"El saber de mis hijos
hará mi grandeza"

UNIVERSIDAD DE SONORA

DEPARTAMENTO DE INVESTIGACION EN FISICA

# ELECTRODINAMICA RELATIVISTA CON EL LENGUAJE DE LAS FORMAS DIFERENCIALES

TESIS

QUE PARA OBTENER EL
GRADO DE MAESTRO EN CIENCIAS

PRESENTA

FRANCISCO MIGUEL VELARDE LOPEZ

DIRECCION: DR. JESUS SANTOS CASTILLO
CODIRECCION: DR. MARTIN GILDARDO GARCIA ALVARADO

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# Contents

Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, ... without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.

Galileo Galilei.

One could perhaps describe the situation by saying that God is a mathematician of a very high order, and He used very advanced mathematics in constructing the universe. Our feeble attempts at mathematics enable us to understand a bit of the universe, and as we proceed to develop higher and higher mathematics we can hope to understand the universe better.

Paul Dirac.

"What led me more or less directly to the special theory of relativity was the conviction that the electromotive force acting on a body in motion in a magnetic field was nothing else but an electric field."

Albert Einstein.

# CHAPTER 1

# Introduction

Albert Einstein published in 1905 the Special Theory of Relativity, which answered to several questions, one of them was the contradiction between Newtonian and Maxwell's Electromagnetic Theory.

According to Newton, all inertial frames are the same, opposed to Maxwell, who stated all inertial observers measure the same value for the speed of light.

It is a well known story that at the beginning of the 20th century, due to the works of Lorentz, Poincare and Einstein, it was established that the system of Maxwell equations was compatible with a new version of spacetime structure serving as arena for physical phenomena, namely Minkowski spacetime. It became also obvious that the classical Lorentz force law needed to be modified in order to leave the theory of classical charged particles and their electromagnetic fields invariant under spacetime transformations in Minkowski spacetime defining a representation of the Poincare group. Such a condition was a necessary one for the theory to satisfy the Principle of Relativity.

The central idea in relativity is that the laws of physics should take on a simple and formally identical form when expressed in any Lorentz inertial frame. This is sometimes called covariance.

It is curious that magnetic forces cannot be included in the Galilean relativity. For if the velocity of a charge is zero in one frame but not zero in another, then does that mean that the particle has a non-zero force or no force? In the rest frame of the constant velocity charge apparently there is no magnetic force, yet in another inertially related frame where the charge is in motion there would be a magnetic force. How can this be? The problem with our thinking is we have not asked how the magnetic field transforms for one thing, but more fundamentally we will find that you cannot separate the magnetic force from the electric force. There is no nice way of addressing this in Newtonian mechanics. It is an inherently relativistic problem, and Einstein attributes it as one of his motivating factors in dreaming up his special relativity.

The most revolutionary quantum leap in the history of theoretical physics is the birth of general relativity and quantum field theory (the standard model of elementary particle). These theories describe nature better than any physicist ever had at hand, although they have not been unified into a coherent picture of the world. One of the main ingredients of these theories is differential geometry. Euclidean geometry was abandoned in favour of differential geometry and classical field theories had to be quantized.

Maxwell's equations in the language of differential geometry lead to a generalization to these new theories, and these equations are a special case of Yang-Mills equations (beyond the scope of this thesis), which is also gauge invariant and describe not only electromagnetism but also the strong and weak nuclear forces.

In recent years has shown the key role that gauge fields play in physics, and the possibility of describing gauge fields in terms of connections on fiber bundles.

Currently gauge theories remain a central part of the proposed Unification Theories for the four fundamental interactions: gravitation, electromagnetism, strong and weak Interaction.

The Salam-Weinberg model is a Unified Theory of Electromagnetism with Weak Interactions is based on two key concepts: non-Abelian gauge fields and symmetry breaking.

To solve the problem posed in classical electromagnetic theory, we will need the tools developed at the turn of the century by a score of mathematicians from France and Germany, the most prominent of which is Elie Cartan. In a series of articles and books extending over fifty years, he systematized these tools and discovered a beautiful mathematical structure, which he called exterior differential calculus, where antisymmetry plays a fundamental role. He has given an overview of the theory in a book [Cartan, 1945], which is still in print. This theory has been very useful in understanding the local geometry of submanifolds and the differential structure of Lie groups, but has had few applications to other areas of mathematics and to physics. It is no exaggeration to claim that it has been little known to analysts and physicists, let alone economists. And yet, I think that the Cartan-Kähler theorem, for instance, is as basic a result for Partial differential equations as the Cauchy-Lipschitz theorem is for ordinary differential equations.

Once the foundation is laid, we can proceed with the mathematical analysis. Thus, our hope is to make mathematics and physics walk hand in hand.

## Objectives

One of the objectives of this thesis is to formulate electrodynamics in a geometrically oriented manner.

Why this geometric approach? The strengths of the geometric approach are more of conceptual nature (but not its only one). It will enable us to understand the framework of physical foundations to formulate electrodynamics. Do we need a metric to formulate electrodynamics? How does the structure of electrodynamics emerge from the condition of relativistic invariance? These types of questions are best addressed in a geometry oriented framework.

Another is to formulate the equations of electrodynamics in a very concise manner. This compact formulation is not only of aesthetic value, rather, it connects to the third and perhaps most important aspect: understand electrodynamics as a representative of a family of theories known as gauge theories.

The interpretation of electrodynamics as a gauge theory has paved the way to one of the most important developments in modern theoretical physics, the understanding of fundamental forces, electromagnetism, weak and strong interactions, and gravity as part of one unifying scheme.

We will here formulate the geometric view of electrodynamics in an axiomatic approach. We want to apply the principle of general covariance which says that every physical quantity must be describable by a (coordinate-free) geometric object, and the laws of physics must all be expressible as geometric relationships between these geometric objects.

Thus form fields are really the natural language in which to write Maxwell's equations.

To formulate them we use exterior calculus because it is the appropriate mathematical framework for handling fields the integrals of which—here charge and flux—possess an invariant meaning.

For the above mentioned (introduction), and due to its increasingly widespread use, this thesis aims to using differential forms, curvature and torsion in electromagnetic theories (gauge) in general.

Last but not least we will show some strong applications of this formulation and of others made by the use of differential forms.

We assume sufficient knowledge of modern differential geometry, differential forms on manifolds and therefore we here use the standard notation of modern differential geometry.

We assume sufficient knowledge of topology.

We also assume sufficient knowledge of special and general relativity.

According to Arnold, [Arnol'd, 1989], p. 163,

"Hamiltonian mechanics cannot be understood without differential forms."

We say here relativistic electrodynamics cannot be understood without differential forms.

# Part I

# SPECIAL ELECTRODYNAMICS

The densities of the electric charge and electric current are assumed to be phenomenologically specified. These quantities will not be resolved any further and will be considered as fundamental for classical electrodynamics.

The two main experimentally well-founded facts of electrodynamics are the conservation of electric charge and conservation of magnetic flux.

The approach is essentially classical. This is a legitimate assumption within the well known and rather wide limits of the idealized representation of electric charges and currents by means of particles and continuous media and of electromagnetic radiation by means of classical waves of electric and magnetic fields.

The word "field" is employed here with its usual meaning in physics: it may be a scalar, a vector, a spinor, a tensor, etc.

It is to be noted that there are electrodynamic phenomena that cannot be satisfactorily explained by the known theories, the theory of relativity included. An example of such a phenomenon is the experiment of G. Sagnac. For a long time this phenomenon has been mentioned in literature, although not enough analysed.

# CHAPTER 2 Motivation

A true vector field keeps its individuality when you change coordinates.

In particular, if a vector field is 0 in one coordinate system, it will be 0 in every coordinate system.

See theorem of zero tensor equation [Bishop and Goldberg, 1968]

This is not true of $\vec{E}$ and $\vec{B}$.

$\vec{E}$ and $\vec{B}$ are not really vector fields.

