UNIVERSITY OF SONORA
FACULTY OF PHYSICS RESEARCH
Hermosillo, Sonora, Mexico

HIGHER DIMENSIONAL SPHERES, BLACK HOLES AND COSMOLOGY

by

GERMAN AVILA SIERRA

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Physics

Professor Juan Antonio Nieto Garcia
Director

Professor Julio C. Saucedo Morales
Co-Director

ABSTRACT

We explain in some detail the geometric structure of spheres in any dimension. Our approach may be helpful for other homogeneous spaces (with other signatures) such as the de Sitter and anti-de Sitter spaces. As a particular case we consider the 1-Sphere. Assuming the light path as a 1-Sphere (circle) in vacuum we develop the corresponding special theory of relativity. We show that the derived metric reproduces time dilation and the length contraction of the special theory of relativity. We argue that this is an interesting result from which one can derive both the Schwarzschild and the de Sitter metrics.

Moreover, using a Lagrangian approach we set the bases for a future work towards a theory for black holes and cosmology models as unified concepts. From this formalism we show that one can derive the field equations for both the FRW-cosmological model and the Schwarzschild black hole solution from a first order Lagrangian of a constrained system, which is derived from the Einstein -Hilbert action.

Keywords: Spheres, Poincaré-Conjecture, Relativity, Cosmological models, Kaluza-Klein theory, Black holes.

July, 2014

To my beloved wife Elia,
and my beloved newborn son German.

ACKNOWLEDGMENTS

This project would not have been possible without the support of many people. Many thanks to my adviser, Professor Juan Antonio Nieto Garcia, who stood by me throughout this process and read my numerous revisions helping me to make some sense of the confusion. Professor Julio C. Saucedo Morales, thank you for your enormous support and inspiration about Science and Nature. Also thanks to my committee members, Professor Martin G. Garcia Alvarado and Professor Sergio Barraza Felix, who offered guidance and support. I specially want to thank Professor Santos Jesus Castillo for being a great adviser in this work as well as inspiration about possibilities and self expression and for sharing with me so much of his time and knowledge during the last year to make this work possible. Thank you for your patience and for your interest in seeing us, your students, grow. Thank you very much. Thanks to the University of Sonora Faculty of Physics Research and the National Council of Science and Technology (CONACYT) for providing me with the financial means to complete this project. And finally, thanks to my wife, my son, my mother, and the rest of my family who endured this long process with me, always offering support and love, and patience as well. Specially thanks to my brother Luis, for your support and for understanding what really is important to me. Dad, wherever you are, thanks for always being a model of passion and love for your career and of course, for your family too. I miss you so much. Thanks.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Higher-dimensional spheres S^d are of great importance in both mathematics and physics. In mathematics, for instance, one can find the statement that the only parallelizable spheres over the real are S^1, S^3 and S^7 (see Refs. [1]-[10]) and also that any simply connected compact manifold over the real must be homeomorphic to S^d (generalized Poincaré-conjecture) (see Refs. [11]-[15]). Topologically, the relevance of several spheres emerges trough the Hopf maps $S^3 \xrightarrow{S^1} S^2$, $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$ [4], with fibers S^1, S^3 and S^7 respectively. Of course these fiber spaces are deeply related to the normed division algebras; real numbers, complex numbers, quaternions and octonions (see Ref. [10]). Surprisingly, there is also an intriguing relation between N-qubit theory and the spheres S^1, S^3 and S^7 [16]-[18] (see also Ref. [19]). Moreover, the sphere S^8 is of great relevance in Bott periodicity theorem (see Ref. [10] and references therein). In physics, one meets with S^3 in the Friedmann-Robertson-Walker cosmological model (see Ref. [20] and references therein), while in supergravity and superstring compactification (see Ref. [21] and references therein) one learns that one of the most interesting candidate for a realistic Kaluza-Klein theory is S^7 or the corresponding squashed sphere S^7 [22].

