Π can be parameterized by triples (γ, σ, P) consisting on a connection γ with vertical distribution $\mathbb{V} = T\mathcal{F}$, a horizontally non-degenerate 2-form σ, and bivector field P. Such triples are called *geometric data*. The Jacobi identity for Π can be expressed in terms of four integrability equations for (γ, σ, P), which arise from bigraded calculus in the manifold.

In the following sections, we prove that the Lichnerowicz - Poisson complex of a coupling Poisson structure Π in a fiber bundle (E, π, B) is isomorphic to a bigraded cochain complex intrinsically defined by the fiber bundle and the geometric data (γ, σ, P) associated to Π. This allows us to study the first cohomology group of coupling Poisson structures in fiber bundles, which is done in Chapter 4.

3.2.1 Coupling bivector fields and geometric data

Let $(M, \mathcal{F})$ be a foliated manifold. Because of Frobenius Theorem, there is a correspondence between regular foliations and involutive distributions, i.e., $\mathbb{V} := T\mathcal{F}$ is an involutive distribution. In this section we present the concepts of coupling Poisson structure and geometric data on pairs $(M, \mathbb{V})$, with $\mathbb{V}$ a regular and involutive distribution in M.

Recall that, for a regular foliation $\mathcal{F}$ (or vertical distribution $\mathbb{V}$), a local vector field $X \in \mathfrak{X}_M$ is called *projectable* if its flow Fl_X^t takes leaves of $\mathcal{F}$ into leaves of $\mathcal{F}$. In infinitesimal terms, this is equivalent to $[X, \Gamma\mathbb{V}] \subset \Gamma\mathbb{V}$. The sheaf of local projectable vector fields is denoted by $\mathfrak{X}_{\mathrm{pr}}(M)$. Some useful properties on projectable vector fields can be found in Section 2.6. In the following sections, we will be focused on the case of coupling Poisson structures on fiber bundles. In that case, the space of projectable vector fields coincides with the *horizontal lifts* of vector fields in the base space.

For the rest of this section, M is a differential manifold and $\mathbb{V}$ is a regular involutive distribution on M.

Definition 3.2.1. *A bivector field $\Pi \in \chi_M^2$ is called a* ***coupling bivector field*** *on $(M, \mathbb{V})$ if*

$$\Pi^\sharp(\mathbb{V}^0) \oplus \mathbb{V} = TM, \tag{3.7}$$

where $\mathbb{V}^0 \subset T^*M$ is the annihilator of $\mathbb{V}$. A ***coupling Poisson structure*** *Π in $(M, \mathbb{V})$ is a bivector field in M which is both a Poisson structure on M and a coupling bivector field on $(M, \mathbb{V})$, i.e., equations* (3.7) *and* $[\Pi, \Pi] = 0$ *are satisfied.*

The notion of coupling Poisson structure associated to an involutive distribution is deeply related to the one of geometric data. Indeed, coupling bivector fields are parameterized by geometric data, and the Jacobi identity can be described by the *integrability equations* for geometric data.

Recall that a generalized connection γ in M is a vector bundle morphism $\gamma : TM \longrightarrow TM$ satisfying $\gamma^2 = \gamma$. For the rest of this section, we only consider connections γ in $(M, \mathbb{V})$ such that $\mathrm{Im}(\gamma) = \mathbb{V}$, i.e., the vertical distribution of γ coincides with the regular involutive distribution $\mathbb{V}$. In this case, the kernel of γ is also a regular distribution $\mathbb{H}$, called *horizontal distribution*, satisfying $TM = \mathbb{H} \oplus \mathbb{V}$. We denote by $\mathfrak{X}_{\mathrm{pr}}^{\mathbb{H}}(M)$ the space of *horizontal projectable vector fields:* $\mathfrak{X}_{\mathrm{pr}}^{\mathbb{H}}(M) := \Gamma\mathbb{H} \cap \mathfrak{X}_{\mathrm{pr}}(M)$. Also, the *curvature 2-form* $R \in \Omega^2(M; \mathbb{V})$ is given by

$$R(X, Y) := \gamma[(\mathrm{Id}_{TM} - \gamma)X, (\mathrm{Id}_{TM} - \gamma)Y]$$

and measures the integrability of $\mathbb{H}$ (see Section 2.5).

Definition 3.2.2. *A triple (γ, σ, P) of* ***geometric data*** *on $(M, \mathbb{V})$ consists on*

- *a connection γ in M with vertical distribution $\mathbb{V}$,*
- *a horizontal 2-form σ: $\sigma \in \Gamma \bigwedge^2 \mathbb{V}^0$,*
- *a vertical bivector field P: $P \in \Gamma \bigwedge^2 \mathbb{V}$.*

Furthermore, the geometric data (γ, σ, P) are said to be ***integrable*** *if they satisfy the following equations:*

$$[P, P] = 0, \tag{3.8}$$
$$\mathcal{L}_X P = 0, \qquad \forall X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M), \tag{3.9}$$
$$R(X, Y) = -P^\sharp \mathrm{d}[\sigma(X, Y)], \qquad \forall X, Y \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M), \tag{3.10}$$
$$\mathrm{d}\sigma(X, Y, Z) = 0, \qquad \forall X, Y, Z \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M). \tag{3.11}$$

Equations (3.8)-(3.11) *are called* ***integrability equations*** *of (γ, σ, P).*

Note that equation (3.8) simply means that P is a *vertical Poisson structure*. In terms of the regular foliation $\mathcal{F}$ associated to $\mathbb{V}$, we say that $(\mathcal{F}, P)$ is a *Poisson foliation*, i.e., P induces a Poisson structure at each leaf of $\mathcal{F}$. Equation (3.9) means that γ is a *Poisson connection* on $(\mathcal{F}, P)$. This is precisely $\mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M) \subset \mathrm{Poiss}(M, P)$.

It can be shown that if γ is a Poisson connection in $(\mathcal{F}, P)$, then the curvature of projectable vector fields values on vertical Poisson vector fields:

$$R(X, Y) \in \mathrm{Poiss}_{\mathbb{V}}(M, P) \qquad \forall X, Y \in \mathfrak{X}_{\mathrm{pr}}(M).$$

Indeed, since $\Gamma\mathbb{V} \subset \mathfrak{X}_{\mathrm{pr}}(M)$, it follows that $(\mathrm{Id}_{TM} - \gamma)\mathfrak{X}_{\mathrm{pr}}(M) \subset \mathfrak{X}_{\mathrm{pr}}(M)$. Taking in account that $\mathfrak{X}_{\mathrm{pr}}(M)$ is a Lie subalgebra of vector fields, we have

$$(\mathrm{Id}_{TM} - \gamma)[(\mathrm{Id}_{TM} - \gamma)X, (\mathrm{Id}_{TM} - \gamma)Y] \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M) \qquad \forall X, Y \in \mathfrak{X}_{\mathrm{pr}}(M).$$

Since γ is a Poisson connection, this implies that the vector fields

$$(\mathrm{Id}_{TM} - \gamma)[(\mathrm{Id}_{TM} - \gamma)X, (\mathrm{Id}_{TM} - \gamma)Y], \qquad [(\mathrm{Id}_{TM} - \gamma)X, (\mathrm{Id}_{TM} - \gamma)Y],$$

are infinitesimal automorphisms. Their difference is precisely $R(X, Y)$, which is clearly a vertical infinitesimal automorphism: $R(X, Y) \in \mathrm{Poiss}_{\mathbb{V}}(M, P)$.

Thus, the vector fields $R(X, Y)$, with $X, Y \in \mathfrak{X}_{\mathrm{pr}}(M)$, are a special family of vertical Poisson vector fields for P. In this sense, equation (3.10) means that those are Hamiltonians for P, and the vector field $R(X, Y)$ has $-\sigma(X, Y)$ as Hamiltonian. Finally, equation (3.11) means that σ is *covariantly constant*. This is clarified in Proposition 3.3.3, where we present an equation in terms of the *covariant exterior differential*, equivalent integrability equation (3.11) for σ.

Now, we present a generalization of the integrability equations (3.9) and (3.10).

Lemma 3.2.3. *Let γ be a Poisson connection for $(\mathcal{F}, P)$, i.e., equation* (3.9) *is satisfied. Then $(\mathcal{L}_Z P)_{0,2} = 0$ for any $Z \in \Gamma\mathbb{H}$.*

Proof. Since horizontal vector fields are locally generated by horizontal projectable vector fields, it is suffices to prove that the identity holds for $Z = fX$, with $f \in C^\infty_M$, and $X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$. Note that $\mathcal{L}_Z P = [fX, P] = f[X, P] + \mathrm{i}_{\mathrm{d}f} P \wedge X$. From equation (3.9), we have $[X, P] = 0$. So, $\mathcal{L}_Z P = \mathrm{i}_{\mathrm{d}f} P \wedge X \in \chi^{1,1}_M$. In particular, the component of $\mathcal{L}_Z P$ of bidegree $(0, 2)$ is zero: $(\mathcal{L}_Z P)_{0,2} = 0$. ■

Lemma 3.2.4. *Let (γ, σ, P) be geometric data in $(M, \mathbb{V})$ satisfying equation* (3.10)*. If $X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$ and $Z \in \Gamma\mathbb{H}$, then*

$$P^\sharp \mathrm{d}\,[\sigma(Z, X)] + P^\sharp \mathcal{L}_Z(\sigma^\flat X) + R(Z, X) = 0.$$

Proof. It is enough to prove this result for $Z = fY$, with $f \in C^\infty_M$ and $Y \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$. First, note that, for any $\alpha \in \Gamma\mathbb{V}^0$ and $W \in \Gamma\mathbb{V}$,

$$\mathcal{L}_Y \alpha(W) = \mathcal{L}_Y(\alpha(W)) - \alpha[Y, W].$$

Since Y is projectable, then $[Y, W]$ is vertical. Therefore, both summands in the right-hand side are zero, due to the horizontality of α. Ths proves that $\mathcal{L}_Y \alpha(W) = 0$ for all $W \in \Gamma\mathbb{V}$. Hence, it follows that $P^\sharp \mathcal{L}_Y \alpha = 0$, due to $P \in \Gamma \bigwedge^2 \mathbb{V}$. In particular, taking $\alpha = \sigma^\flat X$, we have

$$P^\sharp \mathcal{L}_Y(\sigma^\flat X) = 0. \tag{3.12}$$

On the other hand,

$$\mathcal{L}_Z(\sigma^\flat X) = \mathcal{L}_{fY}(\sigma^\flat X) = f\mathcal{L}_Y(\sigma^\flat X) + (\mathrm{i}_Y \sigma^\flat X)\mathrm{d}f = f\mathcal{L}_Y(\sigma^\flat X) + \sigma(X, Y)\mathrm{d}f,$$

which implies $P^\sharp \mathcal{L}_Z(\sigma^\flat X) = fP^\sharp \mathcal{L}_Y(\sigma^\flat X) + P^\sharp[\sigma(X, Y)\mathrm{d}f]$. Because of equation (3.12), we get

$$P^\sharp \mathcal{L}_Z(\sigma^\flat X) = P^\sharp[\sigma(X, Y)\mathrm{d}f]. \tag{3.13}$$

Since the exterior differential d is a graded derivation, it follows that

$$\mathrm{d}\,[\sigma(Z, X)] = \mathrm{d}\,[f\sigma(Y, X)] = \sigma(Y, X)\mathrm{d}f + f\mathrm{d}\,[\sigma(Y, X)]\,. \tag{3.14}$$

Using equations (3.14), (3.10), and (3.13), we get

$$\begin{aligned} P^\sharp \mathrm{d}\,[\sigma(Z, X)] &= P^\sharp[\sigma(Y, X)\mathrm{d}f] + fP^\sharp \mathrm{d}\,(\sigma(Y, X)) \\ &= -P^\sharp[\sigma(X, Y)\mathrm{d}f] - fR(Y, X) \\ &= -P^\sharp \mathcal{L}_Z(\sigma^\flat X) - R(fY, X) \\ &= -P^\sharp \mathcal{L}_Z(\sigma^\flat X) - R(Z, X). \end{aligned}$$

■

3.2.2 Jacobi identity and integrability equations

Let $\mathbb{V}$ be an involutive distribution in the differential manifold M. Among the triples of geometric data (γ,σ,P) in $(M,\mathbb{V})$, some of the most important are those for which σ satisfy that the map $\sigma^\flat : TM \longrightarrow \mathbb{V}^0$, defined by $\sigma^\flat(X) := \mathrm{i}_X\sigma$, is surjective.

Definition 3.2.5. *A horizontal 2-form $\sigma \in \Gamma \bigwedge^2 \mathbb{V}^0$ is said to be **horizontally non-degenerate** if the vector bundle morphism $\sigma^\flat : TM \longrightarrow \mathbb{V}^0$ is surjective.*

If σ is a horizontally non-degenerate 2-form, then for each horizontal distribution $\mathbb{H}$, the restriction $\sigma^\flat|_{\mathbb{H}} : \mathbb{H} \longrightarrow \mathbb{V}^0$ is an isomorphism. In this case, the geometric data (γ,σ,P) defines a unique coupling bivector field Π in M. Furthermore, according to [36, 33, 37], we have the following result.

Theorem 3.2.6. *Let $\mathbb{V}$ be a regular involutive distribution on M. There is a natural correspondence between coupling Poisson structures Π and integrable geometric data (γ,σ,P) in $(M,\mathbb{V})$ such that σ is horizontally non-degenerate.*

For completeness, we prove this theorem, based on the following facts.

Lemma 3.2.7. *There is a correspondence between coupling bivector fields Π and geometric data (γ,σ,P) in $(M,\mathbb{V})$ such that σ is horizontally non-degenerate.*

Proof. If Π is a coupling Poisson structure, then $TM = \Pi^\sharp(\mathbb{V}^0) \oplus \mathbb{V}$. Define $\gamma : TM \longrightarrow TM$ by $\gamma := \mathrm{pr}_{\mathbb{V}}$, i.e., the projection over $\mathbb{V}$ along the previous decomposition. It is clear that γ is a connection in M, which induces a bigrading in M. The bigraded decomposition of Π is $\Pi = \Pi_{2,0}+\Pi_{0,2}$ (there is no (1,1)-component: $\Pi_{1,1} = 0$). The horizontal distribution of γ is $\mathbb{H} = \Pi^\sharp(\mathbb{V}^0) = \Pi^\sharp_{2,0}(\mathbb{V}^0)$. So, $\Pi^\sharp_{2,0} : \mathbb{V}^0 \longrightarrow \mathbb{H}$ is a vector bundle isomorphism. Define the vector bundle morphism $\sigma^\flat : TM \longrightarrow T^*M$ by $\sigma^\flat|_{\mathbb{V}} := 0$ and by $\sigma^\flat|_{\mathbb{H}} := -(\Pi^\sharp_{2,0}|_{\mathbb{V}^0})^{-1}$. It is clear that $\sigma^\flat$ is skew-symmetric. Thus, $\sigma(X,Y) := \sigma^\flat(X)(Y)$ defines a section of $\bigwedge^2 \mathbb{V}^0$. Finally, because of bidegrees, $P := \Pi_{0,2}$ is a vertical bivector field and (γ,σ,P) is a triple of geometric such that σ is horizontally non-degenerate. Conversely, if (γ,σ,P) are geometric data such that σ is horizontally non-degenerate, then $\sigma^\flat|_{\mathbb{H}} : \mathbb{H} \longrightarrow \mathbb{V}^0$ is an isomorphism, where $\mathbb{H}$ is the horizontal distribution of γ. Now, define $\Pi^\sharp_{2,0} : T^*M \longrightarrow TM$ by $\Pi^\sharp_{2,0}|_{\mathbb{H}^0} = 0$ and $\Pi^\sharp_{2,0}|_{\mathbb{V}^0} := -(\sigma^\flat|_{\mathbb{H}})^{-1}$. Finally, if $\Pi_{0,2} := P$, then $\Pi := \Pi_{2,0} + \Pi_{0,2}$ is a coupling bivector field, since $\Pi^\sharp(\mathbb{V}^0) = \Pi^\sharp_{2,0}(\mathbb{V}^0) = \mathbb{H}$. ■

As it follows from the proof of Lemma 3.2.7, a coupling bivector field Π in $(M,\mathbb{V})$ defines intrinsically the horizontal distribution $\mathbb{H} := \Pi^\sharp(\mathbb{V}^0)$. This induces a bigrading in M and a splitting of Π of the form $\Pi = \Pi_{2,0} + \Pi_{0,2}$. In terms of the bigraded decomposition, the characteristic distribution of Π at each $x \in M$ reads $D^\Pi_x = \Pi^\sharp(T^*_xM) = \Pi^\sharp(\mathbb{V}^0_x) \oplus \Pi^\sharp(\mathbb{H}^0_x) = \mathbb{H}_x \oplus \Pi_{0,2}(\mathbb{H}^0_x)$. Since $\Pi_{2,0}$ is regular, the singular points of Π and $\Pi_{0,2}$ coincide. Also, notice that $\mathbb{H} = \Pi^\sharp_{2,0}(T^*M)$. Therefore, $\Pi_{2,0}$ is a Poisson structure if and only if $\mathbb{H}$ is integrable.