If in one coordinate system the charge is at rest and the electric field is 0, then the particle will not be accelerated in those coordinates.

In another system moving at constant velocity with respect to the first (on a train rolling through the laboratory, for instance) it will still not be accelerated.

But it now feels a force from the magnetic field, which must be compensated for by an electric field, which cannot now be zero.

For the following we use

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$
$\beta = \frac{V}{c}$
where $c$ is the velocity of light.

**Theorem 2.1** *Electric and magnetic field under Lorentz transformation.*

$$\vec{E}' = \frac{(\vec{E}\cdot\vec{\beta})\vec{\beta}}{\beta^2} - \gamma(\vec{E}_\perp + \vec{\beta} \times c\vec{B}) \tag{2.1}$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{\gamma+1}\frac{(\vec{B}\cdot\vec{\beta})\vec{\beta}}{\beta^2} \tag{2.2}$$

SI units.

**Remark 1** *To convert in the opposite direction, we can swap primed and unprimed fields and change the sign on v:*

.

**Proof.**

$$\vec{E} = \vec{E}_\parallel + \vec{E}_\perp \tag{2.3}$$
$$\vec{E}' = \vec{E}'_\parallel + \vec{E}'_\perp$$

$$\vec{E}_\parallel := \hat{P}_V\vec{E} = \tfrac{(\vec{E}\cdot\vec{\beta})\vec{\beta}}{\beta^2} \tag{2.4}$$
$$\vec{E}_\perp := \left(1 - \hat{P}_V\right)\vec{E}$$

$$\vec{E}'_{\shortparallel} = \vec{E}_{\shortparallel} = \frac{(\vec{E}\cdot\vec{\beta})\vec{\beta}}{\beta^2}$$
$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{\beta} \times c\vec{B}_{\perp}) \tag{2.5}$$

$$\vec{B}'_{\shortparallel} = \vec{B}_{\shortparallel} = \frac{(\vec{B}\cdot\vec{\beta})\vec{\beta}}{\beta^2}$$
$$c\vec{B}'_{\perp} = \gamma(c\vec{B}_{\perp} - \vec{\beta} \times \vec{E}_{\perp}) \tag{2.6}$$

■

**Remark 2** $E \rightharpoonup B \quad B \rightharpoonup -E$

**Theorem 2.2** *Parallel and orthogonal transformation with respect to $\beta$.*

$$\vec{D}'_{\shortparallel} = \vec{D}_{\shortparallel} = \frac{(\vec{D}\cdot\vec{\beta})\vec{\beta}}{\beta^2}$$
$$c\vec{D}'_{\perp} = \gamma(c\vec{D}_{\perp} + \vec{\beta} \times \vec{H}_{\perp})$$

$$\vec{H}'_{\shortparallel} = \vec{H}_{\shortparallel} = \frac{(\vec{H}\cdot\vec{\beta})\vec{\beta}}{\beta^2}$$
$$\vec{H}'_{\perp} = \gamma(\vec{H}_{\perp} - \vec{\beta} \times c\vec{D}_{\perp})$$

**Theorem 2.3** *Parallel and orthogonal transformation with respect to $\beta$.*

$$\vec{J}'_{\shortparallel} = \gamma\left(\vec{J}_{\shortparallel} - \rho V\right)$$
$$\vec{J}'_{\perp} = \vec{J}_{\perp}$$

$$c\rho = \gamma(c\rho' + \vec{\beta} \cdot \vec{J'})$$

Let us pause for a moment and wonder how the notions "electric" and "magnetic" are attached to certain fields and whether there is a conventional element involved.

We cannot speak of an equivalence of electric and magnetic fields; the expression reciprocity is much more appropriate.

Fundamentally, electricity and magnetism enter into classical electrodynamics in an asymmetric way.

**Theorem 2.4** *Electric and magnetic field under Lorentz transformation.*

$$\begin{aligned} \vec{E}' &= \gamma\hat{\alpha}^{-1} \cdot \vec{E} + \gamma\vec{\beta} \cdot c\vec{B} \\ c\vec{B}' &= -\gamma\vec{\beta} \cdot \vec{E} + \gamma\hat{\alpha}^{-1} c\vec{B} \end{aligned} \tag{2.7}$$

**Theorem 2.5** *For D and H.*

$$\begin{aligned} c\vec{D}' &= \gamma\hat{\alpha}^{-1} \cdot c\vec{D} + \gamma\vec{\beta} \cdot \vec{H} \\ \vec{H}' &= -\gamma\vec{\beta} \cdot c\vec{D} + \gamma\hat{\alpha}^{-1} \cdot \vec{H} \end{aligned} \tag{2.8}$$

**Theorem 2.6** *For $\rho$ and $j$.*

$$\begin{aligned} c\rho' &= \gamma(c\rho - \vec{\beta} \cdot \vec{j}) \\ \vec{j}' &= \hat{\alpha} \cdot \vec{j} - \gamma\beta c\rho \end{aligned} \tag{2.9}$$

All demostrations of this theorems can be found in [Stratton, 1941].

# CHAPTER 3 Differential Forms

The Calculus of Differential Forms, often called Exterior Differential Calculus, a tool which is used in applied mathematics, has been increasing. Like the tensor calculus, its origins are on differential geometry, largely due to the investigations of E. Cartan at the beginning of the last century. It was quickly recognized that the ramifications of this theme is extended far beyond the land of differential geometry, thus it can be considered to belong to analysis itself.

Differential forms in physics, why? In physics, we tend to associate everything that comes with a sense of 'magnitude and direction' with a vector. From a computational point of view, this identification is (mostly) o.k., conceptually, however, it may obscure the true identity of physical objects. Many of our accustomed 'vector fields' are not vector fields at all. And if they are not, they are usually differential forms in disguise.

The use of differential forms should not be viewed as just another formalism of fancy. The technique goes beyond the methods of tensor calculus, and admits the study of topological evolution, while tensor based theories do not.

We present here a quick, simplified view of an involved subject. The aim is to introduce the crucial ideas and those simple notions which are of established physical relevance.

The geometry of a general Riemann-Cartan-Weyl space $M$ is $M$ (a differentiable manifold) carrying a metric g $\in T_2^0(M)$ of signature $(p, q)$, a connection $\nabla$, an orientation $\tau$g. Such a structure is denoted by $(M, g, \nabla, \tau g)$.

The pair $(\nabla, g)$ is called a geometry for $M$.

When M is 4-dimensional (and other requirements) and g has signature (1, 3), the pair (M, g) is called a Lorentzian manifold.

Endow $M$ with a general connection $\nabla$, with a spacetime orientation $\tau$g and with a time orientation $\uparrow$ .

Such a general structure will be denoted by $(M, g, \nabla, \tau g, \uparrow)$ and is said to be a Riemann-Cartan-Weyl spacetime.

Lorentzian spacetimes are structures $(M, g, D, \tau g, \uparrow)$ restricted by the condition that the connection $D$ is metric compatible, i.e. $Dg = 0$ and that the torsion tensor of that connection is zero, i.e. $\Theta[D] = 0$.

A connection satisfying these two requirements is called a Levi-Civita connection (and it is unique).

Moreover, if in addition to the previous requirements, $M = R^4$ and the Riemann curvature tensor of the connection is null (i.e. $R[D] = 0$) the Lorentzian spacetime is called Minkowski spacetime.

That structure is the 'arena' for what is called special relativistic theories.