In particular, the 1-sphere S^1 emerges trough the Hopf map $S^3 \xrightarrow{S^1} S^2$ (see Ref. [4]). Of course this result is closely related to the complex numbers. Similarly, the Hopf maps $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$ are related to quaternions and octonions (see Ref. [8]-[10]). Even more surprising is that S^1 is related to 1-qubit theory (see Refs. [16]-[19]). Physically, S^1 is, of course, connected to one of the fundamental forces, namely electromagnetism. In fact, in terms of a five dimensional theory, S^1 provides one of the most interesting realistic Kaluza-Klein theory via a spontaneous compactification of one space dimension. Of course, cosmologically one meets with $S^3 \times S^1$ in the Friedmann-Robertson-Walker cosmological model type (see Ref. [20] and references therein) in a five dimensional theory.

Historically, the circle is an old mathematical system. But, perhaps, Tales de Mileto (624- 547 BC) gave the first serious steps in the mathematical structure of S^1. In the Elements of Euclides (325-265 BC) one finds some of the properties of S^1. In fact, Menaechmus (380-320 BC) is famous for the discovery that S^1 is a conic section. Moreover, considering a path of a point on the circumference of a circle as the circle rolls without slipping along a straight line one obtains one of the most famous curves in the history of mathematics: the cycloid. The curve was named by Galileo (1564-1642) in 1599. But it seems that Nicholas de Cusa (1401-1464) was the first to study it seriously and Marin Mersenne (1588-1648) gave the formal definition of a cycloid. Personne de Roberval (1602-1675) in 1628 showed the area under the curve being $3\pi a^2$. It is worth mentioning that Torricelli also found the correct value of the area. From modern perspective one can say that the cycloid is the result of a relative motion of a point on the circumference of a circle (see Ref. [21] and references therein). More general a trochoid is the word created by Gilles de Roberval for the curve described by a fixed point as a circle rolls along a straight line in the x-axis direction. As a circle of radius a rolls without slipping along a horizontal direction, the center of the circle moves parallel to the x-axis, and every other point P in the rotating plane rigidly attached to the circle traces the curve called the trochoid. Let b_0 the radius inside the circle ($b_0 < a_0$), on its circumference ($b_0 = a_0$), or outside ($b_0 > a_0$), the trochoid is called curtate, common, or prolate, respectively.

It is worth emphasizing the interesting result that light can travel not only describing a straight line but also in a circular path, that is a S^1 (see Refs. [39] and [41] and references therein). This result must be understood as a possible vacuum topological solution of the Maxwell theory. The idea emerges by combining the Hopf map $S^3 \xrightarrow{S^1} S^2$ with the Maxwell equations. These comments refer to empty space since it is well known that light can travel in a cycloid path in a medium with index n. So our idea of considering light traveling in S^1 is not empty but may be justified by these previous works. The new idea that we would like to add is to see light traveling in S^1 from the point of view of a rest frame: A-observer. So, we are interested to see what will be the scenario for a B-observer which is looking S^1 traveling with constant velocity v along the x-direction.

In this thesis we would like to explain the geometry structure of any higher-dimensional sphere S^d. Our method is straightforward and can be applied to any spacetime with t-time and s-space signatures. We explain how to obtain the de Sitter space in spherical coordinates. In particular, we explain the geometry structure of the 1-dimensional sphere S^1. We discuss how the development of the special theory of relativity can be obtained when one changes the Pythagorean theorem for a horizontal-vertical light paths to a light moving in S^1 at rest: the A-frame or A-observer. The advantage of this is that both the time dilation and the length contraction arise as particular cases of a parent cycloid. Moreover, in a generalized context we show that one may be able to obtain not only the Schwarzschild solution but the de Sitter (or anti-de Sitter) metric as well. As it is known, an object moving in a circle in the rest A-frame must describe a parent cycloid in the non rest B-frame. This is true if the A-system is moving with constant velocity v much less than c. But if the system is moving near the velocity of light the curve must be a generalized parent cycloid. In the case of light moving in S^1 with radius b_0 one must compare it with a reference frame moving also in S^1 but with radius $a_0 < b_0$.