On the other hand, by Theorem 2.14 and the involutivity of $\mathbb{V}$, the bigraded decomposition of the exterior differential has the form $\mathrm{d} = \mathrm{d}_{1,0} + \mathrm{d}_{0,1} + \mathrm{d}_{2,-1}$, with Frölicher - Nijenhuis decompositions $\mathrm{d}_{1,0} = \mathcal{L}_{\mathrm{Id}_{TM}-\gamma} + 2\mathrm{i}_R$, $\mathrm{d}_{0,1} = \mathcal{L}_\gamma - \mathrm{i}_R$, $\mathrm{d}_{2,-1} = -\mathrm{i}_R$. Finally, the following identity is needed, and we can find a proof of it in [33]:

$$[\Pi,\Psi](\alpha,\beta,\theta) = \mathcal{L}_{\Pi^\sharp\theta}\Psi(\alpha,\beta) - \mathrm{d}\theta(\Pi^\sharp\alpha,\Psi^\sharp\beta) + \mathcal{L}_{\Psi^\sharp\theta}\Pi(\alpha,\beta) - \mathrm{d}\theta(\Psi^\sharp\alpha,\Pi^\sharp\beta), \quad (3.15)$$

with α,β,θ being arbitrary 1-forms and Π,Ψ arbitrary bivector fields.

By using the above bigrading decomposition, the condition for a coupling bivector field Π to be a Poisson structure is formulated as follows:

Lemma 3.2.8. *Let $\mathbb{V}$ be an involutive distribution in M. A coupling bivector field Π in $(M,\mathbb{V})$ is a Poisson structure if and only if the following identities hold for any $\alpha,\beta,\theta \in \Gamma\mathbb{V}^0$, $\lambda,\eta \in \Gamma\mathbb{H}^0$:*

1. *$\Pi_{0,2}$ is a Poisson structure,*
2. *$\mathcal{L}_{\Pi^\sharp_{2,0}\theta}\Pi_{0,2}(\lambda,\eta) = 0$,*
3. *$\mathrm{d}_{2,-1}\lambda(\Pi^\sharp_{2,0}\alpha,\Pi^\sharp_{2,0}\beta) = \mathcal{L}_{\Pi^\sharp_{0,2}\lambda}\Pi_{2,0}(\alpha,\beta)$,*
4. *$\mathrm{d}_{1,0}\theta(\Pi^\sharp_{2,0}\alpha,\Pi^\sharp_{2,0}\beta) = \mathcal{L}_{\Pi^\sharp_{2,0}\theta}\Pi_{2,0}(\alpha,\beta)$.*

Proof. Due to the splitting $\Pi = \Pi_{2,0} + \Pi_{0,2}$, the Jacobi identity $[\Pi,\Pi] = 0$ is equivalent to the following bigraded equations:

(i) $[\Pi_{0,2},\Pi_{0,2}]_{0,3} = 0$,

(ii) $[\Pi_{2,0},\Pi_{0,2}]_{1,2} = 0$,

(iii) $[\Pi_{2,0},\Pi_{2,0}]_{2,1} + 2[\Pi_{2,0},\Pi_{0,2}]_{2,1} = 0$, and

(iv) $[\Pi_{2,0},\Pi_{2,0}]_{3,0} = 0$.

We prove that (iii) is equivalent to identity 3 of Lemma 3.2.8. By formula (3.15),

$$\begin{aligned}
[\Pi_{2,0},\Pi_{2,0}]_{2,1}(\alpha,\beta,\lambda) &= 2[\mathcal{L}_{\Pi^\sharp_{2,0}\lambda}\Pi_{2,0}(\alpha,\beta) - \mathrm{d}\lambda(\Pi^\sharp_{2,0}\alpha,\Pi^\sharp_{2,0}\beta)] \\
&= -2\mathrm{d}\lambda(\Pi^\sharp_{2,0}\alpha,\Pi^\sharp_{2,0}\beta) = -2\mathrm{d}_{2,-1}\lambda(\Pi^\sharp_{2,0}\alpha,\Pi^\sharp_{2,0}\beta), \\
2[\Pi_{2,0},\Pi_{0,2}]_{2,1}(\alpha,\beta,\lambda) &= 2(\mathcal{L}_{\Pi^\sharp_{2,0}\lambda}\Pi_{0,2}(\alpha,\beta) - \mathrm{d}\lambda(\Pi^\sharp_{2,0}\alpha,\Pi^\sharp_{0,2}\beta) + \mathcal{L}_{\Pi^\sharp_{0,2}\lambda}\Pi_{2,0}(\alpha,\beta) - \mathrm{d}\lambda(\Pi^\sharp_{0,2}\alpha,\Pi^\sharp_{2,0}\beta)) \\
&= 2\mathcal{L}_{\Pi^\sharp_{0,2}\lambda}\Pi_{2,0}(\alpha,\beta).
\end{aligned}$$

Combining these equations, we get

$$\left([\Pi_{2,0},\Pi_{2,0}]_{2,1} + 2[\Pi_{2,0},\Pi_{0,2}]\right)(\alpha,\beta,\lambda) = 2\mathcal{L}_{\Pi^\sharp_{0,2}\lambda}\Pi_{2,0}(\alpha,\beta) - 2\mathrm{d}_{2,-1}\lambda(\Pi^\sharp_{2,0}\alpha,\Pi^\sharp_{2,0}\beta),$$

proving the desired equivalence. The other equivalences are proved similarly. ∎

Now we are ready to prove Theorem 3.2.6:

Proof of Theorem 3.2.6. Let Π be a coupling bivector field in $(M, \mathbb{V})$, and (γ, σ, P) its associated geometric data, in the sense of Lemma 3.2.7. We now shown that each of the four integrability equations (3.8)-(3.11) for (γ, σ, P) are equivalent to the identities 1, 2, 3, 4 in Lemma 3.2.8 for Π, respectively. Recall that $\Pi^{\sharp}_{2,0}|_{\mathbb{V}^0}$ and $-\sigma^{\flat}|_{\mathbb{H}}$ are isomorphisms inverse of each other and $\Pi_{0,2} = P$,

- Equivalence of (3.8) and identity 1 in Lemma 3.2.8 is clear, since $P = \Pi_{0,2}$.
- Fix $X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$. Since P is vertical, it follows that $\mathcal{L}_X P$ is vertical. So, it is enough to evaluate it on vertical 1-forms λ, μ. On the other hand, $\Pi^{\sharp}_{2,0} : \Gamma\mathbb{V}^0 \longrightarrow \Gamma\mathbb{H}$ is a C^{∞}_M-module isomorphism. Hence, there exists $\theta \in \Gamma\mathbb{V}^0$ such that $X = \Pi^{\sharp}_{2,0}\theta$. Substituting, it follows that $\mathcal{L}_X P(\lambda, \mu) = \mathcal{L}_{\Pi^{\sharp}_{2,0}\theta}\Pi_{0,2}(\lambda, \eta)$. Hence, if $\mathcal{L}_{\Pi^{\sharp}_{2,0}\theta}\Pi_{0,2}(\lambda, \eta) = 0$ for each $\lambda, \eta \in \Omega^{0,1}_M$ and $\theta \in \Gamma\mathbb{V}^0$, then $\mathcal{L}_X P = 0$ for all $X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$. This proves that identity 2 of Lemma 3.2.8 implies equation (3.9). Conversely, if equation (3.9) is satisfied, then $\mathcal{L}_{\Pi^{\sharp}_{2,0}\theta}\Pi_{0,2}(\lambda, \eta) = 0$, $\lambda, \eta \in \Omega^{0,1}_M$ and $\theta \in \Gamma\mathbb{V}^0$, due to Lemma (3.2.3). This proves the converse.
- Let $X, Y \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}$ be projectable vector fields and $\alpha, \beta \in \Gamma\mathbb{V}^0$ such that $\Pi^{\sharp}_{2,0}\alpha = X$, $\Pi^{\sharp}_{2,0}\beta = Y$. Note that for any $\lambda \in \Omega^{0,1}_M$,

$$-\lambda(R(X,Y)) = \mathrm{d}_{2,-1}\lambda(\Pi^{\sharp}_{2,0}\alpha, \Pi^{\sharp}_{2,0}\beta),$$

since $\mathrm{d}_{2,-1} = -\mathrm{i}_R$. On the other hand,

$$\begin{aligned}
\lambda(P^{\sharp}\mathrm{d}[\sigma(X,Y)]) =& \lambda(\Pi^{\sharp}_{0,2}\mathrm{d}[\sigma(X,Y)]) = -\Pi_{0,2}(\mathrm{d}[\sigma(X,Y)], \lambda) - 2\mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}(\sigma(X,Y)) \\
=& -\Pi_{0,2}(\mathrm{d}[\sigma(X,Y)], \lambda) + \mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}(\alpha(Y)) - \alpha[\Pi^{\sharp}_{0,2}\lambda, Y] \\
& - \mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}(\beta(X)) + \beta[\Pi^{\sharp}_{0,2}\lambda, X] \\
=& -\Pi_{0,2}(\mathrm{d}[\Pi_{2,0}(\alpha,\beta)], \lambda) + \mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}\alpha(Y) - \mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}\beta(X) \\
=& \mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}(\Pi_{2,0}(\alpha,\beta)) - \Pi_{2,0}(\mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}\alpha, \beta) - \Pi_{2,0}(\alpha, \mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}\beta) \\
=& \mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}\Pi_{2,0}(\alpha,\beta).
\end{aligned}$$

Now, assuming that identity 3 of Lemma 3.2.8 holds, these equalities imply $\lambda(P^{\sharp}\mathrm{d}[\sigma(X,Y)]) = -\lambda(R(X,Y))\ \forall \lambda \in \Gamma\mathbb{H}^0$. Since $P^{\sharp}\mathrm{d}[\sigma(X,Y)]$ and $R(X,Y)$ are vertical, the integrability equation (3.10) holds. Conversely, if equation (3.10) holds, then our previous equalities imply that

$$\mathrm{d}_{2,-1}\lambda(\Pi^{\sharp}_{2,0}\alpha, \Pi^{\sharp}_{2,0}\beta) = \mathcal{L}_{\Pi^{\sharp}_{0,2}\lambda}\Pi_{2,0}(\alpha,\beta), \tag{3.16}$$

if $\Pi^{\sharp}_{2,0}\alpha, \Pi^{\sharp}_{2,0}\beta \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$. Since horizontal projctable vector fields generate any horizontal vector field and (3.16) is tensorial in α, β, it follows that (3.16) also holds for any $\alpha, \beta \in \Gamma\mathbb{V}^0$. This proves the converse.

- If $X_i \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$, then there exists $\alpha_i \in \Gamma\mathbb{V}^0$ such that $\Pi^{\sharp}_{2,0}(\alpha_i) = X_i$, $\forall i = 0,1,2$. Since $\sigma(\Pi^{\sharp}_{2,0}(\alpha_i), \Pi^{\sharp}_{2,0}(\alpha_j)) = -\alpha_i(\Pi^{\sharp}_{2,0}(\alpha_j)) = \Pi_{2,0}(\alpha_i,\alpha_j)$, we get

$$\begin{aligned}
[\Pi_{2,0},\Pi_{2,0}](\alpha_0,\alpha_1,\alpha_2) &= -2\sum_{(0,1,2)} \Pi_{2,0}(\alpha_0, \mathrm{d}\Pi_{2,0}(\alpha_1,\alpha_2)) - \alpha_2[\Pi^{\sharp}_{2,0}\alpha_0, \Pi^{\sharp}_{2,0}\alpha_1] \\
&= -2\sum_{(0,1,2)} \mathrm{d}\Pi_{2,0}(\alpha_1,\alpha_2)(X_0) - \alpha_2[X_0,X_1] \\
&= -2\sum_{(0,1,2)} \mathcal{L}_{X_0}\left(\sigma(X_1,X_2)\right) + \sigma(X_2,[X_0,X_1]) \\
&= -2\mathrm{d}\sigma(X_0,X_1,X_2),
\end{aligned}$$

  where the first equality follows from (3.15). If identity 4 of Lemma 3.2.8 holds, then $\mathrm{d}\sigma(X_0,X_1,X_2) = 0$, proving that the integrability equation (3.11) holds. Conversely, if equation (3.11) holds, then $[\Pi_{2,0},\Pi_{2,0}](\alpha_0,\alpha_1,\alpha_2) = 0$ for any $\alpha_i \in \Gamma\mathbb{V}^0$ such that $\Pi^{\sharp}_{2,0}\alpha_i \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$. Since $[\Pi_{2,0},\Pi_{2,0}]$ is C^{∞}_M-linear, it follows that $[\Pi_{2,0},\Pi_{2,0}]_{3,0} = 0$, proving the fourth equivalence.

■

3.3 Cochain complexes in Poisson fiber bundles

In this Section we construct a cochain complex induced by integrable geometric data in a fiber bundle. In further sections, we will see that, when the integrable geometric data is induced by a coupling Poisson structure in the bundle, it turns out that the corresponding geometric data define a cochain complex isomorphic to the Lichnerowic - Poisson complex of the coupling structure. Since the cochain complex induced by the geometric data is a bigraded cochain complex, in the sense of Definition 1.2.1, we can apply Theorem 1.2.2 to derive a splitting result for the first Poisson cohomology group.

We begin by introducing the bigraded C^{∞}_B module $\mathcal{V}_E$ of *vertical-valued forms in the base*, which is the tensor product of the exterior algebra of differential forms in the base with the Poisson algebra of vertical multivector fields in the total space, $\mathcal{V}_E = \Omega_B \otimes_{C^{\infty}_B} \chi_{\mathbb{V}}(E)$. This can be endowed with a Poisson algebra structure of degree -1. After that, we study two classes of graded derivations in the Poisson algebra $(\mathcal{V}_E, \wedge, [,])$: the *adjoint operators*, $\mathrm{ad}_\eta : \mathcal{V}_E \longrightarrow \mathcal{V}_E$, $\eta \in \mathcal{V}_E$ given by $\mathrm{ad}_\eta \nu := [\eta,\nu]$, and the *covariant exterior differential* $\partial^{\gamma}_{1,0} : \mathcal{V}_E \longrightarrow \mathcal{V}_E$ of a connection γ, given by a Koszul-Cartan-type formula. We then describe the integrability conditions of the geometric data (γ,σ,P) in terms of the covariant exterior differential $\partial^{\gamma}_{1,0}$ and the curvature Curv^{γ} of the connection γ, the vertical bivector field P and the projection of the horizontal 2-form $\pi^{\gamma}_{*}\sigma$. Finally, we show that the integrability condition for (γ,σ,P) implies that the graded operator $\partial^{\gamma} = \partial^{\gamma}_{1,0} - \mathrm{ad}_{\pi^{\gamma}_{*}\sigma} + \mathrm{ad}_P$ is a coboundary in $\mathcal{V}_E$. Moreover, if σ is horizontally non-degenerate, then the pair $(\mathcal{V}_E, \partial^{\gamma})$ is a Poisson bigraded model for the Lichnerowicz - Poisson complex of the coupling structure associated to (γ,σ,P).

3.3.1 The Poisson algebra of vertical-valued forms

Let (E, π, B) be a fiber bundle and $\mathbb{V}$ its canonical vertical distribution. Recall that projectable functions in the total space are precisely the pull-back of base functions, $C^\infty_{\mathrm{pr}}(E) = \pi^* C^\infty_B$. Hence, $C^\infty_{\mathrm{pr}}(E)$ and C^∞_B are isomorphic $\mathbb{R}$-algebras. Because of the involutivity of $\mathbb{V}$ and the above isomorphism, it follows that the triple $(\chi^\bullet_{\mathbb{V}}(E), \wedge, [,])$ is a Poisson C^∞_B-algebra of degree -1, where $\chi^\bullet_{\mathbb{V}}(E) := \Gamma \bigwedge \mathbb{V}$ are *vertical multivector fields*, $\wedge$ is the exterior product, and $[,]$ is the Schouten - Nijenhuis bracket (see definition and properties of Page 32).

Definition 3.3.1. *Let (E, π, B) be a fiber bundle and $\mathbb{V}$ its vertical distribution. Define the space of* vertical-valued forms in the base *by*

$$\mathcal{V}_E := \Omega_B \otimes_{C^\infty_B} \chi_{\mathbb{V}}(E).$$

Also, for each $n, p, q \in \mathbb{Z}$, define

$$\begin{aligned} \mathcal{V}^{p,q}_E &:= \Omega^p_B \otimes_{C^\infty_B} \chi^q_{\mathbb{V}}(E), \\ \mathcal{V}^n_E &:= \bigoplus_{p+q=n} \mathcal{V}^{p,q}_E. \end{aligned}$$

Here, we put $\mathcal{V}^{p,q}_E = \{0\}$ if $p < 0$ or if $q < 0$.