**Definition 1** *Minkowski metric.*

*We define the inner product by*

$$\boldsymbol{g} = dt \otimes dt - dx \otimes dx - dy \otimes dy - dz \otimes dz$$

**Definition 2** *Differential form.*

*A differential form $\omega$ is a completely antisymmetric tensor and covariant.*

**Notation 1** *We denote this* $\boldsymbol{\omega} \in A^p$ *(with bold for distinction).*

**Definition 3** *Star Hodge operator.*

*The star Hodge operator of $\alpha$ is defined by the relation*

$$\langle \beta, \alpha \rangle vol = \beta \curlywedge ({}^*\alpha) \qquad \forall \beta \in A^r (V^*)$$

*where $\alpha$ is an r-differential form, and vol is the volume elemental form.*

**Notation 2** *${}^*\alpha$ is the star Hodge operator of $\alpha$*

**Notation 3** *$\curlywedge$ is the wedge product.*

**Definition 4** *Exterior derivative.*

*Given* $\mathbf{A}_p = \dfrac{1}{p!} A_{i_1 \cdots i_p} \tilde{b}^{i_1} \curlywedge \cdots \curlywedge \tilde{b}^{i_p} \qquad \mathbf{A} \in A^p$

*We define the function*

$$\boldsymbol{d}: \quad A^k(M) \longrightarrow A^{k+1}(M)$$

*by*

$$\boldsymbol{d}\mathbf{A} = (-1)^r (r+1)! Alt(grad\mathbf{A})$$

**Notation 4** *Alt is the alternating operator.*

**Remark 3** $\boldsymbol{d}\mathbf{A} \in A^{r+1}$

**Theorem 3.1** *Stokes. (Generalized).*
*Given* $\omega \in A^{n-1}$ *in* $M$

$$\int_{\partial M} i^* \omega = \int_M \mathbf{d}\omega$$

*Where* $M$ *is an oriented smooth paracompact n-dimensional manifold and* $\partial M$ *is its boundary.*

**Notation 5** $i$ *is the inclusion map.*

Examples:

$dx \curlywedge dy = dx \otimes dy - dy \otimes dx$

$*dx = dydz$

$*dy = dzdx$

$*dz = dxdy$

$dv = \sum_{i<j} v_{i,j} dx^i \curlywedge dx^j$

For other objects these operations can be extended by linearity and other properties.

# CHAPTER 4

# Maxwell's Equations

Is there something natural that the electric field and the magnetic field together represent?

The answer is yes: there is a 2-form field on $R^4$, namely Faraday tensor.

It is really a natural object, the same in every inertial frame.

Let $(M, g, g)$ be an oriented Lorentzian manifold.

Maxwell equations on $(M, g, \tau g)$ refer to an exterior system of differential equations given by a closed 2-form $\boldsymbol{F} \in \mathrm{A}^2\ \mathrm{T^*M}$ and a exact 3-form $\boldsymbol{J} \in \mathrm{A}^3\ T^*M$.

Then there exists $\boldsymbol{G} \in \mathrm{A}^2 T^*M$ such that $\boldsymbol{dF} = 0$ and $\boldsymbol{dG} = -\boldsymbol{J}$ .

It is postulated that in vacuum there is a relation between $\boldsymbol{G}$ and $\boldsymbol{F}$ (said constitutive relation) given by $\boldsymbol{G} = {}^*F$ .

**Definition 5** *The scalar potential* $\phi$ *is a 0-form.*

**Definition 6** *2-form Faraday tensor.*

$$\boldsymbol{F} = \sum_{\mu<\nu} \boldsymbol{F}_{\mu\nu}\, b^{\mu} \curlywedge\, b^{\nu}$$

$$\boldsymbol{F} \;= \mathbf{E} \curlywedge (dx^0) + \mathbf{B}$$

In coordinates we have

$$\boldsymbol{F}_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -E_z & -cB_y & cB_x & 0 \end{pmatrix}$$

**Definition 7** *2-form Maxwell tensor.*

$$\boldsymbol{G} \;= \boldsymbol{H} \curlywedge (dx^0) - \boldsymbol{D}$$

**Definition 8** *3-form* 4*-current density.*

$$\boldsymbol{J} = \boldsymbol{j} \curlywedge \boldsymbol{d}t - \boldsymbol{\rho}$$

*with* $\boldsymbol{\rho} = \rho dx^1 \curlywedge dx^2 \curlywedge dx^3$

**Remark 4** **F, G, J** *are 4-dimensional.*

**Notation 6** *Electric field 1-form* **E**.

**Notation 7** *Electric displacement 2-form* **D**.

**Notation 8** *Magnetic field 2-form* **B**.

**Notation 9** *Magnetic displacement 1-form* **H**.

**Notation 10** *Current density 3-form* $\boldsymbol{\rho}$.

**Notation 11** *Current density 2-form* $\boldsymbol{j}$.

**Remark 5** *All these quantities are 3-dimensional.*

**Theorem 4.1** *There is a unique 2-form F on $R^4$, called the Faraday two-form such that*

$$E^\flat = -i_{\frac{\partial}{\partial t}}\boldsymbol{F}$$
$$B^\flat = -i_{\frac{\partial}{\partial t}}{}^*\boldsymbol{F}.$$

Here the $^\flat$ is associated with the Euclidean metric in $R^3$ and the $i$ is associated with the Lorentzian metric

in $\mathrm{R}^4$.

**Proof.** If

$$\begin{aligned}\boldsymbol{F} &= F_{xy}dx \curlywedge dy + F_{zx}dz \curlywedge dx + F_{yz}dy \curlywedge dz \\ &\quad + F_{xt}dx \curlywedge dt + F_{yt}dy \curlywedge dt + F_{zt}dz \curlywedge dt,\end{aligned}$$

then ,

$$\begin{aligned}{}^*\boldsymbol{F} &= F_{xy}dz \curlywedge dt + F_{zx}dy \curlywedge dt + F_{yz}dx \curlywedge dt \\ &\quad - F_{xt}dy \curlywedge dz - F_{yt}dz \curlywedge dx - F_{zt}dx \curlywedge dy\end{aligned}$$

and so

$$-i_{\frac{\partial}{\partial t}}\boldsymbol{F} = F_{xt}dx + F_{yt}dy + F_{zt}dz$$

and

$$-i_{\frac{\partial}{\partial t}}{}^{*}\boldsymbol{F} = F_{xy}dz + F_{zx}dy + F_{yz}dx.$$

Thus, $\boldsymbol{F}$ is uniquely determined by equations, namely

$$\begin{aligned}\boldsymbol{F} &= E^1 dx \curlywedge dt + E^2 dy \curlywedge dt + E^3 dz \curlywedge dt\\ &+ B^3 dx \curlywedge dy + B^2 dz \curlywedge dx + B^1 dy \curlywedge dz.\end{aligned}$$

■

**Theorem 4.2** *There is a unique 1-form* $\boldsymbol{J}$ *on* $R^4$*, called source 1-form such that*

$$\rho = i_{\frac{\partial}{\partial t}}\boldsymbol{J}$$
$${}^{*}j^{b} = -i_{\frac{\partial}{\partial t}} * \boldsymbol{J}.$$

**Theorem 4.3** *Charge.*

$q = \frac{1}{4\pi}\oint \boldsymbol{F}^{0j} d^2 S_j$

$q = \int J^0 d^3 x$

**Definition 9** *1-form 4-magnetic potential.*

$$\mathbf{A} = A_{\mu}\boldsymbol{dx}^{\mu}$$
$$\mathbf{A} = \mathbf{A}_0 \curlywedge dx^0 + A$$

**Notation 12** *$A$ is the 3-dimensional 1-form magnetic potential.*

**Theorem 4.4** *The dual tensor of Faraday.*

$$^*\boldsymbol{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\gamma\beta}\boldsymbol{F}^{\gamma\beta}$$
$$^*\boldsymbol{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\gamma\beta}\boldsymbol{F}_{\gamma\beta}$$

**Remark 6** *The dual tensor of Faraday can be obtained from* $\boldsymbol{F}$ *by the following changes* $E \rightharpoonup B \quad B \rightharpoonup -E$

**Lemma 4.5** $^{**}\boldsymbol{F} = -\boldsymbol{F}$

We know that $F_{\mu\nu}$ and $\eta_{\mu\nu}$ are Lorentz invariant, and we can derive some Lorentz scalars from them. The most obvious one is $F_{\mu\nu}\eta^{\mu\nu}$, but as $F$ is antisymmetric and $\eta$ symmetric this evaluates to zero.