One may expect that the above comments about higher dimensional spheres can be linked to cosmological and black holes scenarios. In particular, this may be done by taking advantage of the recently approaches of the Friedman-Robertson-Walker (FRW) cosmological model and the Schwarchild balck-holes solution in terms of a first order Lagrangian of constrained systems, which is obtained from the Einstein-Hilbert action. Traditionally, in (1+3)-dimensions the FRW cosmological model is described by the S^3-sphere while the black-holes solutions is associated with the S^2-sphere. This two particular cases are genaralized for higher dimensions by introducing higher dimensional spheres. In this sense one needs a specific criteria for selecting exceptional spheres and this in turn implies exceptional cosmological and black-holes solutions. The main motivation of this Thesis is to work towards a unified theory for cosmology, black-holes and spheres. For this purpose we first review the higher-dimensional spheres formalism putting special emphasis in the 1-sphere S^1. Moreover, we discuss the possible unification of cosmology and black-holes in higher dimensions. We would expect that the complete formalism presented in this work may be helpful to obtain a unified theory for cosmology, black-holes and spheres as a final goal.

Moreover we generalize such formalism showing that the FRW-cosmological model and the Schwarzchild balck-hole solution arise as a limit cases of our more general first order Lagrangian formalism which is also derived by using higher dimensional Einstein-Hilbert action.

Our approach may be physically interesting for a number of reasons. First it may allow in a consisitent way to unfied the concepts of FRW-cosmology and the black-hole solution. Second, one may use the complete mathematical tools of Lagrangians for constrained systems to study a number of symetries underliying the FRW-cosmology and the Schwarchild solution. Third, a unify treatment of cosmlogical and black-holes may be of particular interest in the context of string theory or M-theory. This is because M-theory predicts among other things that our universe may be a brane world and it appears attractive to explore whether M-theory also considers a brane-world/black hole correspondece.

Technically this thesis is organized as follows. In Chapter 2, we discuss the structure of higher dimensional spheres starting with a constrained system. In particular, we present the Riemann curvature tensor, the Ricci tensor and the scalar curvature for homogeneous spaces (section 2.1). In section 2.2, we derive the De Sitter metric in spherical coordinates. In Chapter 3, section 3.1, we briefly review the usual mechanism to derive the time dilation and length contraction of special relativity. We show that in this case the Pythagorean theorem is a key concept. In section 3.2, we also derive the time dilation and length contraction but now assuming that the light path is a circle. Moreover, we derive the Schwarszchild and the de Sitter formalism from this approach. In Chapter 4, section 4.1, we focus on the particular case of a $(1+D+d)$-cosmological model. In section 4.2, we discuss the Schwarzschild solution from the point of view of a constrained system. In section 4.3 we review the general case of $(n+D+d)$-cosmological model. In section 4.4, we show that the $((1+1)+D+d)$-cosmological model is reduced to the FRW-cosmological model and to the black-hole solution in $(1+D)$-dimensions. Finally, in Chapter 5, we make some final remarks about this work.

CHAPTER 2

HIGHER DIMENSIONAL SPHERES (THE GEOMETRIC STRUCTURE)

2.1 Spheres as a constrained system

Let us consider coordinate x^A with $A = 1, 2, ..., d+1$. We define the S^d sphere, with constant radius r_0, through the constraint

$$x^i x^j \delta_{ij} + x^{d+1} x^{d+1} = r_0^2, \tag{1}$$

where the δ_{ij} is the Kronecker delta and $i = 1, 2, ..., d$.