Observe that, in particular, we have:

$$\begin{aligned} &\mathcal{V}^{0,0}_E = C^\infty_E, \\ &\mathcal{V}^{0,q}_E = \chi^q_{\mathbb{V}}(E) = \{\text{vertical multivector fields}\}, \\ &\mathcal{V}^{p,0}_E \simeq \Gamma \bigwedge^p \mathbb{V}^0 = \{\text{horizontal forms in the total space}\}. \end{aligned}$$

Note that we have a natural grading and bigrading in $\mathcal{V}_E$, given by

$$\mathcal{V}^\bullet_E := \bigoplus_{n\in\mathbb{Z}} \mathcal{V}^n_E, \qquad \mathcal{V}^{\bullet,\bullet}_E := \bigoplus_{n\in\mathbb{Z}} \mathcal{V}^{p,q}_E.$$

Moreover, $\mathcal{V}^{p,q}_E$ is isomorphic to the C^∞_B-module of p-linear alternating applications

$$\eta := \mathfrak{X}_B \times \cdots \times \mathfrak{X}_B \longrightarrow \chi^q_{\mathbb{V}}(E).$$

From this point of view, we naturally endow $\mathcal{V}_E$ with a graded Poisson C^∞_B-algebra structure of degree -1. Indeed, since Ω_B has a graded exterior algebra structure, and $\chi_{\mathbb{V}}(E)$ is a Poisson algebra of degree -1, we can define a product and a bracket on decomposable in $\mathcal{V}_E$ elements by

$$\begin{aligned} (\alpha \otimes A) \wedge (\beta \otimes B) &:= (-1)^{|\beta||A|} (\alpha \wedge \beta) \otimes (A \wedge B), \\ [\alpha \otimes A, \beta \otimes B] &:= (-1)^{|\beta|(|A|-1)} (\alpha \wedge \beta) \otimes [A, B]. \end{aligned}$$

This definition coincides with equation (2.2). So, the triple $(\mathcal{V}_E, \wedge, [,])$ is in fact a graded Poisson algebra of degree -1. Equivalently, these operations are given on homogeneous elements by

$$(\eta \wedge \nu)(u_1, \dots, u_{p+p'}) := (-1)^{p'q} \sum_{\sigma \in S_{(p,p')}} (-1)^\sigma \eta(u_{\sigma(1)}, \dots, u_{\sigma(p)}) \wedge \nu(u_{\sigma(p+1)}, \dots, u_{\sigma(p+p')}),$$

$$[\eta, \nu](u_1, \dots, u_{p+p'}) := (-1)^{p'(q-1)} \sum_{\sigma \in S_{(p,p')}} (-1)^\sigma [\eta(u_{\sigma(1)}, \dots, u_{\sigma(p)}), \nu(u_{\sigma(p+1)}, \dots, u_{\sigma(p+p')})], \tag{3.17}$$

where $\eta \in \mathcal{V}_E^{p,q}$, $\nu \in \mathcal{V}_E^{p',q'}$, and $u_i \in \mathfrak{X}_B$, $i = 1, \dots, p+p'$. For $p = p' = 0$, the bracket $[\eta, \nu]$ coincides with the usual Schouten - Nijenhuis bracket for vertical multivector fields, and for $q = q' = 0$, $[\eta, \nu] = 0$ since the resulting bidegree is $(p + p', -1)$.

Adjoint derivations and the covariant exterior differential. Since $\mathcal{V}_E$ has graded Poisson algebra structure, it is clear that each $\eta \in \mathcal{V}_E^{p,q}$ induces a graded derivation ad_η of bidegree $(p, q-1)$ in $\mathcal{V}_E$, called *adjoint operator*, by $\mathrm{ad}_\eta(\nu) := [\eta, \nu]$. By the Leibniz rule and the Jacobi identity of graded Poisson algebras, we have $\mathrm{ad}_\eta \in \mathrm{Der}^{(p,q-1)}_{C^\infty_B}(\mathcal{V}_E, \wedge, [,])$ and $[\mathrm{ad}_\eta, \mathrm{ad}_\nu] = \mathrm{ad}_{[\eta,\nu]}$ $\forall \eta, \nu \in \mathcal{V}_E$. This identity is a special case of the following general property: for any graded endomorphism D in $\mathcal{V}_E$, $D \in \mathrm{Der}_{\mathbb{R}}(\mathcal{V}_E, [,])$ if and only if

$$[D, \mathrm{ad}_\eta] = \mathrm{ad}_{D\eta} \qquad \forall \eta \in \mathcal{V}_E. \tag{3.18}$$

Another important class of operators in $\mathcal{V}_E$ is represented by the *covariant exterior differentials*. Those are bigraded operators of bidegree (1,0) induced by an Ehresmann connection in (E, π, B).

Definition 3.3.2. *Let γ be an Ehresmann connection in (E, π, B). The **covariant exterior differential** $\partial^\gamma_{1,0} : \mathcal{V}_E \longrightarrow \mathcal{V}_E$ is defined on homogeneous elements by*

$$\partial^\gamma_{1,0}\eta(u_0, \dots, u_p) := \sum_{i=0}^p (-1)^i \mathcal{L}_{\mathrm{hor}^\gamma u_i}(\eta(u_0, \dots \widehat{u}_i \dots, u_p)) + \sum_{i<j} (-1)^{i+j} \eta([u_i, u_j], u_0, \dots \widehat{u}_i \dots \widehat{u}_j \dots, u_p).$$

Note that $\eta(u_0, \dots \widehat{u}_i \dots, u_p)$ is a vertical multivector field and $\mathrm{hor}(u_i)$ is projectable. Therefore, $\mathcal{L}_{\mathrm{hor}(u_i)}(\eta(u_0, \dots \widehat{u}_i \dots, u_p))$ is again a multivector field, showing that, in fact, $\partial^\gamma_{1,0}$ is a bigraded operator in $\mathcal{V}_E$ of bidegree $(1, 0)$.

It follows from Definition 3.3.2 that the covariant exterior differential is a graded derivation of the exterior product and the bracket in $\mathcal{V}_E$: $\partial^\gamma_{1,0} \in \mathrm{Der}^1_{\mathbb{R}}(\mathcal{V}^\bullet_E, \wedge, [,])$. Therefore, equation (3.18) implies

$$[\partial^\gamma_{1,0}, \mathrm{ad}_\eta] = \mathrm{ad}_{\partial^\gamma_{1,0}\eta}, \qquad \forall \eta \in \mathcal{V}_E. \tag{3.19}$$

The covariant exterior differential $\partial^{\gamma}_{1,0}$ is related to the horizontal bigraded component of the exterior differential $\mathrm{d}_{1,0}$ in E, defined by the bigrading of the Ehresmann connection γ. Indeed, for each $\omega \in \Omega^p_{\mathbb{V}^0}(E) \simeq \mathcal{V}^{p,0}_E$,

$$\partial^{\gamma}_{1,0}\pi^{\gamma}_*\omega = \pi^{\gamma}_*\mathrm{d}_{1,0}\omega \tag{3.20}$$

(recall the isomorphism π^{γ}_* given in page 41). In particular, Corollary 2.5.7 implies that $(\partial^{\gamma}_{1,0})^2\pi^{\gamma}_*\omega = \pi^{\gamma}_*(\mathcal{L}_R\omega)$, where $R \in \Omega^2(E;\mathbb{V})$ is the curvature of γ. Thus, $(\partial^{\gamma}_{1,0})^2 = 0$ if and only if $\mathbb{H}$ is integrable. More generally, if $\mathrm{Curv}^{\gamma} \in \mathcal{V}^{2,1}_E$ is the curvature 2-form (as in Definition 2.6.6), then the following identity is satisfied on the whole algebra $\mathcal{V}_E$:

$$(\partial^{\gamma}_{1,0})^2 = \mathrm{ad}_{\mathrm{Curv}^{\gamma}}. \tag{3.21}$$

Also, *Bianchi identity* for γ (equation (2.13)) can be expressed as

$$\partial^{\gamma}_{1,0}\mathrm{Curv}^{\gamma} = 0 \tag{3.22}$$

Finally, observe for any Ehresmann connections γ and $\tilde{\gamma}$ in (E,π,B), we have $\partial^{\tilde{\gamma}}_{1,0} - \partial^{\gamma}_{1,0} = \mathrm{ad}_{\pi^{\gamma}_*\Theta}$, where $\Theta = \gamma - \tilde{\gamma}$ (see Remark 2.6.7).

3.3.2 Coboundary operators from integrable geometric data

We now show that each triple (γ,σ,P) of integrable geometric data in the fiber bundle (E,π,B) defines a coboundary operator in $\mathcal{V}_E$. This is a graded derivation of degree 1 for both of the operations in $\mathcal{V}_E$, and it is defined by its bigraded components. The component of bidegree $(1,0)$ is the covariant exterior differential of γ, the $(0,1)$ component is the adjoint of the vertical bivector field P, and the $(2,-1)$ component is the adjoint of the projection of the horizontal 2-form σ. The integrability condition of (γ,σ,P) results in the fact that ∂^{γ} is a coboundary.

Integrability equations in fiber bundles. Let (E,π,B) be a fiber bundle. Recall that the *vertical subbundle* $\mathbb{V}$ in E is a regular involutive distribution. For the rest of this work, a triple (γ,σ,P) of geometric data on (E,π,B) means a triple of geometric data on $(E,\mathbb{V})$, and a coupling structure Π in (E,π,B) also means a coupling structure on $(E,\mathbb{V})$.

Recall that an Ehresmann connection γ induces the covariant exterior differential $\partial^{\gamma}_{1,0}$, which is a derivation of bidegree $(1,0)$ for both operations $\wedge$ and $[,]$ in the algebra $\mathcal{V}_E$ defined in Section 3.3. Also, the curvature of the connection γ is a vertical-valued 2-form in the base, which measures the integrability of the horizontal distribution: $\mathrm{Curv}^{\gamma} \in \mathcal{V}^{2,1}_E$. Similarly, since P is a vertical bivector field, and σ is a horizontal 2-form, we naturally have $P \in \mathcal{V}^{0,2}_E$ and $\pi^{\gamma}_*\sigma \in \mathcal{V}^{2,0}_E$, where

$$\pi^{\gamma}_* : \Gamma\bigwedge\mathbb{V}^0 \longrightarrow \Omega_B \otimes C^{\infty}_M$$

is the projection defined in Page 41.

The goal of this part is to define an operator by means of geometric data, and show that the operator is a coboundary if the geometric data defining it are integrable. To do this, equivalent conditions to the integrability equations are presented in terms of $\partial^{\gamma}_{1,0}$, Curv^{γ}, P and $\pi^{\gamma}_{*}\sigma$.

Proposition 3.3.3. *Let (γ, σ, P) be geometric data in the fiber bundle (E, π, B). The integrability equations (3.8)-(3.11) for (γ, σ, P) can be written as follows*

- $[P, P] = 0$,
- $\partial^{\gamma}_{1,0}P = 0$,
- $\mathrm{Curv}^{\gamma} = [\pi^{\gamma}_{*}\sigma, P]$,
- $\partial^{\gamma}_{1,0}(\pi^{\gamma}_{*}\sigma) = 0$.

Here, $[,]$ denotes the bracket in $\mathcal{V}_E$ introduced by equation (3.17) and $\partial^{\gamma}_{1,0}$ is the covariant exterior derivative introduced in Definition 3.3.2.

Proof. Recall that the bracket in $\mathcal{V}_E$ coincides on vertical multivector fields with the Schouten - Nijenhuis.

- In virtue of our previous observation, the first equation of this proposition is precisely the first integrability condition fot the geometric data.
- If $X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(E)$, then $X = \mathrm{hor}^{\gamma}(u)$ for some $u \in \mathfrak{X}_B$. Furthermore,
$$\mathcal{L}_X P = \mathcal{L}_{\mathrm{hor}^{\gamma}(u)}P = [\mathrm{hor}^{\gamma}(u), P] = (\partial^{\gamma}_{1,0}P)(u),$$
proving that $\partial^{\gamma}_{1,0}P = 0$ if and only if $\mathcal{L}_X P = 0$ $\forall X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(E)$, which is the second integrability equation of (γ, σ, P).
- If $X, Y \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(E)$, with $X = \mathrm{hor}^{\gamma}(u)$ and $Y = \mathrm{hor}^{\gamma}(v)$, then $\mathrm{Curv}^{\gamma}(u, v) = R(X, Y)$ and $[\pi^{\gamma}_{*}\sigma, P](u, v) = [\sigma\big(\mathrm{hor}^{\gamma}(u), \mathrm{hor}^{\gamma}(v)\big), P] = -P^{\sharp}[\sigma(X, Y)]$, proving that $\mathrm{Curv}^{\gamma} = [\pi^{\gamma}_{*}\sigma, P]$ if and only if $R(X, Y) = -P^{\sharp}[\sigma(X, Y)]$ $\forall X, Y \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(E)$, which is the third integrability equation of (γ, σ, P).
- Finally, recall that $\pi^{\gamma}_{*} : \Gamma \bigwedge^k \mathbb{V}^0 \longrightarrow \mathcal{V}^{k,0}_E$ is an isomorphism and $\partial^{\gamma}_{1,0}\pi^{\gamma}_{*}\sigma = \pi^{\gamma}_{*}\mathrm{d}_{1,0}\sigma$. Therefore, $\partial^{\gamma}_{1,0}\pi^{\gamma}_{*}\sigma = 0$ if and only if $\mathrm{d}_{1,0}\sigma = 0$, which is equivalent to the last integrability equation of (γ, σ, P).

■

Let (γ, σ, P) be geometric data in the fiber bundle (E, π, B). Recall that the adjoint of an element of the Poisson algebra $\mathcal{V}_E$, with respect to the bracket $[,]$, is a derivation for both operations in $\mathcal{V}_E$. In particular, for the vertical bivector field P and the horizontal 2-form σ, we have $\mathrm{ad}_P \in \mathrm{Der}^{0,1}_{\mathbb{R}}(\mathcal{V}_E, \wedge, [,])$ and $\mathrm{ad}_{\pi^{\gamma}_{*}\sigma} \in \mathrm{Der}^{2,-1}_{\mathbb{R}}(\mathcal{V}_E, \wedge, [,])$. If (γ, σ, P) are integrable, then both operators, together with the covariant exterior differential $\partial^{\gamma}_{1,0}$, allow to construct a coboundary operator.

Definition 3.3.4. *Let (γ, σ, P) be geometric data in the fiber bundle (E, π, B). One can associate to the triple (γ, σ, P) an operator $\partial \in \mathrm{End}^1_{\mathbb{R}} \mathcal{V}_E$*

$$\partial^\gamma := \partial_{1,0} + \partial_{0,1} + \partial_{2,-1}$$

whose bigraded components are defined by

$$\partial_{1,0} := \partial^\gamma_{1,0}, \qquad \partial_{2,-1} := -\mathrm{ad}_{\pi^\gamma_* \sigma}, \qquad \partial_{0,1} := \mathrm{ad}_P.$$

Since ∂^γ is the sum of bigraded derivations whose total degree is 1, it follows that ∂^γ is a graded derivation of degree 1 for the Poisson algebra $(\mathcal{V}_E, \wedge, [,])$.

We are now ready to prove that if ∂^γ is defined by integrable geometric data, then ∂^γ is a coboundary operator. So, $(\mathcal{V}_E, \partial^\gamma)$ is a bigraded cochain complex.

Theorem 3.3.5. *Let (γ, σ, P) be integrable geometric data in the fiber bundle (E, π, B), and $\partial^\gamma = \partial^\gamma_{1,0} - \mathrm{ad}_{\pi^\gamma_* \sigma} + \mathrm{ad}_P$ the associated operator in $\mathcal{V}_E$. Then, $(\partial^\gamma)^2 = 0$, and hence, $(\mathcal{V}_E, \partial^\gamma)$ is a bigraded cochain complex associated to (γ, σ, P).*

Proof. As shown in Section 1.2, the coboundary condition for ∂^γ splits into the following equations:

$$\begin{aligned}
\partial^2_{0,1} &= 0,\\
\partial^2_{2,-1} &= 0,\\
\partial_{1,0}\partial_{0,1} + \partial_{0,1}\partial_{1,0} &= 0,\\
\partial_{1,0}\partial_{2,-1} + \partial_{2,-1}\partial_{1,0} &= 0,\\
\partial^2_{1,0} + \partial_{2,-1}\partial_{0,1} + \partial_{0,1}\partial_{2,-1} &= 0.
\end{aligned}$$

Note that $\partial^2_{2,-1} = \frac{1}{2}[\mathrm{ad}_{\pi^\gamma_* \sigma}, \mathrm{ad}_{\pi^\gamma_* \sigma}] = \frac{1}{2}\mathrm{ad}_{[\pi^\gamma_* \sigma, \pi^\gamma_* \sigma]} = 0$. So, second equation is always satisfied. The proof of the remaining equations follows from (3.18) and the integrability equations in Proposition 3.3.3:

$$\begin{aligned}
\partial^2_{0,1} = \frac{1}{2}[\mathrm{ad}_P, \mathrm{ad}_P] &= \frac{1}{2}\mathrm{ad}_{[P,P]} = 0,\\
\partial_{1,0}\partial_{0,1} + \partial_{0,1}\partial_{1,0} = [\partial^\gamma_{1,0}, \mathrm{ad}_P] &= \mathrm{ad}_{\partial^\gamma_{1,0} P} = 0,\\
\partial_{1,0}\partial_{2,-1} + \partial_{2,-1}\partial_{1,0} = -[\partial^\gamma_{1,0}, \mathrm{ad}_{\pi^\gamma_* \sigma}] &= -\mathrm{ad}_{\partial^\gamma_{1,0} \pi^\gamma_* \sigma} = 0,\\
\partial^2_{1,0} + \partial_{2,-1}\partial_{0,1} + \partial_{0,1}\partial_{2,-1} = (\partial^\gamma_{1,0})^2 - [\mathrm{ad}_{\pi^\gamma_* \sigma}, \mathrm{ad}_P] &= \mathrm{ad}_{\mathrm{Curv}^\gamma} - \mathrm{ad}_{[\pi^\gamma_* \sigma, P]} = 0.
\end{aligned}$$

■

Corollary 3.3.6. *If Π is a coupling Poisson structure in the fiber bundle (E, π, B), and (γ, σ, P) are its associated geometric data (in the sense of Theorem 3.2.6), then $\partial^\gamma = \partial^\gamma_{1,0} - \mathrm{ad}_{\pi^\gamma_* \sigma} + \mathrm{ad}_P$ is a coboundary.*

Some versions of Theorem 3.3.5 can also be found in [12, 11, 6, 18].

3.4 Bigraded cohomological models

Let Π be a coupling Poisson structure in the fiber bundle (E, π, B). The Lichnerowicz - Poisson operator $\delta^\Pi : \chi_E \longrightarrow \chi_E$ is a coboundary and a graded derivation of degree 1 for both the exterior product and the Schouten - Nijenhuis bracket. On the other hand, if (γ, σ, P) are the associated geometric data to Π, then are integrbale (Theorem 3.2.6), and its associated operator $\partial^\gamma : \mathcal{V}_E \longrightarrow \mathcal{V}_E$ is a graded derivation a and coboundary (Theorem 3.3.5). Therefore, a coupling Poisson structure induces two cochain complexes: the Lichnerowicz - Poisson complex (χ_E, δ^Π), and the bigraded complex $(\mathcal{V}_E, \partial^\gamma)$ associated to its geometric data. However, it turns out that such complexes are isomorphic [6, 18].