**Lemma 4.6** *Invariants of Faraday tensor.*

$$(\boldsymbol{F} \curlywedge (^*\boldsymbol{F}))_{0123} = \frac{1}{2}\boldsymbol{F}_{ik}\boldsymbol{F}^{ki}$$
$$(\boldsymbol{F} \curlywedge \boldsymbol{F})_{0123} = -\frac{1}{2}\varepsilon^{ijkl}\boldsymbol{F}_{ij}\boldsymbol{F}_{kl}$$

**Theorem 4.7** *Invariants of Faraday tensor.*

$$^*(\boldsymbol{F} \curlywedge {}^*\boldsymbol{F}) = -2(E^2 - B^2) \tag{4.1}$$
$$^*(\boldsymbol{F} \curlywedge \boldsymbol{F}) = -4\vec{E} \cdot \vec{B} \tag{4.2}$$

**Definition 10** *3-form Momentum density.*

$\boldsymbol{p}_\alpha = -\boldsymbol{B} \curlywedge (\vec{e}_\alpha \rfloor \boldsymbol{D})$

*where* $\rfloor$ *is the interior product.*

**Definition 11** *2-form Maxwell stress.*

$S_{a..} = \frac{1}{2}( \ (\vec{e}_\alpha \rfloor \ \boldsymbol{E}) \ \curlywedge \ \boldsymbol{D} - \ (\vec{e}_\alpha \rfloor \boldsymbol{E}) \curlywedge \boldsymbol{D}$
$\quad +(\vec{e}_\alpha \rfloor \ \boldsymbol{H}) \ \curlywedge \ B - \ (\vec{e}_\alpha \rfloor \boldsymbol{B}) \curlywedge \boldsymbol{H} \ )$

**Definition 12** *Kinematic energy-momentum current 3-form.*

$\boldsymbol{T}_\alpha = -p_\alpha - \boldsymbol{dt} \curlywedge \boldsymbol{S}_a$

**Definition 13** *Electromagnetic energy density 3-form.*

$\boldsymbol{w} = \frac{1}{2}(\tilde{E} \ \curlywedge \ \boldsymbol{D} + \tilde{H} \ \curlywedge \ \boldsymbol{B})$

**Theorem 4.8** $T_\alpha = \frac{1}{2}(\boldsymbol{F} \curlywedge (e_\alpha \rfloor \boldsymbol{G}) - \boldsymbol{G} \curlywedge (e_\alpha \rfloor \boldsymbol{F}))$

$\boldsymbol{T}_0 = \boldsymbol{w} - \boldsymbol{dt} \curlywedge \boldsymbol{S}$

**Definition 14** *Force density 4-form.*

$$\boldsymbol{X}_\alpha = -\frac{1}{2}(\boldsymbol{F} \curlywedge (L_{e_\alpha}\boldsymbol{G}) - \boldsymbol{G} \curlywedge (L_{e_\alpha}\boldsymbol{F}))$$

*where* $L$ *is the Lie derivative.*

On the unquantized level, the Maxwellian structure is believed to be of universal validity.

**Theorem 4.9** *Faraday tensor under Lorentz transformation.*

$$\boldsymbol{F}\,' = M\boldsymbol{F}M^{\dagger} \tag{4.3}$$

*where M represents the Lorentz transformation.*

**Axiom 1** *Maxwell equations.*

$$^{*}\boldsymbol{d}\ ^{*}\boldsymbol{F} \quad = \quad \boldsymbol{J} \tag{4.4}$$

$$\boldsymbol{dF} = 0 \tag{4.5}$$

It can be shown that if we use the star operator for the Minkowski metric then 4.5 can be rewritten as the single equation $^{*}\boldsymbol{d}^{*}\boldsymbol{F} \ = {}^{*}J$ .

**Axiom 2** 4-*Maxwell equations.*

$\boldsymbol{dG} \ = \ \boldsymbol{J}$

The equation $\boldsymbol{dF} = 0$ has nothing to do with the metric on Minkowski space at all.

In fact, if $\phi : M^4 \longrightarrow M^4$ is any diffeomorphism at all,

we have $\boldsymbol{dF} = 0 \quad \iff \quad \boldsymbol{d}(\phi^{*}\boldsymbol{F}) = 0$

and so the truth of the equation $\boldsymbol{dF} = 0$ is really a differential topological fact;

a certain form is closed.

The metric structure of Minkowski space is irrelevant.

A curved metric or a non-flat linear connection do not affect, since these geometric objects don't enter the axioms.

The same will not be true for the second pair.

Even if we start out with the form on spacetime it will turn out that the metric will necessarily be implicitly in the differential forms version of the second pair of Maxwell's equations.

**Remark 7** *Note that there is no connection between these equations and the pseudo-Euclidean metric.*

**Remark 8** *Since the metric of spacetime represents Einstein's gravitational potential (and the linear connection is also related to gravitational properties), the three axioms of electrodynamics are not contaminated by gravitational properties, in contrast to what happens in the usual textbook approach to electrodynamics.*

$$\oint_{V_2} \mathbf{F}_{ij} d\tau^{ij} = 0$$

$d\tau^{ij} = \eta^{ijkm} M_k N_m d_2 v$

$$\oint_{V_2} \mathbf{F}^*_{ij} d\tau^{ij} = \frac{1}{3} \int_{V_3} \eta_{\alpha ijk} J^\alpha d\tau^{ijk}$$

$d\tau^{ijk} = \eta^{ijkm} L_m d_3 v$

Here $M$ and $N$ are unit vectors, orthogonal to $V2$ and to one another, and $L$ is a unit vector orthogonal to $V3$;

$d2v$ and $d3v$ are invariant elements of area and 3-volume respectively.

**Remark 9** *We need, in addition, an electromagnetic spacetime relation that expresses the excitation* $\mathbf{G}$ *in terms of the field strength* $\boldsymbol{F}$*, i.e.,* $\mathbf{G} = \mathbf{G}[F]$.

*For classical electrodynamics, this functional becomes the Maxwell-Lorentz spacetime relation.*

What the present discussion does not tell us is how the covariant tensors $F$ and $G$ are connected to each other.

To establish this connection, we need additional structure, viz. a metric.

**Theorem 4.10** *Charge conservation equation. (global)*

$$\oint_{\partial V_4} \boldsymbol{J} = 0$$

For a proof see [Lindell, 2004].

**Theorem 4.11** *Charge conservation equation. (local)*

$$\boldsymbol{dJ} = 0$$

**Proof.** $\oint_{V_4} \boldsymbol{dJ} = 0$ Stokes' theorem,

$\Longrightarrow$ $\boldsymbol{dJ} = 0$ ■

**Theorem 4.12** *Charge conservation equation. (local)*

$$\boldsymbol{d}{}^*J = 0$$

**Proof.** $\boldsymbol{dd}\,{}^*\boldsymbol{F} \;=\; \boldsymbol{d}\,{}^*J \quad\Longrightarrow\quad 0 = \boldsymbol{d}\,{}^*J$ ■

**Theorem 4.13** *Charge conservation equation. (local)*

$$\boldsymbol{d} * \boldsymbol{J} = 0$$

**Proof.** $0 = \int_V \boldsymbol{d}\,{}^*J = \int_{\partial V} {}^*J$
$0 = \boldsymbol{d} * J$ ■

**Theorem 4.14** 4-*Charge conservation equation. (local).*

$$\partial_\mu J^\mu = 0$$

Where $J$ is the 4-electric current density.
**Proof.** From 4-divergence follows directly. ■

**Theorem 4.15** *Existence of* $\bar{\boldsymbol{A}}$.

$$\boldsymbol{dF} = 0 \qquad (4.5)$$
$$\Longrightarrow \exists \bar{\boldsymbol{A}} : \boldsymbol{F} = \boldsymbol{d}\bar{\boldsymbol{A}}$$