We shall be interested in the line element

$$ds^2 \equiv dx^A dx^B \delta_{AB} = dx^i dx^j \delta_{ij} + dx^{d+1} dx^{d+1}. \tag{2}$$

From (1) one obtains

$$x^{d+1} = \pm (r_0^2 - x^i x^j \delta_{ij})^{1/2}. \tag{3}$$

Thus, taking the differential of (3) gives

$$dx^{d+1} = -\frac{(\pm) x^i dx^j \delta_{ij}}{(r_0^2 - x^r x^s \delta_{rs})^{1/2}}. \tag{4}$$

So, substituting (4) into (2) leads to

$$ds^2 \equiv dx^i dx^j \delta_{ij} + \frac{(x^i dx^j \delta_{ij})(x^k dx^l \delta_{kl})}{(r_0^2 - x^r x^s \delta_{rs})}. \tag{5}$$

This expression can be rewritten as

$$ds^2 \equiv dx^i dx^j g_{ij}, \tag{6}$$

where

$$g_{ij} = \delta_{ij} + \frac{x_i x_j}{(r_0^2 - x^r x^s \delta_{rs})}. \tag{7}$$

Here, we used the expression $x^k \delta_{ki} = x_i$.

By using (7) we shall calculate the Christoffel symbols

$$\Gamma^i_{kl} = \frac{1}{2} g^{ij} (g_{jk,l} + g_{jl,k} - g_{kl,j}), \tag{8}$$

and the Riemann tensor

$$R^i_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}. \tag{9}$$

First, let us observe that

$$g_{jk,l} = \frac{1}{(r_0^2 - x^p x^q \delta_{pq})}[\delta_{jl}x_k + \delta_{kl}x_j + \frac{2x_j x_k x_l}{(r_0^2 - x^r x^s \delta_{rs})}]. \tag{10}$$

One finds that the Christoffel symbol (8) becomes

$$\Gamma^i_{kl} = \frac{1}{(r_0^2 - x^r x^s \delta_{rs})} g^{ij}(g_{kl}x_j). \tag{11}$$

But since

$$g^{ij} = (\delta^{ij} - \frac{x^i x^j}{r_0^2}), \tag{12}$$

one finds that

$$\Gamma^i_{kl} = \frac{g_{kl}x^i}{r_0^2}. \tag{13}$$

By substituting this result into (9) yields

$$\begin{aligned} R_{ijkl} = g_{im}R^m_{jkl} = g_{im}[\partial_k(\tfrac{g_{jl}x^m}{r_0^2}) - \partial_l(\tfrac{g_{jk}x^m}{r_0^2}) + (\tfrac{g_{nk}x^m}{r_0^2})(\tfrac{g_{jl}x^n}{r_0^2}) \\ -(\tfrac{g_{nl}x^m}{r_0^2})(\tfrac{g_{jk}x^n}{r_0^2})]. \end{aligned} \tag{14}$$

Simplifying (14) gives

$$R_{ijkl} = g_{im}[\partial_k(\frac{g_{jl}x^m}{r_0^2}) - \partial_l(\frac{g_{jk}x^m}{r_0^2}) + \frac{1}{r_0^4}x^m x^n(g_{nk}g_{jl} - g_{nl}g_{jk})]. \tag{15}$$

Now, taking the derivatives in the first and second terms in (15) one sees that

$$R_{ijkl} = \frac{1}{r_0^2}g_{im}[(g_{jl}\delta^m_k - g_{jk}\delta^m_l) + x^m(g_{jl,k} - g_{jk,l}) + \frac{1}{r_0^2}x^m x^n(g_{nk}g_{jl} - g_{nl}g_{jk})]. \tag{16}$$

Using (10) one notes that (16) becomes

$$\begin{aligned} R_{ijkl} = \tfrac{1}{r_0^2}(g_{ik}g_{jl} - g_{il}g_{jk}) + \tfrac{g_{im}x^m}{r_0^2(r_0^2 - x^r x^s \delta_{rs})}(\delta_{jk}x_l - \delta_{jl}x_k) \\ + \tfrac{g_{im}}{r_0^4}x^m x^n(g_{nk}g_{jl} - g_{nl}g_{jk}). \end{aligned} \tag{17}$$