In this Section, we give a complete proof of this fact. Moreover, such isomorphism only depends on the horizontal component of the coupling Poisson structure. This cochain complex isomorphism allows us to study the Poisson cohomology by means of the bigraded decomposition of the operator ∂^γ.

Recall that each $\sigma \in \Gamma \bigwedge^2 \mathbb{V}^0$ defines a C^∞_E-linear morphism $\flat_\sigma : \Gamma \bigwedge TE \longrightarrow \Gamma \bigwedge T^*E$ given by $(\flat_\sigma A)(X_1, \dots, X_k) := A(\sigma^\flat X_1, \dots, \sigma^\flat X_k)$. If, additionally, σ is horizontally non-degenerate, then $\flat_\sigma : \Gamma \bigwedge \mathbb{H} \longrightarrow \Gamma \bigwedge \mathbb{V}^0$ is an isomorphim. Furthermore, we can extend this map to an isomorphism from multivector fields χ_M to vertical-valued differential forms $\mathcal{V}_E$ by tensor product.

Definition 3.4.1. *For each $\sigma \in \Gamma \bigwedge^2 \mathbb{V}^0$, the linear mapping of bigraded C^∞_B-modules $\flat_\sigma : \chi_E \longrightarrow \mathcal{V}_E$ is defined by*

$$(\flat_\sigma A)(u_1, \dots, u_p; \mu_1, \dots, \mu_q) := (-1)^p A\big(\sigma^\flat \mathrm{hor}(u_1), \dots, \sigma^\flat \mathrm{hor}(u_p), \mu_1, \dots, \mu_q\big),$$

for any homogeneous element $A \in \chi^{p,q}_E$, $u_i \in \mathfrak{X}_B$ and $\mu_j \in \Omega^{0,1}_E$. This definition does not depend on the choice of the connection used to calculate the horizontal lift.

So, our goal is to prove the following result (see also [6, 18]).

Theorem 3.4.2. *Let (E, π, B) be a fiber bundle, Π a coupling Poisson structure and (γ, σ, P) its associated geometric data. Let $\delta^\Pi : \chi_E \longrightarrow \chi_E$ be the Lichnerowicz - Poisson operator of Π, and $\partial^\gamma := \partial^\gamma_{1,0} - \mathrm{ad}_{\pi^\gamma_* \sigma} + \mathrm{ad}_P$, the coboundary operator associated to (γ, σ, P). The map $\flat_\sigma : (\chi_E, \delta^\Pi) \longrightarrow (\mathcal{V}_E, \partial^\gamma)$ is a cochain complex isomorphism, i.e., the following diagram commutes:*

$$\begin{tikzcd}
\chi^k_E \arrow[r, "\delta^\Pi"] \arrow[d, "\flat_\sigma"'] & \chi^{k+1}_E \arrow[d, "\flat_\sigma"] \\
\mathcal{V}^k_E \arrow[r, "\partial^\gamma"'] & \mathcal{V}^{k+1}_E
\end{tikzcd}$$

Thus, $(\mathcal{V}_E, \partial^\gamma)$ is a bigraded model for the Lichnerowicz - Poisson complex.

To prove this theorem, we need the following result:

Proposition 3.4.3. *Let $\sigma \in \Gamma \bigwedge^2 \mathbb{V}^0$ be a horizontally non-degenerate 2-form. The map $\flat_\sigma : (\chi_E, \wedge) \longrightarrow (\mathcal{V}_E, \wedge)$ is an exterior algebra isomorphism.*

Proof. By purpose of calculus, fix a horizontal distribution $\mathbb{H}$. Since horizontal lifts generate at each point the tangent space, and $\sigma^\flat : \mathbb{H} \longrightarrow \mathbb{V}^0$ is an isomorphism, then $\mathbb{V}^0_p = \{\sigma^\flat \mathrm{hor}(u)(p) \mid u \in \mathfrak{X}_B\}$ $\forall p \in E$.Hence, any $A \in \chi^{p,q}_E$ is determined by its values on the image of projectable vector fields $\sigma^\flat\mathrm{hor}(u_i)$ and vertical forms μ_j. This proves the injectivity of $\flat_\sigma$, and the surjectivity is proved similarly. On the other hand, if $A \in \chi^{p,q}_E$, $B \in \chi^{p',q'}_E$, $u_1, \ldots, u_{p+p'} \in \mathfrak{X}_B$ and $\mu_1, \ldots, \mu_{q+q'} \in \Omega^{0,1}_E$, then

$$
\begin{aligned}
&\flat_\sigma(A \wedge B)(u_1, \ldots, u_{p+p'}; \mu_1, \ldots, \mu_{q+q'}) = \\
&(-1)^{p+p'}(A \wedge B)\big(\sigma^\flat\mathrm{hor}(u_1), \ldots, \sigma^\flat\mathrm{hor}(u_{p+p'}), \mu_1, \ldots, \mu_{q+q'}\big) = \\
&(-1)^{p+p'+p'q} \sum_{\sigma \in S_{(p,p')}, \tau \in S_{(q,q')}} (-1)^\sigma(-1)^\tau A\big(\sigma^\flat\mathrm{hor}(u_{\sigma(1)}), \ldots, \sigma^\flat\mathrm{hor}(u_{\sigma(p)}), \mu_{\tau(1)}, \ldots, \mu_{\tau(q)}\big) \\
&\qquad B\big(\sigma^\flat\mathrm{hor}(u_{\sigma(p+1)}), \ldots, \sigma^\flat\mathrm{hor}(u_{\sigma(p+p')}), \mu_{\tau(q+1)}, \ldots, \mu_{\tau(q+q')}\big) = \\
&(-1)^{p+p'+p'q} \sum_{\sigma \in S_{(p,p')}, \tau \in S_{(q,q')}} (-1)^\sigma(-1)^\tau(-1)^p \flat_\sigma(A)\big(u_{\sigma(1)}, \ldots, u_{\sigma(p)}; \mu_{\tau(1)}, \ldots, \mu_{\tau(q)}\big) \\
&\qquad (-1)^{p'}\flat_\sigma(B)\big(u_{\sigma(p+1)}, \ldots, u_{\sigma(p+p')}; \mu_{\tau(q+1)}, \ldots, \mu_{\tau(q+q')}\big) = \\
&[\flat_\sigma(A) \wedge \flat_\sigma(B)](u_1, \ldots, u_{p+p'}; \mu_1, \ldots, \mu_{q+q'}).
\end{aligned}
$$

■

Since $\flat_\sigma : (\chi_E, \wedge) \longrightarrow (\mathcal{V}_E, \wedge)$ is an exterior algebra isomorphism, it induces an isomorphism of $\mathbb{R}$-derivations $\flat^*_\sigma : \mathrm{Der}_\mathbb{R}(\mathcal{V}_E, \wedge) \longrightarrow \mathrm{Der}_\mathbb{R}(\chi_E, \wedge)$ given by $\flat^*_\sigma(D) := \flat^{-1}_\sigma \circ D \circ \flat_\sigma$. Thus, the proof of Theorem 3.4.2 is reduced to show that δ^Π and $\flat^{-1}_\sigma \circ \partial^\gamma \circ \flat_\sigma$ are the same graded derivation in $(\chi_E, \wedge)$.

Proof of Theorem 3.4.2. Recall that graded derivations in $(\chi_E, \wedge)$ are determined by their action on functions and vector fields. So it is enough to show that the identity $\flat_\sigma \circ \delta^\Pi = \partial^\gamma \circ \flat_\sigma$ holds on C^∞_E, $\Gamma\mathbb{V}$ and $\Gamma\mathbb{H}$ by straightforward calculation.

- If $f \in C^\infty_E$, then $(\partial^\gamma \circ \flat_\sigma)(f) = \partial^\gamma f = \partial^\gamma_{1,0} f + \mathrm{ad}_P f = \pi^\gamma_* \mathrm{d}_{1,0} f + [P, f]$ and $(\flat_\sigma \circ \delta^\Pi)(f) = \flat_\sigma([\Pi_{2,0}, f] + [\Pi_{0,2}, f]) = \flat_\sigma[\Pi_{2,0}, f] + [P, f]$. To prove the equality $\flat_\sigma[\Pi_{2,0}, f] = \pi^\gamma_* \mathrm{d}_{1,0} f$, just evaluate in $u \in \mathfrak{X}_B$:
$$
\begin{aligned}
\flat_\sigma[\Pi_{2,0}, f](u) &= (\mathrm{i}_{\mathrm{d}f}\Pi_{2,0})\sigma^\flat\mathrm{hor}(u) = \Pi_{2,0}(\mathrm{d}f, \sigma^\flat\mathrm{hor}(u)) \\
&= -\Pi_{2,0}(\sigma^\flat\mathrm{hor}(u), \mathrm{d}_{1,0} f) = \mathrm{d}_{1,0} f(\mathrm{hor}(u)) = (\pi^\gamma_* \mathrm{d}_{1,0} f)(u).
\end{aligned}
$$
- If $W \in \Gamma\mathbb{V}$, then $(\partial^\gamma \circ \flat_\sigma)(W) = \partial^\gamma(W) = \partial^\gamma_{1,0} W + \mathrm{ad}_P W - \mathrm{ad}_{\pi^\gamma_*\sigma} W$. By bidegrees, we must prove $\partial^\gamma_{1,0} W = \flat_\sigma[\Pi, W]_{1,1}$, $\mathrm{ad}_P W = \flat_\sigma[\Pi, W]_{0,2}$, and $-\mathrm{ad}_{\pi^\gamma_*\sigma} W = \flat_\sigma[\Pi, W]_{2,0}$. The second equality is evident since $[\Pi_{2,0}, W]$ has no vertical part. The remaining equalities are verified as follows:
$$
\begin{aligned}
\flat_\sigma[\Pi, W]_{1,1}(u; \lambda) &= -\, [\Pi, W](\sigma^\flat\mathrm{hor}(u), \lambda) \\
&= -\, \mathcal{L}_W(\Pi(\sigma^\flat\mathrm{hor}(u), \lambda)) - \sigma^\flat\mathrm{hor}(u)[W, \Pi^\sharp\lambda] + \lambda[W, \Pi^\sharp\sigma^\flat\mathrm{hor}(u)] \\
&= \lambda[\mathrm{hor}(u), W] = \lambda(\partial^\gamma_{1,0} W(u)) = \partial^\gamma_{1,0} W(u; \lambda),
\end{aligned}
$$

$$\begin{aligned}\flat_\sigma[\Pi, W]_{2,0}(u,v) =& \mathcal{L}_W(\Pi(\sigma^\flat \mathrm{hor}(u), \sigma^\flat \mathrm{hor}(v))) \\ &+ \sigma^\flat \mathrm{hor}(u)[W, \Pi^\sharp\sigma^\flat \mathrm{hor}(v)] - \sigma^\flat \mathrm{hor}(v)[W, \Pi^\sharp \sigma^\flat \mathrm{hor}(u)] \\ =& \mathcal{L}_W(\sigma(\mathrm{hor}(u), \mathrm{hor}(v))) = [W, \pi_*^\gamma \sigma(u,v)] = -\mathrm{ad}_{\pi_*^\gamma\sigma} W(u,v).\end{aligned}$$

- Let $X \in \Gamma\mathbb{H}$ be a horizontal vector field. Then $(\partial^\gamma \circ \flat_\sigma)(X) = \partial^\gamma_{1,0}\flat_\sigma(X) + \mathrm{ad}_P\flat_\sigma(X)$. By bidegrees, must prove

$$\partial^\gamma_{1,0}\flat_\sigma(X) = \flat_\sigma[\Pi, X]_{2,0}, \quad \mathrm{ad}_P\flat_\sigma(X) = \flat_\sigma[\Pi, X]_{1,1}, \quad 0 = \flat_\sigma[\Pi, X]_{0,2}. \tag{3.23}$$

The last equality in (3.23) follows from Lemma 3.2.3. The first identity in (3.23) is proved as follows,

$$\begin{aligned}&\partial^\gamma_{1,0}\flat_\sigma(X)(u,v) = \mathcal{L}_{\mathrm{hor}(u)}(\flat_\sigma(X)(v)) - \mathcal{L}_{\mathrm{hor}(v)}(\flat_\sigma(X)(u)) - \flat_\sigma(X)[u,v] = \\ &- \mathcal{L}_{\mathrm{hor}(u)}(\sigma(\mathrm{hor}(v), X)) - \mathcal{L}_{\mathrm{hor}(v)}(\sigma(X, \mathrm{hor}(u))) + \sigma(\mathrm{hor}[u,v], X) = \\ &- \mathcal{L}_{\mathrm{hor}(u)}(\sigma(\mathrm{hor}(v), X)) - \mathcal{L}_{\mathrm{hor}(v)}(\sigma(X, \mathrm{hor}(u))) + \sigma([\mathrm{hor}(u), \mathrm{hor}(v)], X) = \\ &\mathcal{L}_X(\sigma(\mathrm{hor}(u), \mathrm{hor}(v))) - \sigma([\mathrm{hor}(v), X], \mathrm{hor}(u)) - \sigma([X, \mathrm{hor}(u)], \mathrm{hor}(v)) = \\ &\mathcal{L}_X(\sigma(\mathrm{hor}(u), \mathrm{hor}(v))) + \sigma^\flat\mathrm{hor}(u)[X, \Pi^\sharp\sigma^\flat\mathrm{hor}(v)] - \sigma^\flat\mathrm{hor}(v)[X, \Pi^\sharp\sigma^\flat\mathrm{hor}(u)] = \\ &[\Pi, X](\sigma^\flat\mathrm{hor}(u), \sigma^\flat\mathrm{hor}(v)) = \flat_\sigma[\Pi, X]_{2,0}(u,v),\end{aligned}$$

where in the third equality it has been used that $\mathrm{hor}[u,v] - [\mathrm{hor}(u), \mathrm{hor}(v)]$ is vertical, and fourth equality follows from $\mathrm{d}\sigma(\mathrm{hor}(u), \mathrm{hor}(v), X) = 0$. Finally, the second identity in (3.23) is proved as follows,

$$\begin{aligned}&\mathrm{ad}_P\flat_\sigma(X)(u;\lambda) = \lambda\big([P, \flat_\sigma(X)](u)\big) = -\lambda[P, \flat_\sigma(X)(u)] = \lambda[P, \sigma(\mathrm{hor}(u), X)] = \\ &- \lambda\big(P^\sharp \mathrm{d}[\sigma(\mathrm{hor}(u), X)]\big) = \lambda\big(R(\mathrm{hor}(u), X)\big) - \lambda\big(P^\sharp\mathcal{L}_X\sigma^\flat\mathrm{hor}(u)\big) = \\ &- \lambda[X, \mathrm{hor}(u)] + \mathcal{L}_X\sigma^\flat\mathrm{hor}(u)(P^\sharp\lambda) = \mathcal{L}_X\sigma^\flat\mathrm{hor}(u)(P^\sharp\lambda) - \lambda[X, \mathrm{hor}(u)] = \\ &\mathcal{L}_X(\sigma^\flat\mathrm{hor}(u)(P^\sharp\lambda)) - \sigma^\flat\mathrm{hor}(u)[X, \Pi^\sharp\lambda] + \lambda[X, \Pi^\sharp\sigma^\flat\mathrm{hor}(u)] = \\ &- \mathcal{L}_X(\Pi(\sigma^\flat\mathrm{hor}(u), \lambda)) - \sigma^\flat\mathrm{hor}(u)[X, \Pi^\sharp\lambda] + \lambda[X, \Pi^\sharp\sigma^\flat\mathrm{hor}(u)] = \\ &- [\Pi, X](\sigma^\flat\mathrm{hor}(u), \lambda) = \flat_\sigma[\Pi, X]_{1,1}(u;\lambda),\end{aligned}$$

where, in the fifth equality, Lemma 3.2.4 has been applied.

■

Corollary 3.4.4. *Let (E, π, B) be a fiber bundle. Let also Π be a coupling Poisson structure and (γ, σ, P) its associated geometric data. If ∂^γ is the coboundary operator defined by (γ, σ, P) as in Definition 3.3.4, then we have a cohomology isomorphism*

$$(\flat_\sigma)_* : \mathcal{H}^k_{LP}(E, \Pi) \longrightarrow \mathcal{H}^k_{\partial^\gamma}.$$

This result allows us to reduce the study of Poisson cohomology of coupling structures in fiber bundles to the cohomology of a bigraded cochain complex, in the sense of Chapter 1.

Chapter 4

Geometric Splitting of First Poisson Cohomology

In this chapter we present some splitting-type results for the first cohomology group of Poisson structures. We begin by the case of regular Poisson manifolds (Theorem 4.1.4). We apply there a global scheme which can be used only for regular Poisson structures (see also [38, 32, 39]). Since our original results are motivated by the singular case, we also present a splitting theorem for the first cohomology group of coupling Poisson structures in fiber bundles (Theorem 4.2.5).

Given a closed symplectic leaf S of the Poisson manifold (M, Ψ), there is a tubular neighborhood of S diffeomorphic to the *normal bundle* E over S. Furthermore, it is well-known [36] that, under such identification, the Poisson structure Ψ is isomorphic to a coupling structure Π in the bundle E. In other words, *the study of coupling Poisson structures in fiber bundles is related to the semilocal study of Poisson structures (around symplectic leaves).* Therefore, Theorem 4.2.5 is a first step in the study of the first Poisson cohomology group of singular structures.