**Remark 10** $\boldsymbol{F} \;=\; \square \curlywedge \boldsymbol{A}$

**Remark 11** *The potential A is not measurable.*

**Theorem 4.16** *Lorentz force 4-form.*

$$\begin{aligned} f_\alpha &= (e_\alpha \rfloor \boldsymbol{F}) \curlywedge \boldsymbol{J} \\ &= -\boldsymbol{F} \curlywedge (e_\alpha \rfloor \boldsymbol{J}) \end{aligned}$$

**Remark 12** *It is an equation that is free from the metric and the connection.*

**Theorem 4.17**

$$\oint_{C_2} \boldsymbol{F} = 0$$

Faraday's induction law results from if one chooses $C_2'$

**Theorem 4.18** *4EDM2*

$$\boldsymbol{dF} = 0$$

See 4.5

**Proof.** $\oint_{C_2} \boldsymbol{F} = 0$

$\oint_{V_3} \boldsymbol{dF} = 0$ by Stokes' theorem,

with $C_2 = \partial V_3$

$\Longrightarrow \quad \boldsymbol{dF} = 0$ ■

# CHAPTER 5

# Equivalence theorems

Today is usually spoken of as anything evident of the four Maxwell equations, but you should know that the real number which contains the treaty is thirteen.

The final synthesis and theoretical clarification of the equations representing the four were due to the work, first independently and then jointly, is due to Heaviside and Hertz.

These are Maxwell's equations in 3-dimensional form.

| | |
|---|---|
| $\boldsymbol{dE} = -\frac{\partial}{\partial t}\boldsymbol{B}$ | Faraday law |
| $\boldsymbol{dH} = \frac{\partial}{\partial t}\boldsymbol{D} + \boldsymbol{j}$ | Oersted-Ampere law |
| $\boldsymbol{dD} = \boldsymbol{\rho}$ | Gauss law |
| $\boldsymbol{dB} = 0$ | |

$$\oint_P \mathbf{E} = -\frac{d}{dt}\int_A \mathbf{B}$$
$$\oint_P \mathbf{H} = \frac{d}{dt}\int_A \mathbf{D} + \int_A \boldsymbol{j}$$

$$\oint_P \boldsymbol{D} = \int_V \boldsymbol{\rho}$$
$$\oint_P \boldsymbol{B} = 0$$

Let's see their equivalents.

**Theorem 5.1** *Equivalence.*

$$\boldsymbol{dF} = 0 \quad \Longleftrightarrow \quad \begin{cases} \boldsymbol{d\bar{E}} = -\frac{\partial}{\partial t}\boldsymbol{B} \\ \boldsymbol{dB} = 0 \end{cases}$$

*:4.5*

**Proof.**

$$\boldsymbol{dF} = \boldsymbol{d}_S\boldsymbol{B} + dt \curlywedge \partial_t\boldsymbol{B} + (\boldsymbol{d}_S E + dt \curlywedge \partial_t\boldsymbol{\bar{E}}) \curlywedge dt$$

$$= \boldsymbol{d}_S\boldsymbol{B} + (\partial_t\boldsymbol{B} + \boldsymbol{d}_S E) \curlywedge dt = 0$$

$$\Longrightarrow \qquad \boldsymbol{d}_S\boldsymbol{B} = 0$$

$$\partial_t\boldsymbol{B} + \boldsymbol{d}_S E = 0$$

where $\mathrm{d}_S$ is the spatial exterior derivative. ■

**Theorem 5.2** *Equivalence.*

$$\boldsymbol{dG} = \boldsymbol{J} \quad \Longleftrightarrow \quad \begin{array}{ll} \boldsymbol{dD} = \boldsymbol{\rho} & \textit{Gauss law} \\ \frac{\partial}{\partial t}\boldsymbol{D} = \boldsymbol{d}\tilde{H} - \boldsymbol{J} & \textit{Ampere law} \end{array}$$

**Theorem 5.3** *Equivalence.*

$$\partial_\nu \boldsymbol{F}^{\mu\nu} = \frac{4\pi}{c} J^\mu \quad \Longleftrightarrow \quad \begin{array}{l} \vec{\nabla} \cdot \vec{E} = 4\pi\rho \\ \vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \end{array}$$

**Theorem 5.4** *Equivalence.*

$$\partial_\nu \boldsymbol{F}^{\nu\mu} = \frac{1}{\epsilon_0} J^\mu \quad \Longleftrightarrow \quad \begin{array}{l} \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \end{array}$$

*:4.4*

**Proof.** $*F = *_S E - *_S B \curlywedge dt$

$*_S d_S *_S E = \rho \qquad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho$

$-\partial_t E + *_S \boldsymbol{d}_S *_S B = j \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ ∎

**Theorem 5.5** *Equivalence.*

$$\boldsymbol{dA} = \boldsymbol{F} \iff \begin{array}{l} \operatorname{curl} \vec{A} = \vec{B} \\ \operatorname{grad} \mathbf{A}_0 - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E} \end{array}$$

**Proof.** $\boldsymbol{F}_{\nu\lambda} = 2\partial_{[\nu}\mathbf{A}_{\lambda]}$

$\partial_{[\kappa}\partial_\nu \mathbf{A}_{\lambda]} = 0$

$\lambda\mu\nu = 123 \iff \operatorname{div} B = 0$

$E_\alpha = \boldsymbol{F}_{0\alpha} \qquad \alpha = 1, 2, 3$

set any index, e.g., $\lambda$, equal to zero $\iff$ $\operatorname{curl} \vec{E} + \frac{\partial B}{\partial x^0} = 0$ ∎

**Theorem 5.6** *Maxwell's equations are equivalent to the equations* $\boldsymbol{dF} = 0$ *and* $\delta\boldsymbol{F} = \boldsymbol{J}$ *on the manifold* $R^4$ *endowed with the Lorentz metric.*

**Proof.** A straightforward computation shows that

$$\begin{aligned}\boldsymbol{dF} &= (\operatorname{curl} E + \frac{\partial B}{\partial t})_x dy \curlywedge dz \curlywedge dt + (\operatorname{curl} E + \frac{\partial B}{\partial t})_y dz \curlywedge dx \curlywedge dt \\ &+ (\operatorname{curl} E + \frac{\partial B}{\partial t})_z dx \curlywedge dy \curlywedge dt + (\operatorname{div} B) dx \curlywedge dy \curlywedge dz.\end{aligned}$$

Thus $\boldsymbol{dF} = 0$ is equivalent to such equations.

Since the index of the Lorentz metric is 1, we have $\delta = {*d*}$ . Thus,

$$\begin{aligned}\boldsymbol{\delta F} &= *\boldsymbol{d}*\boldsymbol{F} = *d(-E^1 dy \curlywedge dz - E^2 dz \curlywedge dx - E^3 dx \curlywedge dy \\ &+ B^1 dx \curlywedge dt + B^2 dy \curlywedge dt + B^3 dz \curlywedge dt) \\ &= *(-(\operatorname{div} E) dx \curlywedge dy \curlywedge dz + \operatorname{curl} B - \frac{\partial B}{\partial t} x dy \curlywedge dz \curlywedge dt + \\ &+ (\operatorname{curl} B - \frac{\partial E}{\partial t})_y dz \curlywedge dx \curlywedge dt + \operatorname{curl} B - \frac{\partial E}{\partial t}_z dx \curlywedge dy \curlywedge dt) \\ &= (\operatorname{curl} B - \frac{\partial E}{\partial t})_x dx + (\operatorname{curl} B - \frac{\partial E}{\partial t})_y dy \\ &+ (\operatorname{curl} B - \frac{\partial E}{\partial t})_z dz - (\operatorname{div} E) dt.\end{aligned}$$

Thus $\boldsymbol{F} = \boldsymbol{\delta J}$ if and only if the equations hold. ■

**Theorem 5.7** *Equivalence.*

$$\boldsymbol{dG} + 4\pi\boldsymbol{\rho} = 0 \qquad \Longleftrightarrow \qquad \begin{aligned} \boldsymbol{d\rho} &= 0 \\ \operatorname{div} \vec{J} + \tfrac{\partial \rho}{\partial t} &= 0 \end{aligned}$$

**Proof.** $G^{kl} := \frac{1}{2}\epsilon^{klmn} G_{mn}$

$\partial_j G^{ij} = J^i$

$$\boldsymbol{F}^{0\alpha} = -\boldsymbol{F}_{0\alpha} \qquad \boldsymbol{F}^{\alpha\beta} = F_{\alpha\beta} \qquad\qquad \alpha, \beta = 1, 2, 3 \ \blacksquare$$

# CHAPTER 6

# Energy and momentum

The concept of energy and momentum are very important concepts in physics, let's study them in relation to electrodynamics.