But, one finds that

$$
\begin{aligned}
&\tfrac{1}{r_0^2(r_0^2-x^rx^s\delta_{rs})}(\delta_{jk}x_l-\delta_{jl}x_k)+\tfrac{1}{r_0^4}x^n(g_{nk}g_{jl}-g_{nl}g_{jk})\\
&=\tfrac{1}{r_0^2(r_0^2-x^rx^s\delta_{rs})}(\delta_{jk}x_l-\delta_{jl}x_k)+\tfrac{1}{r_0^4}[x_k(1+\tfrac{1}{(r_0^2-x^rx^s\delta_{rs})}x_nx^n)\delta_{jl}\\
&\qquad -x_l(1+\tfrac{1}{(r_0^2-x^rx^s\delta_{rs})}x_nx^n)\delta_{jk}]\\
&=\tfrac{1}{r_0^2(r_0^2-x^rx^s\delta_{rs})}(\delta_{jk}x_l-\delta_{jl}x_k)+\tfrac{1}{r_0^2(r_0^2-x^rx^s\delta_{rs})}(\delta_{jl}x_k-\delta_{jk}x_l)=0.
\end{aligned}
\tag{18}
$$

Thus, (17) is reduced to

$$
R_{ijkl}=\frac{1}{r_0^2}(g_{ik}g_{jl}-g_{il}g_{jk}). \tag{19}
$$

We recognize in this expression the typical form of the Riemann tensor for any homogeneous space.

From (19) one learns that the Ricci tensor $R_{jl}=g^{ik}R_{ijkl}$ is given by

$$
R_{jl}=\frac{(d-1)g_{jl}}{r_0^2}, \tag{20}
$$

while the scalar curvature $R=g^{jl}R_{jl}$ becomes

$$
R=\frac{d(d-1)}{r_0^2}. \tag{21}
$$

Usually the theory is normalized in the sense of setting $r_0^2=\frac{1}{k}=1$. In this case (21) leads to

$$
R=kd(d-1). \tag{22}
$$

As we shall explain in the next section the above procedure can be generalized to no compact spacetime. In such case one obtains that (22) can be generalized in such a way that $k=\{-1,0,1\}$.

2.2 Higher dimensional De Sitter space-time

Now suppose that instead of the constraint (1) we have

$$x^i x^j \eta_{ij} + x^{d+1} x^{d+1} = r_0^2, \tag{23}$$

where we changed the Euclidean metric $\delta_{ij} = (1, 1, ..., 1)$ by the Minkowski metric $\eta_{ij} = (-1, 1, ..., 1)$. Note that in this case the constant r_0^2 can be positive, negative or zero. Accordingly, the line element (2) is now given by

$$ds^2 \equiv dx^A dx^B \eta_{AB} = dx^i dx^j \eta_{ij} + dx^{d+1} dx^{d+1}. \tag{24}$$

It is not difficult to see that all steps to calculate the Christoffel symbols and the Riemann tensor components of the previous section are exactly the same. At the end, it can be shown that such quantities are

$$\Gamma^i_{kl} = \frac{g_{kl} x^i}{r_0^2} \tag{25}$$

and

$$R_{ijkl} = \frac{1}{r_0^2}(g_{ik} g_{jl} - g_{il} g_{jk}), \tag{26}$$

respectively. Here, the metric g_{ij} is now given by

$$g_{ij} = \eta_{ij} + \frac{x_i x_j}{(r_0^2 - x^r x^s \eta_{rs})}. \tag{27}$$

where $x_i \equiv \eta_{ij} x^j$.