The main result of Chapter 3 implies that the cohomology groups of the Lichnerowicz - Poisson complex (χ_E, δ^{Π}) defined by the coupling Poisson structure are isomorphic to the cohomology groups of the bigraded cochain complex induced by the geometric data (Theorem 3.4.2). Since this second complex is bigraded, in the sense of Definition 1.2.1, we can apply the main result of Chapter 1 (Theorem 1.2.2) to obtain a splitting for the first cohomology group of this bigraded cochain complex. Combining these results, we get the desired splitting for the first cohomology group of a coupling Poisson structure. Such splitting has a natural geometric interpretation.

We present geometric conditions that simplify the calculus of the first Poisson cohomology group. As an application of Corollary 1.2.3, we study the case when the first vertical Poisson cohomology is trivial. Moreover, we present some examples in which the first Poisson cohomology group is simpler, which are not attainable in the general abstract scheme presented in Chapter 1. These examples arise when the Poisson structure on each fiber does not admit global non-trivial Casimir functions. We present some examples for which the Poisson structure on the fiber is a Lie-Poisson structure. The symplectic foliation on the fiber of most of such examples are *open book*, which allows to prove that they do not admit Casimir functions.

4.1 Regular case

In this part we present a geometric splitting for the first cohomology group of a regular Poisson manifold (M, Π). Since the characteristic distribution is regular and involutive, the $\mathbb{R}$-space of *tangent vector fields* is a Lie subalgebra of vector fields. Moreover, tangent multivector fields form a graded Poisson subalgebra of multivector fields with the Schouten - Nijenhuis bracket. This implies that tangent multivector fields form a cochain subcomplex of the Lichnerowicz - Poisson complex, which is called *tangential Poisson complex* [8]. Furthermore, the tangential Poisson complex is isomorphic to the *leafwise de Rham complex*, which is well understood in some particular but important cases [32].

On the other hand, we present a bigrading for the Lichnerowicz - Poisson complex. To do this, we fix a generalized connection γ in the Poisson manifold (M, Π) such that the horizontal distribution of γ and the characteristic distribution of Π coincide: $\mathbb{H} = D^{\Pi}$. With such bigrading, the the Lichnerowicz - Poisson complex (χ_M, δ^{Π}) is a bigraded cochain complex, in the sense of Definition 1.2.1.

Finally, we derive the following splitting for the first Poisson cohomology group:

$$\mathcal{H}^1_{LP}(M, \Pi) \simeq H^1_{\mathrm{d}_{\mathcal{S}}} \oplus \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \mid \mathcal{L}_Y \omega \text{ is } \mathrm{d}_{\mathcal{S}}\text{-exact}\}.$$

The first factor is precisely the leafwise de Rham cohomology and corresponds to the tangential Poisson cohomology. The second factor consists on the trivial deformations of the leafwise symplectic 2-form ω. This splitting is derived as consequence of Theorem 1.2.2 of Chapter 1, applied to the Lichnerowicz - Poisson complex with the bigrading given as explained in above.

4.1.1 Tangential Poisson complex

Let (M, Π) be a regular Poisson manifold with $\mathrm{rank}\Pi = k$. In this case, the characteristic distribution D^{Π} is a subbundle of TM. So, the symplectic foliation $(\mathcal{S}, \omega)$ is a regular foliation. Moreover, since $D^{\Pi} = T\mathcal{S}$ is an involutive distribution, the algebra of **tangent multivector fields** $\Gamma \bigwedge T\mathcal{S} \simeq \Gamma \bigwedge D^{\Pi}$ is a graded Poisson subalgebra of multivector fields $(\chi_M, \wedge, [,])$, with the exterior product and the Schouten - Nijenhuis bracket. In particular, since $\Pi \in \Gamma \bigwedge^2 D^{\Pi}$, we get that tangent multivector fields define a cochain subcomplex of the Lichnerowicz - Poisson complex:

$$\delta^{\Pi}\left(\Gamma \bigwedge D^{\Pi}\right) \subset \Gamma \bigwedge D^{\Pi}.$$

The pair $\left(\Gamma \bigwedge D^{\Pi}, \delta^{\Pi}\right)$ is called **tangential Poisson complex**. The spaces of cocycles, coboundaries and cohomologies are respectively denoted by $\mathcal{Z}^k_{\tan}(M, \Pi)$, $\mathcal{B}^k_{\tan}(M, \Pi)$ and $\mathcal{H}^k_{\tan}(M, \Pi)$.

Recall that, in general, the map $\sharp_{\Pi} : \Omega_M \longrightarrow \chi_M$ is a cochain complex morphism, which induces a cohomology morphism from the de Rham to the Poisson complex by $\sharp_\Pi^*[\alpha] := [\sharp_\Pi \alpha]$. In the regular case, the map $\sharp_\Pi^* : H^k_{dR}(M) \longrightarrow \mathcal{H}^k_{\tan}(M,\Pi)$ values on the tangential Poisson cohomology group.

Also, recall that, associated to each regular foliation on a differential manifold, there is a cochain complex called **leafwise de Rham complex**. For the symplectic foliation $(\mathcal{S},\omega)$, the *foliated exterior differential* $\mathrm{d}_\mathcal{S}$ is a coboundary operator acting on *leafwise differential forms* $\Gamma \bigwedge T^*\mathcal{S}$ by

$$\begin{aligned} \mathrm{d}_\mathcal{S}\lambda(Y_0,\dots,Y_p) \;:=\; & \sum_{i=0}^{p}(-1)^i(\imath_* Y_i)(\lambda(Y_0,\dots,\widehat{Y_i},\dots,Y_p)) \\ & + \sum_{i<j}(-1)^{i+j}\lambda([Y_i,Y_j],Y_1,\dots,\widehat{Y_i},\dots,\widehat{Y_j},\dots,Y_p), \end{aligned}$$

for all $Y_0,\dots,Y_p \in \Gamma(T\mathcal{S})$. Indeed, it can be shown that the triple $(\Gamma \bigwedge T\mathcal{S}, \imath, [,])$ is a Lie algebroid. So, $\mathrm{d}_\mathcal{S}$ is a Lie algebroid differential, and the pair $(\Gamma \bigwedge T^*\mathcal{S}, \mathrm{d}_\mathcal{S})$ is the *leafwise de Rham complex.*

Since Π is a section of $\bigwedge^2 T\mathcal{S}$, the maps $\Pi^\sharp$ and $\sharp_\Pi$, defined in page 44, can be naturally defined for leafwise differential forms in the same fashion:

$$\langle \beta, \Pi^\sharp \alpha\rangle := \Pi(\alpha,\beta), \qquad\qquad \sharp_\Pi\lambda(\alpha_1,\dots,\alpha_k) := \lambda(\Pi^\sharp\alpha_1,\dots,\Pi^\sharp\alpha_k).$$

On the other hand, the symplectic structure ω is a leafwise differential 2-form, i.e., $\omega \in \Gamma \bigwedge^2 T^*\mathcal{S}$, which is given by $\omega(X_f,X_g) := \Pi(\mathrm{d}f,\mathrm{d}g)$. Moreover, ω induces the following maps

$$\omega^\flat : \Gamma(T\mathcal{S}) \longrightarrow \Gamma(T^*\mathcal{S}), \qquad\qquad \flat_\omega : \Gamma \bigwedge T\mathcal{S} \longrightarrow \Gamma \bigwedge T^*\mathcal{S},$$

by $\omega^\flat(X) := -\mathrm{i}_X\omega$ and $\flat_\omega A(X_1,\dots,X_k) := A(\omega^\flat X_1,\dots,\omega^\flat X_k)$. It is clear that

$$\begin{aligned} (\omega^\flat \circ \Pi^\sharp)(\alpha) = -\alpha, &\qquad (\sharp_\Pi \circ \flat_\omega)(A) = (-1)^{|A|}A, \\ (\Pi^\sharp \circ \omega^\flat)(X) = -X, &\qquad (\flat_\omega \circ \sharp_\Pi)(\lambda) = (-1)^{|\lambda|}\lambda. \end{aligned}$$

Thus, $\sharp_\Pi$ and $\flat_\omega$ are isomorphisms. Furthermore, both are exterior algebra isomorphisms, which is easy to verify by simple calculation.

Proposition 4.1.1. *The map $\sharp_\Pi : (\Gamma \bigwedge T^*\mathcal{S}, \mathrm{d}_\mathcal{S}) \longrightarrow \left(\Gamma \bigwedge T\mathcal{S}, \delta^\Pi\right)$ is a cochain complex isomorphism. In particular, there is a cohomology isomorphism $(\sharp_\Pi)_*$: $H^k_{\mathrm{d}_\mathcal{S}} \longrightarrow H^k_{\tan}(M,\Pi)$.*

Proof. Recall that δ^Π and $\mathrm{d}_\mathcal{S}$ are graded derivations of their respective exterior algebras. So, we just need to prove that $\delta^\Pi \circ \sharp_\Pi = \sharp_\Pi \circ \mathrm{d}_\mathcal{S}$ holds on C^∞_M and $\Gamma(T^*\mathcal{S})$. First note for $f \in C^\infty_M$ that $(\delta^\Pi \circ \sharp_\Pi)(f) = [\Pi, f] = -X_f = -\Pi^\sharp \mathrm{d}_\mathcal{S} f = \sharp_\Pi \mathrm{d}_\mathcal{S} f = (\sharp_\Pi \circ \mathrm{d}_\mathcal{S})(f)$. On the other hand, $\Gamma(T^*\mathcal{S})$ is locally generated by elements of the form $f\mathrm{d}_\mathcal{S} g$. Moreover, $\delta^\Pi \circ \sharp_\Pi$ and $\sharp_\Pi \circ \mathrm{d}_\mathcal{S}$ are graded derivations coinciding in f and vanishing in $\mathrm{d}_\mathcal{S} g$. Therefore, both operators coincide in $\Gamma(T^*\mathcal{S})$, as desired. ■

4.1.2 The bigraded Lichnerowicz - Poisson complex

By means of a generalized connection, the above isomorphism can be naturally extended to the whole algebra of multivector fields. Recall that a generalized connection γ in M is a vector bundle endomorphism $\gamma : TM \longrightarrow TM$ of constant rank such that $\gamma^2 = \gamma$. The kernel and the image of γ are regular distributions, respectively called *horizontal* and *vertical* distributions: $\mathbb{H} := \ker(\gamma)$, $\mathbb{V} := \mathrm{Im}(\gamma)$. It turns out that $TM := \mathbb{H} \oplus \mathbb{V}$.

Let (M, Π) be a regular Poisson manifold. It turns out that the symplectic foliation $(\mathcal{S}, \omega)$ is regular. Now, fix a generalized connection γ in M such that *the horizontal distribution of* γ *coincides with the characteristic distribution of* Π: $\mathbb{H} := D^{\Pi}$. This induces a *bigrading* in M (see Section 2.5). In particular, multivector fields become a bigraded algebra with the exterior product. The bigrading in this case is given by

$$\chi_M^{\bullet,\bullet} = \bigoplus_{p,q\in\mathbb{Z}} \chi_M^{p,q},$$

where $\chi_M^{p,q} := \Gamma\left(\bigwedge^p \mathbb{H} \wedge \bigwedge^q \mathbb{V}\right)$. Such bigrading has the following properties:

- The horizontal distribution $\mathbb{H}$ is involutive.
- The bivector field Π is a section of $\bigwedge^2 \mathbb{H}$.

The first property implies that the curvature of γ is zero: $R = 0$. Thus, the bigraded decomposition of the exterior differential is $\mathrm{d} = \mathrm{d}_{1,0} + \mathrm{d}_{0,1} + \mathrm{d}_{-1,2}$, and the components have Frölicher - Nijenhuis decompositions $\mathrm{d}_{1,0} = \mathcal{L}_{\mathrm{Id}_{TM}-\gamma} - \mathrm{i}_{R'}$, $\mathrm{d}_{0,1} = \mathcal{L}_{\gamma} + 2\mathrm{i}_{R'}$, $\mathrm{d}_{-1,2} = -\mathrm{i}_{R'}$, due to Theorem 2.5.6. The second property means that, with this bigrading, Π has bidegree $(2, 0)$.

Observe that the differential operators i_{Π} and $\mathrm{d}_{-1,2}$ commute with each other. Indeed, since $\mathrm{i}_{\Pi} \in \mathcal{D}_2^{-2,0}(M)$ and $\mathrm{d}_{-1,2} \in \mathcal{D}_1^{-1,2}$, we have $[\mathrm{i}_{\Pi}, \mathrm{d}_{-1,2}] \in \mathcal{D}_2^{-3,2}(M)$. If the degree of $\alpha \in \Omega_M$ is equal or less than 2, then, in particular, its horizontal bidegree is at most 2, and the horizontal bidegree of $[\mathrm{i}_{\Pi}, \mathrm{d}_{-1,2}](\alpha)$ is negative. Hence, $[\mathrm{i}_{\Pi}, \mathrm{d}_{-1,2}]$ is a differential operator of order equal or less than 2 vanishing in forms of degree equal or less than 2. This implies $[\mathrm{i}_{\Pi}, \mathrm{d}_{-1,2}] = 0$, due to Proposition 2.3.4.

As consequence of our above discussion, the bigraded decomposition of the Koszul - Brylinski operator is $\mathcal{L}_{\Pi} = [\mathrm{i}_{\Pi}, \mathrm{d}] = [\mathrm{i}_{\Pi}, \mathrm{d}_{1,0}] + [\mathrm{i}_{\Pi}, \mathrm{d}_{0,1}]$. Now, fix $Q \in \chi_M^{h,k}$ $(h+k = q)$ and $\varphi \in \Omega_M^{r,s}$ $(r+s = q+1)$. By the definition of the Schouten - Nijenhuis bracket,

$$\begin{aligned}\langle \varphi, \delta^{\Pi} Q\rangle = \langle \varphi, [\Pi, Q]\rangle &= (-1)^{\frac{q(q+1)}{2}} \mathrm{i}_{[\Pi,Q]}\varphi = (-1)^{\frac{q(q+1)}{2}} [\mathcal{L}_{\Pi}, \mathrm{i}_Q]\varphi \\ &= (-1)^{\frac{q(q+1)}{2}} [[\mathrm{i}_{\Pi}, \mathrm{d}_{0,1}], \mathrm{i}_Q]\varphi + (-1)^{\frac{q(q+1)}{2}} [[\mathrm{i}_{\Pi}, \mathrm{d}_{1,0}], \mathrm{i}_Q]\varphi.\end{aligned}$$

Observing that each summand in this equation is a bigraded operator in χ_M, we arrive at the following result.

Theorem 4.1.2. *Let (M, Π) a regular Poisson manifold with characteristic distribution D^Π. Fix a connection γ such that its horizontal distribution is D^Π,*

$$\mathbb{H} = D^\Pi, \qquad \mathbb{V} = \mathrm{Im}(\gamma), \qquad TM = \mathbb{H} \oplus \mathbb{V}. \tag{4.1}$$

The Lichnerowicz - Poisson operator has the bigraded decomposition $\delta^\Pi = \delta^\Pi_{1,0} + \delta^\Pi_{2,-1}$, where $\delta^\Pi_{1,0}$ and $\delta^\Pi_{2,-1}$ are graded commutative coboundary operators given by

$$\langle \varphi, \delta^\Pi_{1,0} Q \rangle := (-1)^{\frac{q(q+1)}{2}} [[\mathrm{i}_\Pi, \mathrm{d}_{0,1}], \mathrm{i}_Q] \varphi,$$
$$\langle \varphi, \delta^\Pi_{2,-1} Q \rangle := (-1)^{\frac{q(q+1)}{2}} [[\mathrm{i}_\Pi, \mathrm{d}_{2,-1}], \mathrm{i}_Q] \varphi.$$

Remark 4.1.3. *A version of Theorem 4.1.2 is given in [32]. The idea of the proof is very similar: to use a bigrading of the Lichnerowicz - Poisson complex associated to a distribution complementary to the characteristic distribution D^Π. However, the bigrading in [32] is different to* (4.1)*, since it is chosen the characteristic distribution as the vertical distribution.*

4.1.3 First cohomology of regular Poisson structures

Consider the bigrading in M given by the generalized connection γ induced by the decomposition (4.1). Let us apply Theorem 1.2.2 to the Lichnerowicz-Poisson complex in the regular case. For $\partial = \delta^\Pi$, take $\partial_{1,0} = \delta^\Pi_{1,0}$, $\partial_{0,1} = 0$, $\partial_{2,-1} = \delta^\Pi_{2,-1}$. Since $\partial_{0,1} = 0$, the intrinsic objects defined in Chapter 1 can be easily computed. In this case, operator $\overline{\partial}_{1,0}$ is given by the restriction of δ^Π to

$$C^{\bullet,0} = \chi^{\bullet,0}_M = \Gamma \bigwedge \mathbb{H} = \Gamma \bigwedge D^\Pi,$$

which are precisely *tangent multivector fields*. Therefore, the complex $(\mathcal{Z}^{\bullet,0}_{\partial_{0,1}}, \overline{\partial}_{1,0})$ is the *tangential Poisson complex*: $(\Gamma \bigwedge D^\Pi, \delta^\Pi)$ (see [32, 8]).