**Theorem 6.1** *Lorentz force.*

$$f = q(\tilde{E} - \vec{v} \rfloor \boldsymbol{B})$$

**Definition 15** *Stress-energy tensor.*

$$\boldsymbol{T} = \boldsymbol{F} \cdot \boldsymbol{F} - \frac{1}{4}\boldsymbol{g}\langle \boldsymbol{F}, \boldsymbol{F} \rangle$$

**Remark 13** *It is symmetric.*

The contribution of such a Maxwell field to the energy-momentum tensor is given by

$\boldsymbol{T}^S = -\epsilon_0(\vec{E} \otimes \vec{E} - \frac{1}{2}E^2) - \dfrac{1}{\mu_0}(\vec{B} \otimes \vec{B} - \frac{1}{2}B^2)$ spacial part

$\mathbf{T}_{00} = \frac{1}{8\pi}(E^2 + H^2)$

$\mathbf{T}_{0j} = \frac{1}{4\pi}(E \times B)^j$

**Theorem 6.2** *The quantity* $w = \mathbf{T}_{00}$ *is the energy density of the electromagnetic field* $\mathbf{F}_{ik}$ *,*

*The vector* $S_\alpha = -\mathbf{T}_{0\alpha}$ *is Poynting vector,*

**Definition 16** *2-form of Poynting.*

$$\boldsymbol{S} = \boldsymbol{E} \curlywedge \boldsymbol{H}$$

**Theorem 6.3** *Poynting.*

$$\boldsymbol{dS} = -\frac{\partial \boldsymbol{w}}{\partial t} - \boldsymbol{E} \curlywedge \boldsymbol{j}$$

*where* $\boldsymbol{w}$ *is 3-dimensional energy density 3-form and* $j$ *is 3-dimensional current density 2-form.*

**Proof.** Using Maxwell's laws, we can derive a conservation law for electromagnetic energy.

$\boldsymbol{dS} = \boldsymbol{d}(\boldsymbol{E} \curlywedge \boldsymbol{H}) = (\boldsymbol{dE}) \curlywedge \boldsymbol{H} - \boldsymbol{E} \curlywedge (\boldsymbol{dH})$

Using Ampere's and Faraday's laws, this can be written

$\boldsymbol{dS} = -\frac{\partial}{\partial t}\mathbf{B} \curlywedge \boldsymbol{H} - \boldsymbol{E} \curlywedge \frac{\partial}{\partial t}\mathbf{D} - \boldsymbol{E} \curlywedge \boldsymbol{j}$

the theorem follows. ■

**Theorem 6.4** *Maxwell's equations imply that*

$$\mathbf{T}^{\mu\nu}{}_{,\nu} = -F^{\mu\alpha} J_{\alpha}$$

**Theorem 6.5** *Conservation of energy-momentum.*

$$\nabla \cdot \boldsymbol{T} = 0$$

*In the absence of charges/currents, the energy-momentum tensor is conserved.*

This is equivalent to

| | |
|---|---|
| $\frac{d}{dt}E_{field} = 0$ | $\frac{d}{dt}P_{field} = 0$ |
| energy | momentum |

# CHAPTER 7 Electromagnetic action

In the spirit of field theory, all information is contained in the fields, which are the degrees of freedom.

**Definition 17** *Electromagnetic Lagrangian function.*

$$L = -\frac{1}{2}\boldsymbol{F} \curlywedge *\boldsymbol{F} - \boldsymbol{J} \curlywedge \boldsymbol{A}$$

We derive Maxwell's equations by using a mixed variational principle, similar to the Hamilton–Pontryagin principle [Yoshimura and E.Marsden, 2006] for classical Lagrangian mechanics.

To do this, we treat $A$ and $F$ as separate fields, while $G$ acts as a Lagrange multiplier, weakly enforcing the constraint $\boldsymbol{F} = \boldsymbol{dA}$.

Define the extended action to be

$$S[A, F, G] = \int_X (-\frac{1}{2}F \curlywedge *F + \ A \curlywedge J \ + \ \ (F - dA) \curlywedge G)$$

Then, taking the variation of the action along some $\alpha$,$\phi$ ,$\gamma$ (vanishing on $\partial X$), we have.

$$dS[A,F,G]\cdot(\alpha,\phi,\gamma) = \int_X(-\phi \curlywedge {}^*F + \alpha \curlywedge J + (\phi - d\alpha)\curlywedge G + (F - dA)\curlywedge\gamma)$$
$$= \int_X(\alpha \curlywedge (J - dG) + \phi \curlywedge (G - *F) + (F - dA)\curlywedge\gamma)$$

Therefore, setting this equal to zero, we get the equations

$$\boldsymbol{J} = \boldsymbol{dG}$$
$$\boldsymbol{G} = {}^*\boldsymbol{F}$$
$$\boldsymbol{F} = \boldsymbol{dA}$$

This approach provides some additional insight into the geometric structure of electromagnetics:

The gauge condition $\boldsymbol{F} = \boldsymbol{dA}$ and constitutive relations $\boldsymbol{G} = {}^*\boldsymbol{F}$ are explicitly included in the equations of motion, as a direct result of the variational principle.

**Theorem 7.1** *Equation of motion.*

$$m_0 c \frac{\partial \boldsymbol{v}_\nu}{\partial \mathfrak{s}} = \frac{q}{c}\boldsymbol{F}_{\mu\nu}\boldsymbol{v}^\mu \tag{7.1}$$
$$m_0 \mathfrak{c} \frac{\partial \boldsymbol{u}^\nu}{\partial \mathfrak{s}} = \frac{q}{\mathfrak{c}} F^{\nu\mu}\boldsymbol{u}_\mu \tag{7.2}$$

**Proof.** $\delta S = \delta \int \left(-m_0 c\, \partial\mathfrak{s} - \frac{q}{\mathfrak{c}}\vec{\boldsymbol{A}}\cdot\partial\vec{\boldsymbol{x}}\right)$ $\qquad S = S_{\text{particle}} + S_{\text{EM}}$

$$= -\int \left(m_0 c \frac{\partial\vec{x}\cdot\partial\delta\vec{\boldsymbol{x}}}{\partial s} + \frac{q}{c}\vec{\boldsymbol{A}}\cdot\partial\delta\vec{\boldsymbol{x}} + \frac{q}{c}\delta\vec{\boldsymbol{A}}\cdot\partial\vec{\boldsymbol{x}}\right)$$
$$= \int \left(-m_0 c \frac{\partial \boldsymbol{v}_\rho}{\partial \mathfrak{s}} + \frac{q}{c}\left(\partial_\rho A_\mu - \partial_\mu A_\rho\right)\boldsymbol{v}^\mu\right)\delta\boldsymbol{x}^\rho \partial\mathfrak{s} = 0$$

$$\implies \qquad m_0 c \frac{\partial \boldsymbol{v}_\nu}{\partial \mathfrak{s}} = \frac{q}{c}\left(\partial_\nu \mathbf{A}_\mu - \partial_\mu \mathbf{A}_\nu\right) \boldsymbol{v}^\mu$$

$$\text{take} \qquad \boldsymbol{F}_{\mu\nu} := \partial_\nu \mathbf{A}_\mu - \partial_\mu \mathbf{A}_\nu \qquad \text{Faraday Tensor} \ \blacksquare$$