The subspace $\mathcal{A}$ is given by transversal vector fields preserving the foliation, or *vertical projectable vector fields*

$$\mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) = \{X \in \Gamma\mathbb{V} \mid [X, \Gamma\mathbb{H}] \subset \Gamma\mathbb{H}\}.$$

Indeed, note that

$$\begin{aligned}\mathcal{A} &= \{Y \in C^{0,1} \mid \partial_{1,0}(Y) = 0\} = \{Y \in \Gamma\mathbb{V} \mid \delta^\Pi_{1,0}(Y) = 0\} \\ &= \{Y \in \Gamma\mathbb{V} \mid [\Pi, Y]_{1,1} = 0\} = \left\{Y \in \Gamma\mathbb{V} \mid [\Pi, Y] \in \Gamma \wedge^2 D^\Pi\right\},\end{aligned}$$

which proves $\mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \subset \mathcal{A}$. Conversely, if $Y \in \mathcal{A}$, then

$$[Y, X_f] = -[Y, [\Pi, f]] = -[[Y, \Pi], f] - [\Pi, [Y, f]] = \mathrm{i}_{\mathrm{d}f}[Y, \Pi] + \mathrm{i}_{\mathcal{L}_Y f}\Pi.$$

The right-hand side summands are sections to D^Π. Thus, $[Y, X_f] \in \Gamma D^\Pi$. Since D^Π is generated by Hamiltonian vector fields, this implies that $Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H})$, proving that $\mathcal{A} \subset \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H})$. Thus, $\mathcal{A} = \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H})$ is the space of transversal vector fields preserving the foliation. This *is not* a Lie subalgebra of vector fields excepting the case in which vertical distribution $\mathbb{V}$ is involutive, i.e., if the generalized connection γ is flat: $[\gamma, \gamma]_{FN} = 0$.

The morphism $\rho : \mathcal{A} \longrightarrow \mathcal{H}^2_{\tan}(M, \Pi)$ is given by $\rho(Y) = [\delta^{\Pi}_{2,-1}(Y)]$. Therefore,

$$\begin{aligned}\ker \rho &= \{Y \in \mathcal{A} \mid \delta^{\Pi}_{2,-1}(Y) \text{ is } \delta^{\Pi}_{1,0}\text{-exact}\} = \{Y \in \mathcal{A} \mid \mathcal{L}_Y \Pi \text{ is } \delta^{\Pi}_{1,0}\text{-exact}\} \\ &= \{Y \in \mathcal{A} \mid \mathcal{L}_Y \Pi = \delta^{\Pi}(Z), Z \in \Gamma D^{\Pi}\} = \{Y \in \mathcal{A} \mid \sharp_{\Pi}(\mathcal{L}_Y \omega) = \delta^{\Pi}(\sharp_{\Pi}(\alpha)), \alpha \in \Omega^1_M\} \\ &= \{Y \in \mathcal{A} \mid \sharp_{\Pi}(\mathcal{L}_Y \omega) = \sharp_{\Pi}(\mathrm{d}_{\mathcal{S}} \alpha), \alpha \in \Omega^1_M\} = \{Y \in \mathcal{A} \mid \mathcal{L}_Y \omega = \mathrm{d}_{\mathcal{S}} \alpha, \alpha \in \Omega^1_M\} \\ &= \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \mid \mathcal{L}_Y \omega \text{ is } \mathrm{d}_{\mathcal{S}}\text{-exact}\}.\end{aligned}$$

The above discussion allows us to prove the following result.

Theorem 4.1.4. *Let Π be a regular Poisson structure in a manifold M with characteristic distribution D^{Π}. Fix a distribution $\mathbb{V} \subset TM$ complementary to the characteristic distribution $\mathbb{H} = D^{\Pi}$. The first Poisson cohomology group splits as*

$$\mathcal{H}^1_{LP}(M, \Pi) \simeq H^1_{\mathrm{d}_{\mathcal{S}}} \oplus \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \mid \mathcal{L}_Y \omega \textit{ is } \mathrm{d}_{\mathcal{S}}\textit{-exact}\}.$$

Proof. Because of Theorem 4.1.2, the Lichnerowicz - Poisson complex (χ_M, δ^{Π}) is a bigraded cochain complex in the sense of Definition 1.2.1 with the bigrading induced by γ. Applying Theorem 1.2.2 to the bigraded cochain complex (χ_M, δ^{Π}), we arrive to the following short exact sequence:

$$0 \to \mathcal{H}^1_{\partial_{1,0}} \hookrightarrow \mathcal{H}^1_{\delta^{\Pi}} \to \frac{\ker \rho}{\mathcal{B}^1_{\partial_{0,1}}} \to 0. \tag{4.2}$$

Since $\partial_{0,1} = 0$, we have $\mathcal{B}^1_{\partial_{0,1}} = \{0\}$. On the other hand, our previous analysis shows that $\mathcal{H}^1_{\partial_{1,0}}$ is precisely the first tangential Poisson cohomology: $\mathcal{H}^1_{\tan}(M, \Pi)$. Furthermore, Proposition 4.1.1 implies that $\mathcal{H}^1_{\tan}(M, \Pi)$ is isomorphic to the first leafwise de Rham cohomology group of the symplectic foliation $\mathcal{S}$:

$$\mathcal{H}^1_{\tan}(M, \Pi) \simeq H^1_{\mathrm{d}_{\mathcal{S}}}.$$

Finally, our previous analysis also shows that

$$\ker \rho = \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \mid \mathcal{L}_Y \omega \text{ is } \mathrm{d}_{\mathcal{S}}\text{-exact}\}.$$

Therefore, the sequence in (4.2) is equivalent to the following,

$$0 \to H^1_{\mathrm{d}_{\mathcal{S}}} \hookrightarrow \mathcal{H}^1_{LP}(M, \Pi) \to \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \mid \mathcal{L}_Y \omega \text{ is } \mathrm{d}_{\mathcal{S}}\text{-exact}\} \to 0.$$

Since we are in the category of $\mathbb{R}$-vector spaces, this implies

$$\mathcal{H}^1_{LP}(M, \Pi) \simeq H^1_{\mathrm{d}_{\mathcal{S}}} \oplus \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \mid \mathcal{L}_Y \omega \text{ is } \mathrm{d}_{\mathcal{S}}\text{-exact}\},$$

as desired. ∎

The following consequence of 4.1.4 can be found, for example, in [38, 31, 32].

Corollary 4.1.5. *Let (S, ω) be a symplectic manifold of finite Betti numbers, and N any differential manifold with the trivial Poisson structure. Take $M = S \times N$ as product of Poisson manifolds. Then*

$$\mathcal{H}^1_{LP}(M, \Pi) \simeq (H^1_{dR}(S) \otimes C^\infty_N) \oplus \mathfrak{X}_N$$

Proof. In this particular case, the leaves of symplectic foliation of M are $S \times \{n\}$, with $n \in N$, and the leafwise symplectic structure at each leaf is ω in $S \times \{n\}$. Since S has finite Betti numbers,

$$H^1_{\mathrm{d}_\mathcal{S}} \simeq H^1_{dR}(S) \otimes C^\infty_N.$$

On the other hand, let $\sigma \in \Omega^2_M$ be defined by $\mathrm{i}_Y\sigma = 0$ if $Y \in \Gamma\mathbb{V}$ and by $\sigma(X_f, X_g) = \omega(X_f, X_g)$. Then,

$$\mathcal{L}_Y\sigma = \mathrm{i}_Y\mathrm{d}\sigma + \mathrm{d}\mathrm{i}_Y\sigma = 0,$$

so $\mathcal{L}_Y\omega = 0$. Thus $\{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \mid \mathcal{L}_Y\omega \text{ is } \mathrm{d}_\mathcal{S}\text{-exact}\} = \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(M, \mathbb{H}) \simeq \mathfrak{X}_N$. ■

Notice that the scheme presented in this section only works for regular Poisson structures. Indeed, the choice of the characteristic distribution as the horizontal distribution is only possible if the Poisson structure is regular. In the following sections we apply the results of Chapter 3 to the case of coupling Poisson structures in fiber bundles whose characteristic distribution may be singular.

4.2 The case of coupling Poisson structures

In this part, we get a splitting-type result for the first cohomology of a coupling Poisson structure Π in a fiber bundle (E, π, B). To do this, we apply the results of Chapter 1 to the bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$ defined by the geometric data (γ, σ, P) associated to Π.

First of all, we apply the cochain complex isomorphism given in Theorem 3.4.2 to describe the infinitesimal automorphisms of a coupling Poisson structure Π in terms of the geometric data (γ, σ, P). Since $(\mathcal{V}_E, \partial^\gamma)$ is a bigraded cochain complex, the equations for an infinitesimal automorphism of Π are bigraded equations for two parameters in $\mathcal{V}_E$: a vertical vector field $W \in \Gamma\mathbb{V}$ and a horizontal 1-form $\theta \in \Gamma\mathbb{H}$.

Also, for the bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$, we give a geometric description of the objects introduced in Chapter 1. We begin by describing geometrically the coboundary spaces of $\partial_{0,1} = \mathrm{ad}_P$. Then, we show that the subspace $\mathcal{A}^\gamma$ is a Lie $\mathbb{R}$-subalgebra of vertical infinitesimal automorphisms, and we also describe the morphism ρ. From this geometric point of view, we are able to present conditions which simplify the calculus of the first Poisson cohomology group of Π.

4.2.1 Infinitesimal Poisson automorphisms

Let (E, π, B) a fiber bundle and Π a coupling Poisson structure on E,

$$\Pi^\sharp(\mathbb{V}^0) \oplus \mathbb{V} = TE, \qquad \mathbb{V} = \ker(\pi_*). \tag{4.3}$$

Let also (γ, σ, P) be the *integrable geometric data* associated to the coupling structure Π (in the sense of Theorem 3.2.6). The splitting (4.3) induces a bigrading in M such that the coupling Poisson structure and the *exterior differential* have the bigraded decompositions $\Pi = \Pi_{2,0} + \Pi_{0,2}$ and $\mathrm{d} = \mathrm{d}_{1,0} + \mathrm{d}_{0,1} + \mathrm{d}_{2,-1}$.

On the other hand, the involutivity of $\mathbb{V}$ implies that the $\mathbb{R}$-space of *vertical multivector fields* $\chi_{\mathbb{V}}(E) := \Gamma \bigwedge \mathbb{V}$ has a Poisson algebra structure induced by the exterior product and by the Schouten - Nijenhuis bracket. Taking in account that Ω_B also has exterior algebra structure, the $\mathbb{R}$-space of *vertical-valued forms in base* $\mathcal{V}_E = \Omega_B \otimes_{C^\infty_B} \chi_{\mathbb{V}}(E)$ can be naturally endowed with a Poisson algebra structure: $(\mathcal{V}_E, \wedge, [,])$. The *adjoint* operator of $\eta \in \mathcal{V}_E$ respect to the bracket $[,]$,

$$\begin{aligned} \mathrm{ad}_\eta : \mathcal{V}_E &\longrightarrow \mathcal{V}_E, \\ \nu &\longmapsto \mathrm{ad}_\eta(\nu) := [\eta, \nu], \end{aligned}$$

is a graded derivation of both operations $\wedge$ and $[,]$. On the other hand, the *covariant exterior differential* $\partial^\gamma_{1,0}$ associated to an Ehresmann connection γ is also a graded derivation of both operations $\wedge$ and $[,]$ (see Definition 3.3.2). Because of Theorem 3.3.5, the integrable geometric data (γ, σ, P) define a coboundary operator ∂^γ in terms of their bigraded components by $\partial^\gamma := \partial^\gamma_{1,0} - \mathrm{ad}_{\pi^\gamma_* \sigma} + \mathrm{ad}_P$. Moreover, Theorem 3.4.2 says that the given bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$ is isomorphic to the Lichnerowicz - Poisson complex (χ_E, δ^Π) induced by the coupling structure Π. This isomorphism is induced by σ in a natural fashion, and is denoted by $\flat_\sigma$.

We now apply Theorem 3.4.2 to get the following immediate consequence.

Lemma 4.2.1. *If $A \in \chi_E$, then $\delta^\Pi(A) = 0$ if and only if $\partial^\gamma \flat_\sigma(A) = 0$.*

Further, this allow us to re-write conditions for a function and a vector field to be 0 and 1-cocycles of the Lichnerowicz - Poisson complex in terms of the geometric data (γ, σ, P).

Lemma 4.2.2. *Let Π be a coupling Poisson structure on (E, π, B) and (γ, σ, P) its associated geometric data. A smooth function $f \in C^\infty_E$ is Casimir for Π if and only if the following conditions are satisfied:*

1. *f is $\mathrm{d}_{1,0}$-closed: $\mathrm{d}f \in \Gamma\mathbb{H}^0$.*
2. *f is a Casimir function for P.*

Proof. Due to Lemma 4.2.1, $[\Pi, f] = 0$ if and only if $\partial^\gamma f = \partial^\gamma\big(\flat_\sigma(f)\big) = 0$. From the proof of Theorem 3.4.2, this condition is equivalent to $\pi^\gamma_* \mathrm{d}_{1,0} f + [P, f] = 0$. Splitting in bidegrees, and recalling that π^γ_* is an isomorphism, we get $\mathrm{d}_{1,0} f = 0$ and $[P, f] = 0$. ■

Proposition 4.2.3. *Let Π be a coupling Poisson structure on a fiber bundle (E, π, B) and (γ, σ, P) its associated geometric data. A vector field $Z + W \in \mathfrak{X}_E$, with $W \in \Gamma\mathbb{V}$ and $Z = \Pi^\sharp_{2,0}\theta$, is an infinitesimal Poisson automorphism for Π if and only if*

1. $\mathcal{L}_W\sigma = \mathrm{d}_{1,0}\theta$,
2. $[X, W] = P^\sharp \mathrm{d}\,[\theta(X)] \;\; \forall X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M)$,
3. *W is an infinitesimal Poisson automorphism of P, $\mathcal{L}_W P = 0$.*

Proof. Recall that $\theta = -\sigma^\flat Z$. It follows from Corollary 4.2.1 that $Z + W$ is an infinitesimal automorphism for Π if and only if $\partial^\gamma \flat_\sigma(Z + W) = \partial^\gamma(-\pi^\gamma_*\theta + W) = 0$. On the other hand, by the bigraded decomposition $\partial^\gamma = \partial^\gamma_{1,0} + \mathrm{ad}_P - \mathrm{ad}_{\pi^\gamma_*\sigma}$, it follows that $\partial^\gamma(-\pi^\gamma_*\theta + W) = 0$ is equivalent to the following three equations:

$$\partial^\gamma_{1,0}\pi^\gamma_*\theta = -[\pi^\gamma_*\sigma, W], \tag{4.4}$$
$$\partial^\gamma_{1,0}W = [P, \pi^\gamma_*\theta], \tag{4.5}$$
$$[P, W] = 0. \tag{4.6}$$

Note that the left-hand side of equation (4.4) is $\partial^\gamma_{1,0}\pi^\gamma_*\theta = \pi^\gamma_*\mathrm{d}_{1,0}\theta$. Since π^γ_* is an isomorphism, to prove the equivalence between 1 and (4.4), it suffices to show that $[\pi^\gamma_*\sigma, W] = -\pi^\gamma_*\mathcal{L}_W\sigma$. Indeed, if $u, v \in \mathfrak{X}_B$, then

$$\begin{aligned}
[\pi^\gamma_*\sigma, W](u, v) &= [\pi^\gamma_*\sigma(u, v), W] \\
&= [\sigma(\mathrm{hor}(u), \mathrm{hor}(v)), W] = -\mathcal{L}_W\big(\sigma(\mathrm{hor}(u), \mathrm{hor}(v))\big) \\
&= -(\mathcal{L}_W\sigma)(\mathrm{hor}(u), \mathrm{hor}(v)) - \sigma([W, \mathrm{hor}(u)], \mathrm{hor}(v)) - \sigma(\mathrm{hor}(u), [W, \mathrm{hor}(v)]) \\
&= -(\mathcal{L}_W\sigma)(\mathrm{hor}(u), \mathrm{hor}(v)) \\
&= -(\pi^\gamma_*\mathcal{L}_W\sigma)(u, v),
\end{aligned}$$

proving that equation (4.4) is equivalent to $\mathcal{L}_W\sigma = \mathrm{d}_{1,0}\theta$. Now, let us prove the equivalence between 2 and equation (4.5). Evaluating $u \in \mathfrak{X}_B$ on each side of (4.5) gives

$$\partial^\gamma_{1,0}W(u) = [\mathrm{hor}(u), W],$$

$$[P, \pi^\gamma_*\theta](u) = -[P, \pi^\gamma_*\theta(u)] = P^\sharp\mathrm{d}[\theta(\mathrm{hor}(u))].$$

Since any projectable field is an horizontal lift and conversely, we have that equation (4.5) is equivalent to

$$[X, W] = P^\sharp\mathrm{d}[\theta(X)] \qquad \forall X \in \mathfrak{X}^{\mathbb{H}}_{\mathrm{pr}}(M),$$

as desired. Finally, the integrability condition for (γ, σ, P) implies that P is a Poisson bivector field. Since W is vertical, $[P, W]$ in $\mathcal{V}_E$ equals the Schouten - Nijenhuis bracket $[P, W]$ in χ_E. Therefore, equation (4.6) is precisely the condition for W to be an infinitesimal automorphism for P. ∎

4.2.2 First cohomology of coupling Poisson structures

Let Π be a coupling Poisson structure on a fiber bundle (E, π, B) and (γ, σ, P) its associated geometric data. Define the graded operator $\partial^\gamma : \mathcal{V}_E \longrightarrow \mathcal{V}_E$ by its bigraded components

$$\partial_{1,0} = \partial^\gamma_{1,0}, \quad \partial_{0,1} = \mathrm{ad}_P, \quad \partial_{2,-1} = -\mathrm{ad}_{\pi^\gamma_* \sigma}.$$

Because of Theorem 3.3.5, $(\mathcal{V}_E, \partial^\gamma)$ is a bigraded cochain complex. Furthermore, Theorem 3.4.2 implies that the first Poisson cohomology group of Π is isomorphic to the first cohomology group of the bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$. In order to apply our results of Chapter 1, we need to study the cocycles of $\partial_{0,1} = \mathrm{ad}_P$.