**Remark 14** *Theorem 7.1 provides an alternative way of defining the Faraday tensor through the 4-magnetic potential.*

**Theorem 7.2** *Equation of motion for charge in an e.m. field.*

$\vec{\boldsymbol{f}} = \boldsymbol{F} \cdot \vec{J}$

$\frac{d\mathbf{p}}{d\tau} = q\mathbf{F}(\mathbf{u})$

$f = \frac{q}{c}\boldsymbol{F} \cdot v$

**Remark 15** *This matrix equation demonstrates the unity of the electric and magnetic fields.*

*Neither one by itself, E or B, is a frame-independent, geometric entity. But merged together into a single entity, F, they acquire a meaning and significance that transcends coordinates and reference frames.*

In the definition of the action, if individual particles are to be represented by charge distributions rather than as point particles, then the summation over particles in the second term has to be replaced by a volume integral.

That is, the factor e is replaced by $dV$. Then the factor dxk can be replaced by $J^k dt$.

The volume element $dV$, taken with the element cdt, form the same element $d\Omega$, used in the part of the action ascribable to just the fields.

**Theorem 7.3** *4EDM1.*

$\partial_\nu \boldsymbol{F}^{\mu\nu} = \frac{4\pi}{c} J^\mu$

See 4.4

**Proof.** $\delta S = \int -\frac{1}{2} F^{\nu\mu} \delta F_{\nu\mu} - J^\mu \delta A_\mu dx^4$

$\int -\frac{1}{2} F^{\nu\mu} (\frac{\partial \delta A_\nu}{\partial x^\mu} - \frac{\partial \delta A_\mu}{\partial x^\nu}) - J^\mu \delta A_\mu dx^4$

$\int -F^{\nu\mu} \frac{\partial \delta A_\nu}{\partial x^\mu} - J^\mu \delta A_\mu dx^4$

$\int (\frac{\partial F^{\nu\mu}}{\partial x^\mu} - J^\mu) \delta A_\mu dx^4$

$\implies \frac{\partial F^{\nu\mu}}{\partial x^\mu} = J^\mu$ ■

## 7.1 Gauge Transformations

The term gauge transformations was introduced by Hermann Weyl in 1918 in an attempt, geometrically difficult and far from physics, to build a unified theory of electromagnetic field and gravitational field, the two long-range force fields.

**Theorem 7.4** *Gauge transformations for A.*

$$\boldsymbol{A}_{new} = \boldsymbol{A}_{old} + \boldsymbol{d\phi}$$

*arbitrary function $\phi$ leave $\boldsymbol{F}$ unaffected.*

**Theorem 7.5** *One can adjust the gauge so that*

$\nabla \cdot \boldsymbol{A} = 0$ *("Lorentz gauge")*

$\Box \boldsymbol{A} = -4\pi J.$

Here $\Box$ is the wave operator ("d'Alembertian"):
$\Box A = A^{\alpha,\mu}{}_{,\mu}\vec{e}_\alpha$

# CHAPTER 8

# Constitutive Equations

**Definition 18** *Electromagnetic susceptibility.*

$\mathbf{G}^{\lambda\nu} = \frac{1}{2}\chi^{\lambda\nu\sigma\kappa}\boldsymbol{F}_{\sigma\kappa}$

Describes in a generally covariant manner the electromagnetic behavior of a general linear medium with instantaneous and local interaction between the fields.

The number of components of the constitutive tensor is restricted by the symmetry conditions.

**Theorem 8.1** *Skew symmetry properties.*

*The skew symmetry of the fields $\mathbf{G}^{\lambda\nu}$, $\mathbf{F}_{\sigma\kappa}$ implies skew-symmetry in the two index pairs $\lambda\nu, \sigma\kappa$.*

$$\chi^{\lambda\nu\sigma\kappa} = -\chi^{\nu\lambda\sigma\kappa} \tag{8.1}$$
$$\chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\nu\kappa\sigma}$$

**Theorem 8.2** *Skew symmetry properties.*

$$\chi^{\lambda\nu\sigma\kappa} = \chi^{\sigma\kappa\lambda\nu}$$

**Remark 16** *The constitutive tensor can be represented by a 6-by-6 symmetric matrix (21 elements).*

If $\partial_\nu \chi^{[\lambda\nu\sigma\kappa]} = 0$ on a Cartesian Lorentz frame $\Longrightarrow$ $\chi^{[\lambda\nu\sigma\kappa]} = 0$
the number of independent elements: 20.
where [] is the index antisimmetrization operator.

**Remark 17** *The equation has the generally invariant form, because the divergence of any alternating contra-variant density of weight + 1 is known to be a natural invariant.*

# Part II

# GENERAL ELECTRODYNAMICS

Sometimes it is necessary to introduce locally arbitrary (co)frames of reference that are no-inertial, that is, accelerated in general.

Then the components of the electromagnetic field with respect to a coframe $b$, the physical components emerge and the Maxwell equations can be expressed in terms of these physical components.

$$\begin{aligned} \boldsymbol{J} &= \frac{1}{3}\boldsymbol{J}_{\alpha\beta\gamma} b^{\alpha} \curlywedge b^{\beta} \curlywedge b^{\gamma} \\ \boldsymbol{F} &= \frac{1}{2}\boldsymbol{F}_{\alpha\beta} b^{\alpha} \curlywedge b^{\beta} \\ \boldsymbol{G} &= \frac{1}{2}\boldsymbol{G}_{\alpha\beta} b^{\alpha} \curlywedge b^{\beta} \end{aligned} \tag{8.2}$$

current, the excitation, and the field strength

anholonomicity

$$\begin{aligned} \partial_{[\alpha}\boldsymbol{F}_{\beta\gamma]} - C_{[\alpha\beta}{}^{\gamma}\boldsymbol{F}_{\gamma|\delta]} &= 0 \\ \partial_{[\alpha}\boldsymbol{G}_{\beta\gamma]} - C_{[\alpha\beta}{}^{\gamma}\boldsymbol{G}_{\gamma|\delta]} &= \frac{1}{3}\boldsymbol{J}_{\alpha\beta\gamma} \end{aligned} \tag{8.3}$$

$$\begin{aligned} &\partial_{\beta}\boldsymbol{H}'^{\alpha\beta} - C_{\beta\gamma}{}^{\alpha}\boldsymbol{H}'^{\beta\gamma} = \tfrac{1}{3}\boldsymbol{J}'^{\alpha} \\ &\partial_{[\alpha}\boldsymbol{F}_{\beta\gamma]} - C_{[\alpha\beta}{}^{\gamma}\boldsymbol{F}_{\gamma|\delta]} = 0 \end{aligned}$$

The terms with the $C'$s emerge if Maxwell's equations are referred to an arbitrary local (co)frame. Only if we restrict ourselves to natural (or coordinate) frames is $C = 0$, and Maxwell's equations display their conventional form.

This representation of electrodynamics can be used in special or in general relativity.

If one desires to employ a laboratory frame of reference, then this is the way to do it:

The object of anholonomicity in the lab frame has to be calculated.

By substituting, we find the Maxwell equations in terms of the components Fap and so on of the electromagnetic field quantities with respect to the lab frame -and these are the quantities one observes in the laboratory.

Therefore the Fap and so on are called physical components.

# CHAPTER 9
# Electrodynamics in the presence of gravity

These are the basic equations of electrodynamics in the presence of gravity.