Lemma 4.2.4. *For the bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$, the coboundary spaces of $\partial_{0,1} = \mathrm{ad}_P$ are $\mathcal{Z}^{p,0}_{\partial_{0,1}} = \Omega^p_B \otimes \mathrm{Casim}(E, P)$.*

Proof. Fix $\eta \in \mathcal{V}^{p,0}_E$. Then,

$$\partial_{0,1}\eta(u_1, \ldots, u_p) = \mathrm{ad}_P \eta(u_1, \ldots, u_p) = [P, \eta](u_1, \ldots, u_p) = (-1)^p [P, \eta(u_1, \ldots, u_p)],$$

where in the right-hand side we have the Schouten = Nijenhuis bracket. Therefore, $\eta \in \mathcal{Z}^{p,0}_{\partial_{0,1}}$ if and only if $\eta(u_1, \ldots, u_p) \in \mathrm{Casim}(E, P)$ for all $u_i \in \mathfrak{X}_B$. This is precisely $\eta \in \Omega^p_B \otimes \mathrm{Casim}(E, P)$, as desired. ∎

The results of Chapter 1 imply that the covariant exterior derivative of γ, restricted to

$$\overline{\partial}^\gamma_{1,0} := \partial^\gamma_{1,0}|_{\Omega_B \otimes \mathrm{Casim}(E,P)}$$

is a coboundary operator in $\Omega_B \otimes \mathrm{Casim}(E, P)$. This fact can also be verified directly by equations (3.21) and (3.10).

Now, we need to study the subspace $\mathcal{A}^\gamma$. By definition of $\partial_{1,0} = \partial^\gamma_{1,0}$ and $\partial_{0,1} = \mathrm{ad}_P$, the space $\mathcal{A}^\gamma$ is given by

$$\mathcal{A}^\gamma = \{Y \in \mathrm{Poiss}_{\mathbb{V}}(E, P) \mid \exists \beta_Y \in \Omega_B \otimes C^\infty_E \ : \ [Y, \mathrm{hor}^\gamma u] = P^\sharp \beta_Y(u) \ \forall u \in \mathfrak{X}_B\}.$$

Note that $\mathcal{A}^\gamma$ is a Lie subalgebra of $\mathrm{Poiss}_{\mathbb{V}}(E, P)$. Indeed, if $Y, Z \in \mathcal{A}^\gamma$ and $u \in \mathfrak{X}_B$, then $[Z, \mathrm{hor}^\gamma u]$ and $[Y, \mathrm{hor}^\gamma u]$ are Hamiltonian. Since $\mathrm{Ham}(E, P)$ is an ideal in $\mathrm{Poiss}_{\mathbb{V}}(E, P)$, we have that

$$[[Y, Z], \mathrm{hor}^\gamma u] = [Y, [Z, \mathrm{hor}^\gamma u]] + [[Y, \mathrm{hor}^\gamma u], Z]$$

is also a Hamiltonian vector field. Furthermore, the Lie algebra $\mathcal{A}^\gamma$ has $\mathrm{Ham}(E, P)$ as an ideal. Equivalently, a partitions of unity argument shows that $\mathcal{A}^\gamma$ can be defined by

$$\mathcal{A}^\gamma = \{Y \in \mathrm{Poiss}_{\mathbb{V}}(E, P) \mid [Y, \mathrm{hor}^\gamma u] \in \mathrm{Ham}_{\mathbb{V}}(E, P) \ \forall u \in \mathfrak{X}_B\}.$$

To describe the morphism ρ, first note that

$$\begin{aligned}
\partial_{2,-1}Y(u,v) &= -[\pi_*^\gamma\sigma, Y](u,v) = -[\pi_*^\gamma\sigma(u,v), Y] \\
&= \mathcal{L}_Y(\pi_*^\gamma\sigma(u,v)) = \mathcal{L}_Y(\sigma(\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v)) \\
&= \mathcal{L}_Y\sigma(\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v) + \sigma([Y, \mathrm{hor}^\gamma u], \mathrm{hor}^\gamma v) + \sigma(\mathrm{hor}^\gamma u, [Y, \mathrm{hor}^\gamma v]) \\
&= \mathcal{L}_Y\sigma(\mathrm{hor}^\gamma u, \mathrm{hor}^\gamma v) \\
&= (\pi_*^\gamma \mathcal{L}_Y\sigma)(u,v).
\end{aligned}$$

Thus, $\rho : \mathcal{A}^\gamma \longrightarrow H^2_{\partial^\gamma_{1,0}}$ is defined by $\rho(Y) := [\partial^\gamma_{1,0}\beta_Y + \pi_*^\gamma(\mathcal{L}_Y\sigma)]$. Hence,

$$\ker\rho = \{Y \in \mathcal{A}^\gamma \mid \partial^\gamma_{1,0}\beta_Y + \pi^\gamma_*(\mathcal{L}_Y\sigma) \text{ is } \overline{\partial}^\gamma_{1,0} - \text{exact}\}.$$

Theorem 4.2.5. *Let* Π *be a coupling Poisson structure on a fiber bundle* (E,π,B)*. If* (γ,σ,P) *is the geometric data associated to* Π*, then the first Poisson cohomology group of* Π *is isomorphic to*

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq \mathcal{H}^1_{\overline{\partial}^\gamma_{1,0}} \oplus \frac{\ker\rho}{\mathrm{Ham}(E,P)}. \tag{4.7}$$

Proof. Let $(\mathcal{V}_E, \partial^\gamma)$ be the bigraded cochain complex defined in E by (γ,σ,P). Recall that ∂^γ has the following bigraded decomposition $\partial^\gamma = \partial_{1,0} + \partial_{0,1} + \partial_{2,-1}$, where

$$\partial_{1,0} = \partial^\gamma_{1,0}, \qquad \partial_{0,1} = \mathrm{ad}_P, \qquad \partial_{2,-1} = -\mathrm{ad}_{\pi^\gamma_*\sigma}.$$

Therefore, we can apply Theorem 1.2.2 to the bigraded cochain complex $(\mathcal{V}_E, \partial^\gamma)$. In particular, equation (1.14) reads

$$0 \to \mathcal{H}^1_{\overline{\partial}^\gamma_{1,0}} \hookrightarrow \mathcal{H}^1_{\partial^\gamma} \to \frac{\ker\rho}{\mathrm{Ham}(E,P)} \to 0.$$

Since we are in the category of $\mathbb{R}$-vector spaces, this sequence is equivalent to

$$\mathcal{H}^1_{\partial^\gamma} \simeq \mathcal{H}^1_{\overline{\partial}^\gamma_{1,0}} \oplus \frac{\ker\rho}{\mathrm{Ham}(E,P)}.$$

Finally, by Corollary 3.4.4 of Theorem 3.4.2, we know that

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq \mathcal{H}^1_{\partial^\gamma},$$

which completes the proof. ■

Note that, in the right-hand side of splitting (4.7), obtained for the first Poisson cohomology group, the first factor *do* depends on the choice of the Ehresmann connection γ. Indeed, recall that the correspondence between coupling structures Π and geometric data (γ,σ,P) is bijective. This means that if we change the Ehresmann connection γ, then the coupling structure Π is also changed.

We now consider a situation in which the first factor of $\mathcal{H}^1_{LP}(E,\Pi)$ does not depend on the choice of the Ehresmann connection γ.

Let $(E,\pi,B;P)$ a Poisson fiber bundle and γ a *Poisson connection*, i.e., such that horizontal lifts are infinitesimal automorphisms, $\mathcal{L}_{\mathrm{hor}^\gamma u}P = 0$ $\forall u \in \mathfrak{X}_B$. In other words, the vertical Poisson structure is invariant under parallel transport. Consider the covariant exterior differential $\partial^\gamma_{1,0} : \mathcal{V}_E \longrightarrow \mathcal{V}_E$ of γ, given by

$$\begin{aligned}\partial^\gamma_{1,0}\eta(u_0,\dots,u_p) &:= \sum_{i=0}^{p}(-1)^i\mathcal{L}_{\mathrm{hor}^\gamma u_i}(\eta(u_0,\dots\widehat{u}_i\dots,u_p))\\ &\quad+\sum_{i<j}(-1)^{i+j}\eta([u_i,u_j],u_0,\dots\widehat{u}_i\dots\widehat{u}_j\dots,u_p).\end{aligned}$$

Note that the second summand does not depend on the choice of the connection γ.

Proposition 4.2.6. *Let $(E,\pi,B;P)$ be a Poisson fiber bundle and γ a Poisson connection. Assume that* $\mathrm{Poiss}_{\mathbb{V}}(E,P) = \mathrm{Ham}(E,P)$. *Then*

$$\overline{\partial}^\gamma_{1,0} : \Omega_B \otimes \mathrm{Casim}(E,P) \longrightarrow \Omega_B \otimes \mathrm{Casim}(E,P)$$

does not depend on the choice of the Poisson connection γ.

Proof. Let $\tilde\gamma$ be another Poisson connection. For each $u \in \mathfrak{X}_B$, the horizontal lifts $\mathrm{hor}^\gamma(u)$ and $\mathrm{hor}^{\tilde\gamma}(u)$ are Poisson vector fields, which are π-related to u. Thus, the difference is a vertical Poisson vector field: $\mathrm{hor}^\gamma(u) - \mathrm{hor}^{\tilde\gamma}(u) \in \mathrm{Poiss}_{\mathbb{V}}(E,P)$. Furthermore, since $\mathrm{Poiss}_{\mathbb{V}}(E,P) = \mathrm{Ham}(E,P)$, it follows that $\mathrm{hor}^\gamma(u) - \mathrm{hor}^{\tilde\gamma}(u)$ is Hamiltonian. In particular, $\mathcal{L}_{\mathrm{hor}^\gamma(u)-\mathrm{hor}^{\tilde\gamma}(u)}K = 0$ for each $K \in \mathrm{Casim}(E,P)$. Therefore, if $\eta \in \Omega_B \otimes \mathrm{Casim}(E,P)$, then

$$\begin{aligned}\partial^\gamma_{1,0}\eta(u_0,\dots,u_p) &= \sum_{i=0}^{p}(-1)^i\mathcal{L}_{\mathrm{hor}^\gamma u_i}(\eta(u_0,\dots\widehat{u}_i\dots,u_p))\\ &\quad+\sum_{i<j}(-1)^{i+j}\eta([u_i,u_j],u_0,\dots\widehat{u}_i\dots\widehat{u}_j\dots,u_p)\\ &= \sum_{i=0}^{p}(-1)^i\mathcal{L}_{\mathrm{hor}^{\tilde\gamma} u_i}(\eta(u_0,\dots\widehat{u}_i\dots,u_p))\\ &\quad+\sum_{i<j}(-1)^{i+j}\eta([u_i,u_j],u_0,\dots\widehat{u}_i\dots\widehat{u}_j\dots,u_p)\\ &= \partial^{\tilde\gamma}_{1,0}\eta(u_0,\dots,u_p),\end{aligned}$$

proving that $\overline{\partial}^\gamma_{1,0} : \Omega_B \otimes \mathrm{Casim}(E,P) \longrightarrow \Omega_B \otimes \mathrm{Casim}(E,P)$ does not depend on the choice of γ. ■

We conclude that the assertion of Proposition 4.2.6 is also true if $\mathrm{Poiss}(E_b,P_b) = \mathrm{Ham}(E_b,P_b)$ for every $b \in B$.

4.2.3 Regular coupling Poisson structures

In this part we apply Theorem 4.2.5 to a regular coupling structure. More precisely, if B is a regular symplectic leaf of a Poisson manifold, then there is a tubular neighborhood of B which is diffeomorphic to the normal bundle E over B, and the Poisson structure in E is a coupling structure with the following geometric data:

- The horizontal distribution $\mathbb{H}$ is the characteristic distribution. Thus, the Ehresmann connection is flat.
- The vertical Poisson structure is zero: $P = 0$.
- The horizontal 2-form σ is projectable.

Furthermore, we compare the result obtained here with Theorem 4.1.4.

Recall that, according to [36], coupling Poisson structures arise in the semilocal study of Poisson structures. Indeed, if S is a closed symplectic leaf in the Poisson manifold (M, Ψ), then there exists a tubular neighborhood U of S such that U is diffeomorphic to a vector bundle E over S. Moreover, under such diffeomorphism, the Poisson structure Ψ is isomorphic to a coupling structure Π in E.

On the other hand, assume that S is *regular*, i.e., the tubular neighborhood U of S can be chosen such that the Poisson structure is regular in U. Moreover, a model for the regular Poisson structure in the tubular neighborhood of S can be given in the following terms:

- The Poisson structure Ψ can be viewed as a regular coupling structure Π in a fiber bundle (E, π, S), where S is a regular symplectic leaf.
- The characteristic and the horizontal distributions coincide: $\mathbb{H} = \Pi^\sharp(T^*E)$.
- The connection γ is flat: $\mathrm{Curv}^\gamma = 0$.
- The vertical part of Π is zero: $P = \Pi_{0,2} = 0$.

This choice on the geometric data is motivated by the following well-known fact: any regular Poisson structure around a closed symplectic leaf is isomorphic to a coupling Poisson structure with the above geometric data [36].

Moreover, on each symplectic leaf, the horizontal Poisson structure Π restricts to a nondegenerate Poisson structure. Also, we have the nondegenerate Poisson bivector $\Pi_S \in \Gamma \bigwedge^2 TS$ in the base related to Π:

$$\pi^*(\Pi_S(\mathrm{d}_B f, \mathrm{d}_B g)) := \Pi(\mathrm{d}\pi^* f, \mathrm{d}\pi^* g).$$

Therefore, Π is projectable. So, the horizontally nondegenerate 2-form $\sigma \in \Gamma \bigwedge^2 \mathbb{V}^0$ projects to the symplectic form $\omega_B \in \Omega_B^2$ in the base: $\pi_*^\gamma \sigma = \omega_B \otimes \mathbb{1}$. In other words $\sigma \in \Omega^2_{\mathrm{pr}}(E)$. Finally, note that

$$\partial^\gamma_{1,0} \pi^\gamma_* \sigma = \pi^\gamma_* \mathrm{d}_{1,0} \sigma = \mathrm{d}_B \omega \otimes \mathbb{1} = 0,$$

proving that the fourth integrability equation for (γ, σ, P) is satisfied. In this special case, $\mathrm{Poiss}_{\mathbb{V}}(E,P) = \Gamma\mathbb{V}$ and $\mathcal{A}^{\gamma} = \{Y \in \Gamma\mathbb{V} \mid [\mathrm{hor}^{\gamma}(u), Y] = 0 \quad \forall u \in \mathfrak{X}_B\}$; equivalently,

$$\mathcal{A}^{\gamma} = \{Y \in \Gamma\mathbb{V} \mid [Y, \Gamma\mathbb{H}] \subset \Gamma\mathbb{H}\} = \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(E, \mathbb{H}),$$

which is the space of transversal vector fields preserving the foliation. Moreover, since γ is flat, the bigraded decomposition of the exterior differential in E has the form $\mathrm{d} = \mathrm{d}_{1,0} + \mathrm{d}_{0,1}$, where $\mathrm{d}_{1,0}$ and $\mathrm{d}_{0,1}$ are commutative coboundary operators. Hence, the *horizontal forms* $\Omega_{\mathbb{V}^0}(E) = \Gamma \bigwedge \mathbb{V}^0$ with the horizontal component $\mathrm{d}_{1,0}$ define a cochain complex which is isomorphic to the *leafwise de Rham complex* induced by the symplectic foliation $(\mathcal{S}, \omega)$ of Π. Taking into account that the projection of horizontal forms satisfies (3.20), we get a cochain complex isomorphism, $\partial^{\gamma}_{1,0}\pi^{\gamma}_{*}\sigma = \pi^{\gamma}_{*}\mathrm{d}_{1,0}\sigma$. In particular, $\mathcal{H}^2_{\partial^{\gamma}_{1,0}} \simeq H^2_{\mathrm{d}_{\mathcal{S}}}$.

Finally, since $P = 0$, for any $Y \in \mathcal{A}^{\gamma}$ we can take as choice of β_Y the zero vector valued form. Hence, $\rho : \mathcal{A}^{\gamma} \longrightarrow \mathcal{H}^2_{\mathrm{d}_{1,0}}$ simply reads $\rho(Y) = [\mathcal{L}_Y\sigma] \in \mathcal{H}^2_{\mathrm{d}_{1,0}}$. The cochain complex isomorphism $(\Omega_{\mathbb{V}^0}(E), \mathrm{d}_{1,0}) \to (\Gamma \bigwedge \mathbb{H}^*, \mathrm{d}_{\mathcal{S}})$ maps σ to ω, but also $\mathcal{L}_Y\sigma$ to $\mathcal{L}_Y\omega$. Taking into account that $\mathrm{Ham}(E,P) = \{0\}$, we get the following consequence of Theorem 4.2.5.

Proposition 4.2.7. *Under above assumptions, we have the following isomorphism:*

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq H^1_{\mathrm{d}_{\mathcal{S}}} \oplus \{Y \in \mathfrak{X}^{\mathbb{V}}_{\mathrm{pr}}(E, \mathbb{H}) \mid \mathcal{L}_Y\omega \textit{ is } \mathrm{d}_{\mathcal{S}} - \textit{exact}\}.$$

Remark 4.2.8. *Note that this result, obtained for regular coupling Poisson structures, follows from Theorem 4.1.4 formulated for any regular Poisson structure.*

4.3 Examples

In this part we study the first cohomology of two classes of coupling Poisson structures in fiber bundles. The first class consists on coupling structures in which the first cohomology group of its vertical part has trivial cohomology. The coupling structures of the second class are those for which the only Casimir function of their vertical part are the projectable functions.

In both families, the splitting - type formula (4.7) for the first Poisson cohomology group $\mathcal{H}^1_{LP}(E,\Pi)$ simplifies. In the first case, the resulting formula is a direct consequence of Corollary 1.2.3. On the contrary, the resulting formula for the second case cannot be obtained in this manner, since the notion of projectable function cannot be extended to an arbitrary bigraded cochain complex.

Trivial cohomology for the vertical Poisson structure

We study the first Poisson cohomology group of a coupling Poisson structure with vertical trivial cohomology: $\mathrm{Poiss}_{\mathbb{V}}(E,P) = \mathrm{Ham}(E,P)$. In this case, the second factor of splitting (4.7) simplifies, and the first factor do not depend on the choice of the connection γ.

Let (γ, σ, P) be the geometric data associated to a coupling Poisson structure Π on a fiber bundle (E, π, B). Assume that the first vertical Poisson cohomology is trivial, that is,

$$\mathrm{Poiss}_{\mathbb{V}}(E, P) = \mathrm{Ham}(E, P). \tag{4.8}$$

Since, in general, we have

$$\mathrm{Ham}(E, P) \subset \ker \rho \subset \mathcal{A}^{\gamma} \subset \mathrm{Poiss}_{\mathbb{V}}(E, P),$$

equation (4.8) implies $\mathrm{Ham}(E, P) = \ker \rho$. So, $\frac{\ker \rho}{\mathrm{Ham}(E,P)}$. As consequence of Theorem 4.2.5, we get the following result.

Proposition 4.3.1. *Let* Π *be a coupling Poisson structure such that condition* (4.8) *holds. Then*

$$\mathcal{H}^1_{LP}(E, \Pi) \simeq \mathcal{H}^1_{\overline{\partial}^{\gamma}_{1,0}}.$$

Corollary 4.3.2. *Let* Π_1 *and* Π_2 *be two coupling Poisson structures in the fiber bundle* (E, π, B) *with geometric data* (γ_1, σ_1, P) *and* (γ_2, σ_2, P)*, respectively. If the vertical first cohomology of* P *is trivial, i.e.,* $\mathrm{Poiss}_{\mathbb{V}}(E, P) = \mathrm{Ham}(E, P)$*, then*

$$\mathcal{H}^1_{LP}(E, \Pi_1) = \mathcal{H}^1_{LP}(E, \Pi_2).$$

Proof. Because of Proposition 4.3.1, we have

$$\mathcal{H}^1_{LP}(E, \Pi_1) = \mathcal{H}^1_{\overline{\partial}^{\gamma_1}_{1,0}}, \qquad \mathcal{H}^1_{LP}(E, \Pi_2) = \mathcal{H}^1_{\overline{\partial}^{\gamma_2}_{1,0}}.$$

On the other hand, since (γ_1, σ_1, P) and (γ_2, σ_2, P) are geometric data induced by coupling Poisson structures, the integrability equations are satisfied; in particular, γ_1 and γ_2 are Poisson connections in the Poisson bundle $(E, \pi, B; P)$. Finally, Proposition 4.2.6 implies that $\overline{\partial}^{\gamma_1}_{1,0} = \overline{\partial}^{\gamma_2}_{1,0}$. Thus, $\mathcal{H}^1_{LP}(E, \Pi_1) = \mathcal{H}^1_{LP}(E, \Pi_2)$. ∎

Note that a necessary condition for the property $\mathrm{Poiss}_{\mathbb{V}}(E, P) = \mathrm{Ham}(E, P)$ is that on each fiber E_b, $b \in B$, the equality $\mathrm{Poiss}(E_b, P_b) = \mathrm{Ham}(E_b, P_b)$ holds. The converse still is an open problem.

Free Casimir function case. Examples

The previous particular case presented is a direct consequence of Corollary 1.2.3, since the condition (4.8) can be expressed by the bigraded cochain complex. The following examples cannot be obtained in this manner, since the notion of projectable function cannot be generalized to the algebraic case without an additional structure on the bigraded cochain complex.

Let Π be a coupling Poisson structure in the fiber bundle (E, π, B) with integrable geometric data (γ, σ, P), such that every Casimir function for P is projectable:

$$\mathrm{Casim}(E, P) = C^{\infty}_{\mathrm{pr}}(E). \tag{4.9}$$

In this case, the cochain complex given by the coboundary operator

$$\overline{\partial}^{\gamma}_{1,0} : \Omega^p_B \otimes C^\infty_{\mathrm{pr}}(E) \longrightarrow \Omega^p_B \otimes C^\infty_{\mathrm{pr}}(E)$$

is isomorphic to the de Rham complex in the base space.

Lemma 4.3.3. *If* $\mathrm{Casim}(E,P) = C^\infty_{\mathrm{pr}}(E)$*, then*

$$\pi_* : \Omega_B \otimes C^\infty_{\mathrm{pr}}(E) \longrightarrow \Omega_B$$

defines a cochain complex isomorphism from $(\Omega_B \otimes C^\infty_{\mathrm{pr}}(E), \overline{\partial}^{\gamma}_{1,0})$ *onto* (Ω_B, d_B)*, i.e., the following diagram commutes*

$$\begin{tikzcd}
\Omega_B \otimes C^\infty_{\mathrm{pr}}(E) \arrow[r, "\overline{\partial}^{\gamma}_{1,0}"] \arrow[d, "\pi_*"'] & \Omega_B \otimes C^\infty_{\mathrm{pr}}(E) \arrow[d, "\pi_*"] \\
\Omega_B \arrow[r, "\mathrm{d}_B"'] & \Omega_B
\end{tikzcd}$$

Proof. This follows from definition of $\overline{\partial}^{\gamma}_{1,0}$ and the fact that $\pi_* : C^\infty_{\mathrm{pr}}(E) \longrightarrow C^\infty_B$ is an isomorphism. ■

In particular, $\overline{\partial}^{\gamma}_{1,0}$ identifies with the exterior differential d_B in the base B by $\overline{\partial}^{\gamma}_{1,0}(\theta \otimes \mathbb{1}) = (\mathrm{d}_B\theta) \otimes \mathbb{1}$. Hence, we have $\mathcal{H}^k_{\overline{\partial}^{\gamma}_{1,0}} \simeq H^k_{dR}(B)$.

Proposition 4.3.4. *If* (γ, σ, P) *are the geometric data associated to the coupling Poisson structure* Π *in the fiber bundle* (E, π, B) *such that condition* (4.9)*, i.e., every Casimir function for* P *is projectable, then*

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq H^1_{dR}(B) \oplus \frac{\ker\rho}{\mathrm{Ham}(E,P)}$$

In particular, if B *is simply connected, then*

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq \frac{\ker\rho}{\mathrm{Ham}(E,P)}.$$

Moreover, if, in addition, $H^2_{dR}(B) = \{0\}$*, then*

$$\mathcal{H}^1_{LP}(E,\Pi) \simeq \frac{\mathcal{A}^\gamma}{\mathrm{Ham}(E,P)}.$$

The last isomorphism is consequence of the definition of ρ, since ρ values on $\mathcal{H}^2_{\overline{\partial}^{\gamma}_{1,0}} \simeq H^2_{dR}(B)$.

Remark 4.3.5. *Suppose that the following conditions are satisfied:*

$$\mathrm{Casim}(E,P) = \pi^* C^\infty_B, \qquad H^1_{dR}(B) = 0, \qquad \frac{\mathrm{Poiss}_{\mathbb{V}}(E,P)}{\mathrm{Ham}(E,P)} = 0.$$

As direct consequence of Proposition 4.3.4, we conclude that the first Poisson cohomology is trivial. It will be interesting to find some examples of singular Poisson structures in which the above conditions hold, or prove that the above conditions are not compatible in the singular case.

Below, we give some examples of condition (4.9).

Open Book foliations.

Consider a vector bundle (E, π, B), with a coupling Poisson structure Π whose associated geometric data is (γ, σ, P). Assume that the Poisson fiber bundle (E, P) is locally trivial, with typical fiber given by the co-algebra $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. We will consider some particular examples of Lie algebras $\mathfrak{g}$ with the following property: the symplectic foliation of the Lie-Poisson structure on $\mathfrak{g}^*$ is an open *book foliation* [29]. In these cases, condition (4.9) is automatically satisfied. So, Corollary 4.3.4 can be applied.

1. Consider the Lie algebra $\mathfrak{g}$ given by the bracket relations:

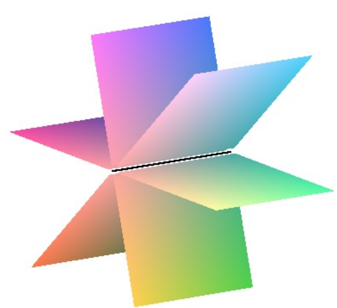
$$[e_1, e_2] = e_2, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -e_3.$$

If (x_1, x_2, x_3) denote the coordinates along the fiber, then the Poisson structure P on g^* has the form $P = x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1}$. The set of 0-dimensional symplectic is $l := \{(r, 0, 0) \mid r \in \mathbb{R}\}$. The complement $N = \mathfrak{g}^* \backslash l$ of this set consists on the points of maximal rank, i.e., regular points. The following Hamiltonian vector fields

$$X_{x_1} = x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}, \qquad X_{x_2} = -x_2 \frac{\partial}{\partial x_1}, \qquad X_{x_3} = -x_3 \frac{\partial}{\partial x_1},$$

span the subspace $\{e_1, x_2 e_2 + x_3 e_3\} \subset T_x \mathfrak{g}^*$ at each $x \in N$. Therefore, the 2-dimensional symplectic leaf passing for $p = (p_1, p_2, p_3) \in N$ consists of its image by the flow of $\frac{\partial}{\partial x_1}$ and $x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$:

$$S_p = \{(p_1 + t_1, p_2 e^{t_2}, p_3 e^{t_2}) \mid (t_1, t_2) \in \mathbb{R}^2\}.$$



On the other hand, if $K \in \mathrm{Casim}(\mathfrak{g}^*, P)$ is a Casimir function, then it is constant along S_p. Also, note that $\{(r, p_2 e^{-k}, p_3 e^{-k})\}_{k \in \mathbb{N}}$ is a sequence of points in S_p that converges to $(r, 0, 0)$. By the continuity of K, the value of K on any $x \in S_p$ equals to $K(r, 0, 0)$. Since this holds for any $p \in N$ and any $r \in \mathbb{R}$, it follows that K must be constant in all $\mathfrak{g}^*$. This analysis shows that any Casimir function for P must be constant along each fiber $E_b \simeq \mathfrak{g}^*$. Therefore, condition (4.9) follows.

2. Another example of vector bundle of rank 3 with property (4.9) is when the typical fiber is the co-algebra of the 3-dimensional Lie algebra:

$$[e_1, e_2] = \alpha e_2 + e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = e_2 - \alpha e_3.$$

for $\alpha > 0$. The set of 0-dimensional symplectic leaves is $\{(r, 0, 0) \mid r \in \mathbb{R}\}$, and the 2-dimensional symplectic leaves are parameterized by

$$S_p = \{(p_1+t_1, e^{\alpha t_2}(p_2 \cos t_2+p_3 \sin t_2), e^{\alpha t_2}(-p_2 \sin t_2+p_3 \cos t_2)) \mid (t_1, t_2) \in \mathbb{R}^2\}.$$

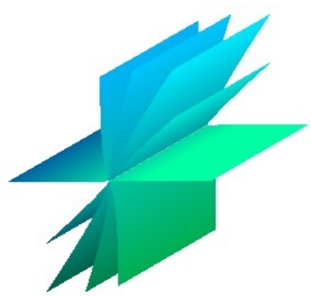
The sequence of points in S_p, $\{(r, e^{-\alpha k}(p_2 \cos k - p_3 \sin k), e^{-\alpha k}(p_2 \sin k + p_3 \cos k))\}_{k\in\mathbb{N}}$ converges to $(r, 0, 0)$ and, by the same arguments as above, condition (4.9) holds.

3. Consider the co-algebra of the Lie algebra

$$[e_1, e_2] = e_2 + e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -e_3.$$

In this case, the set of 0-dimensional symplectic leaves is $\{(r, 0, 0) \mid r \in \mathbb{R}\}$. The 2-dimensional symplectic leaves are parameterized by

$$S_p = \{(p_1 + t_1, (p_2 + t_2 p_3)e^{t_2}, p_3 e^{t_2}) \mid (t_1, t_2) \in \mathbb{R}^2\}.$$



Also, the sequence in S_p

$$\{(r, (p_2 - kp_3)e^{-k}, p_3 e^{-k})\}_{k\in\mathbb{N}}$$

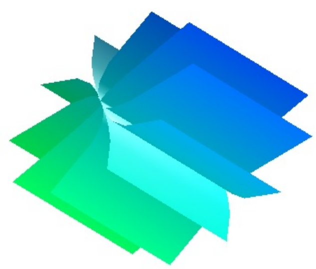
converges to $(r, 0, 0)$. So, condition (4.9) is satisfied.

4. The last example of symplectic book foliation in the fiber is given as follows. If $\alpha > 1$, then the co-algebra of the Lie algebra

$$[e_1, e_2] = \alpha e_2 - e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = e_2 - \alpha e_3,$$

has the following properties. The set of 0-dimensional symplectic leaves is $\{(r, 0, 0) \mid r \in \mathbb{R}\}$, and the 2-dimensional symplectic leaves are parameterized by

$$\begin{aligned} x_1 &= p_1 + t_1, \\ x_2 &= \frac{1}{2}p_2(e^{(\alpha-1)t_2} + e^{(\alpha+1)t_2}) + \frac{1}{2}p_3(e^{(\alpha-1)t_2} - e^{(\alpha+1)t_2}), \\ x_3 &= \frac{1}{2}p_2(e^{(\alpha-1)t_2} - e^{(\alpha+1)t_2}) + \frac{1}{2}p_3(e^{(\alpha-1)t_2} + e^{(\alpha+1)t_2}). \end{aligned}$$



The sequence $(p^n)_{n\in\mathbb{N}}$ in S_p, $p^n = (x_1^n, x_2^n, x_3^n)$, is given by

$$\begin{aligned} x_1^n &= r, \\ x_2^n &= \tfrac{1}{2}p_2(e^{-(\alpha-1)n} + e^{-(\alpha+1)n}) + \tfrac{1}{2}p_3(e^{-(\alpha-1)n} - e^{-(\alpha+1)n}), \\ x_3^n &= \tfrac{1}{2}p_2(e^{-(\alpha-1)n} - e^{-(\alpha+1)n}) + \tfrac{1}{2}p_3(e^{-(\alpha-1)n} + e^{-(\alpha+1)n}). \end{aligned}$$

and converges to $(r, 0, 0)$. Hence, condition (4.9) follows.

One more example

Here we consider an example of a Lie algebra $\mathfrak{g}$ such that the corresponding symplectic foliation in the co-algebra $\mathfrak{g}^*$ is not an open book foliation, but at the same time does not admit global non-trivial Casimir function.

If $0 < \alpha < 1$ is irrational, a Casimir function of the co-algebra of the Lie algebra

$$[e_1, e_2] = \alpha e_2 - e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = e_2 - \alpha e_3$$

must satisfy the following equations:

$$(\alpha x_2 - x_3)\frac{\partial K}{\partial x_2} - (x_2 - \alpha x_3)\frac{\partial K}{\partial x_3} = 0, \qquad \frac{\partial K}{\partial x_1} = 0.$$

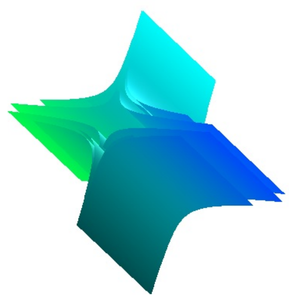
By applying the method of the characteristics, we show that K must have the following form in the regular domain:

$$K = \kappa\left(\frac{|x_2 - x_3|^{\alpha-1}}{|x_2 + x_3|^{\alpha+1}}\right).$$

However, because of the irrationality of α, the numbers $\alpha - 1$ and $\alpha + 1$ are rationally independent. So, K is not of class C^∞ if κ is non-constant. Therefore, in this case, a global Casimir function in the fiber does not exists. So, (4.9) is satisfied.

The set of 0-dimensional symplectic leaves is $\{(r, 0, 0) \mid r \in \mathbb{R}\}$, and the 2-dimensional symplectic leaves are parameterized by

$$\begin{aligned}
x_1 &= p_1 + t_1,\\
x_2 &= \frac{1}{2}p_2(e^{(\alpha-1)t_2} + e^{(\alpha+1)t_2}) + \frac{1}{2}p_3(e^{(\alpha-1)t_2} - e^{(\alpha+1)t_2}),\\
x_3 &= \frac{1}{2}p_2(e^{(\alpha-1)t_2} - e^{(\alpha+1)t_2}) + \frac{1}{2}p_3(e^{(\alpha-1)t_2} + e^{(\alpha+1)t_2}).
\end{aligned}$$



Proposition 4.3.6 (Uniqueness)**.** *Every 3-dimensional Lie-Poisson structure admitting no global non-trivial Casimir function is isomorphic to one of the co-algebras given above.*

Proof. In [26], the Bianchi classification of 3-dimensional linear Poisson structures is presented. For most of the 3-dimensional Lie - Poisson structures presented, there is exhibited a global Casimir function, except for the examples described in above, which do not admit Casimir functions besides constants. It is left to show that the co-algebra of

$$[e_1, e_2] = \alpha e_2 - e_3, \qquad [e_2, e_3] = 0, \qquad [e_3, e_1] = e_2 - \alpha e_3,$$

when $0 < \alpha = \frac{p}{q} < 1$, admits a global Casimir function. Indeed, since $p < q$,

$$K(x) = (x_2 + x_3)^{p+q}(x_2 - x_3)^{q-p}$$

is a polynomial which is also a global Casimir function of class C^∞. ■

Therefore, the examples we presented in this section are essentially the only examples of Lie-Poisson vector bundles $(E, \pi, B; P)$ of rank 3 such that condition (4.9) is satisfied.

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