In any other frame these equations will have the same form, but with commas replaced by ;

**Axiom 3** $F^{\mu\nu}{}_{;\nu} = \mu_0 J^\mu$

**Axiom 4** $F_{[jk;i]} = 0$

where ; is the covariant derivative. (Ricci calculus)

**Remark 18** *The electrodynamic equations are all obtained from special relativity by the comma-goes-to-semicolon rule.*

**Theorem 9.1** *Charge conservation.*

$J^{\alpha}_{;\alpha} = 0$

Give the fields that generate mass-energy, and their time-rates of change, and give 3-geometry of space and its time-rate of change, all at one time, and solve for the 4-geometry of spacetime at that one time. And only then let the equations for geometrodynamics and field dynamics go on to predict for all time, in and by themselves, needing no further prescriptions from outside (needing only work!), both the spacetime geometry and the flow of mass-energy throughout this spacetime.

We emphasize that the statements can be assumed to be strictly valid only as long as the e.m. field is treated as a classical, not quantized, field. When a quantum point of view is adopted, however, the expression is recovered only as a first approximation because, in general, according to a quantum description of the e.m. field in a space with torsion, one may always expect an interaction between the photons and the torsionic background.

In fact, a photon, with a process of the second order in the perturbative development of the e.m. interaction, can virtually disintegrate into an electron-positron pair (vacuum polarization effect); because these particles are massive fermions, which couple to torsion, they feel the presence of a torsionic background.

As a consequence, the e.m. field is also affected by torsion ; even if torsion does not directly interact with the photon field, it does interact with the virtual pairs produced in vacuum by a "physical" photon.

# CHAPTER 10

# Applications

A structure-preserving can be developed by discretization of the Lagrangian framework for electromagnetism, combining techniques from variational integrators and discrete differential forms.

This leads to a general family of variational, multisymplectic numerical methods for solving Maxwell's equations that automatically preserve key symmetries and invariants.

The Yee's finite-difference time-domain (FDTD) scheme, along with a number of related methods, are multisymplectic and derive from a discrete Lagrangian variational principle.

The Yee scheme (also known as finite-difference time-domain, or FDTD) was introduced in Yee (1966) and remains one of the most successful numerical methods used in the field of computational electromagnetics, particularly in the area of microwave problems. Although it is not a "high-order" method, it is still preferred for many applications because it preserves important structural features of Maxwell's equations that other methods fail to capture.

Among these distinguishing attributes are that the Gauss constraint $\vec{\nabla} \cdot \vec{D} = \rho$ is exactly conserved in a discrete sense, and electrostatic solutions of the form $\vec{E} = -\vec{\nabla}\phi$

indeed remain stationary in time.

These desirable properties are direct consequences of the variational and discrete differential structure of the Yee scheme, which mirrors the geometry of Maxwell's equations.

Geometric numerical integrators have been used primarily for the simulation of classical mechanical systems, where features such as symplecticity, conservation of momentum, and conservation of energy are essential.

Among these, variational integrators are developed by discretizing the Lagrangian variational principle of a system, and then requiring that numerical trajectories satisfy a discrete version of Hamilton's stationary-action principle. These methods are automatically symplectic, and they exactly preserve discrete momenta associated to symmetries of the Lagrangian. In addition, variational integrators can be seen to display good long-time energy behavior, without artificial numerical damping.

Building on this, Lew,Marsden, Ortiz, and West (2003) introduced asynchronous variational integrators (AVIs), with which it becomes possible to choose a different time step size for each element of the spatial mesh, while still preserving the same variational and geometric structure as uniform-time-stepping schemes.

These methods were implemented and shown to be not only practical, but in many cases superior to existing methods for problems such as nonlinear elastodynamics.

While there have been attempts to apply the existing AVI theory to computational electromagnetics, these efforts encountered a fundamental obstacle. The key symmetry of Maxwell's equations is not rotational or translational symmetry, as in mechanics, but a differential gauge symmetry.

This can be done by viewing the objects of electromagnetism not as vector fields, but as differential forms in 4-dimensional spacetime. Using a discrete exterior calculus (called DEC) as the framework to discretize these differential forms, we find that the resulting variational integrators automatically respect discrete differential identities, such as $d^2 = 0$ (which encapsulates the previous div-curl-grad relations) and Stokes' theorem.

Consequently, they also respect the gauge symmetry of Maxwell's equations, and therefore preserve the associated discrete momentum. One of its notable features is that the electric field E and magnetic field H do not live at the same discrete space or time locations, but at separate nodes on a staggered lattice. The reason why this particular setup leads to improved numerics is not obvious: if we view E and H simply as vector fields in 3-space—the exact same type of mathematical object—why shouldn't they live at the same points? Indeed, many finite element method (FEM) approaches do exactly this, resulting in a "nodal" discretization. However, from the perspective of differential forms in spacetime, it becomes clear that the staggered-grid approach is more faithful to the structure of Maxwell's equations: as we will see, $E$ and $H$ come from objects that are dual to one another (the spacetime forms $F$ and $G = {}^*F$), and hence they naturally live on two staggered, dual meshes. We see that the differential forms version is not equivalent to an arbitrary vector field discretization, but rather implies a particular choice of discrete objects. Mathematical tools developed by Weyl and Whitney in the 1950s, in the context of algebraic topology, turned out to provide the necessary foundations on which robust numerical techniques for electromagnetism can be built. The particular "flavor" of discrete differential forms and operators is known as discrete exterior calculus, or DEC for short.

The single consumer model
Application to the one consumer model
The household model
Markets

# Part III

# CONCLUSIONS

The topological universality of the Maxwell Faraday and Maxwell Ampere equations is an artifact of C2 differential forms on a domain of dimension 4.

Starting with a 1-form of (electromagnetic) Action, the Maxwell Faraday equations become a consequence of the Poincare lemma.

Starting from an 3-form density, the Maxwell Ampere equations become a consequence of the topological constraint that the 3- form density is exact.

The conservation of charge current is a consequence of the Poincare lemma.

Geometrical structure constraining the deduced 2-form and the induced 2-form establish equivalence classes of constitutive equations.

The structure of electrodynamics has nothing to do with Poincare or Lorentz covariance. The transformations involved are diffeomorphisms and frame transformations alone. As long as a 4-dimensional connected, Hausdorff, paracompact and oriented differentiable manifold is used as the spacetime, the axioms stay covariant and remain the same. In particular, arbitrary frames, holonomic and anholonomic ones, can be used for the evaluation.

If one desires to generalize special relativity to general relativity theory or to the Einstein-Cartan theory of gravity (a viable alternative to Einstein's theory formulated in a non-Riemannian spacetime), then the Maxwellian structure is untouched by it.

We use Cartan's calculus to reformulate the general variational principle and conservation laws in terms of exterior forms. In applying this method to relativistic electrodynamics, we not only benefit from the great economy of Cartan's formalism but also gain a deeper understanding of fundamental results.

Starting from a spacetime and only considering a basic physical quantity, 4-potential, we derived mathematically all electromagnetism. Not only that, but from this point of view there is no such electric field or a magnetic field at each point of spacetime, but a geometric object (in the same sense that a vector is a geometric object), the Faraday tensor, that truly unifies electromagnetism.

From the point of view preRE different inertial observers measure different amounts of electric and magnetic field, the key is that in RE their sum must be the same in all reference systems. What is an electric field in one, in another is a sum of electric and magnetic, which exerts the same force.

This formulation indicates that, again, we had which were wrong fundamental physical concepts and using the proper ones physics is much simpler. This is commonly called unification.

Before there was the next problem. If a charge is at rest, generates an electric field only. What happens if I see this charge from a moving reference frame? Observe a magnetic field, or only an electric field?

How can it be that something apparently inherent to the charge, as the magnetic field, is born only by seeing from a different frame of reference?

Now we have the answer. There are no electric or magnetic fields. There is only the Minkowski tensor.

If we make a transformation between the reference system, obviously changes the tensor components, appreciating in some reference systems purely electric or magnetic fields, and in other, fields mixed. But all that happens is that we are observing the same (tensor) electromagnetic field from different systems of reference.

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