It is worth mentioning that one can even consider a flat metric $\eta_{ij} = (-1, ..., -1, ...1, 1)$ with t-times and s-space coordinates and the procedure is exactly the same, that is, equations (23)-(27) are exactly the same, with the exception that now one must take the corresponding flat metric η_{ij}.

Just to show that our result agree with the de Sitter space-time in spherical coordinates let us consider the reduced spacetime

$$ds^2 \equiv dx^i dx^j g_{ij}, \tag{28}$$

obtained by using (23). Substituting (27) into (28) yields

$$ds^2 \equiv (\eta_{ij} + \frac{x_i x_j}{(r_0^2 - x^r x^s \eta_{rs})}) dx^i dx^j. \tag{29}$$

This expression can be rewritten as

$$ds^2 \equiv \frac{1}{(r_0^2 - x^r x^s \eta_{rs})}[(r_0^2 - x^m x^n \eta_{mn})\eta_{ij} + x_i x_j]dx^i dx^j. \tag{30}$$

By expanding $x^m x^n \eta_{mn} = -x^0 x^0 + x^a x^b \delta_{ab}$, with a, b running from 1 to $d-1$, one learns that (30) leads to

$$\begin{aligned} ds^2 \equiv \tfrac{1}{(r_0^2 + x^0 x^0 - x^e x^f \delta_{ef})}[&(r_0^2 + x^0 x^0 - x^a x^b \delta_{ab})(-dx^0 dx^0 + dx^c dx^d \delta_{cd}) \\ &+ x^0 x^0 dx^0 dx^0 - 2x^0 dx^0 x^a dx^b \delta_{ab} + x^a x^c dx^b dx^d \delta_{ab}\delta_{cd}]. \end{aligned} \tag{31}$$

Now, considering that $r^2 = x^a x^b \delta_{ab}$ one finds that (31) can be written as

$$\begin{aligned} ds^2 \equiv \tfrac{1}{(r_0^2 + x^0 x^0 - r^2)}[&(r_0^2 + x^0 x^0 - r^2)(-dx^0 dx^0 + dr^2 + r^2 d\Omega^{d-2}) \\ &+ x^0 x^0 dx^0 dx^0 - 2x^0 dx^0 r dr + r^2 dr^2]. \end{aligned} \tag{32}$$

where, $d\Omega^{d-2}$ is a volume element in $d-2$ dimensions. This can be simplified in the form

$$\begin{aligned} ds^2 \equiv \tfrac{1}{(r_0^2 + x^0 x^0 - r^2)}&[-(r_0^2 - r^2) dx^0 dx^0 + (r_0^2 + x^0 x^0) dr^2 - 2x^0 dx^0 r dr] \\ &+ r^2 d\Omega^{d-2}. \end{aligned} \tag{33}$$

Now, with the intention of getting a line element with the same form as the line element for black holes, we considered the change of variable

$$x^0 = f(t)(r_0^2 - r^2)^{1/2}. \tag{34}$$

Also

$$dx^0 = f'(t)(r_0^2 - r^2)^{1/2} dt - \frac{f(t) r dr}{(r_0^2 - r^2)^{1/2}}. \tag{35}$$

Consequently, one obtains

$$dx^0 dx^0 = f'^2(t)(r_0^2 - r^2) dt^2 - 2f'(t) f(t) r dr dt + \frac{f^2(t) r^2 dr^2}{(r_0^2 - r^2)} \tag{36}$$

and

$$x^0 dx^0 = f(t) f'(t)(r_0^2 - r^2) dt - f^2(t) r dr. \tag{37}$$

Substituting (34), (36) and (37) into (33) yields

$$ds^2 \equiv \tfrac{1}{(r_0^2-r^2)(1+f^2)}[-(r_0^2 - r^2)(f'^2(t)(r_0^2 - r^2)dt^2 - 2f'(t)f(t)rdrdt$$

$$+\tfrac{f^2(t)r^2dr^2}{(r_0^2-r^2)}) + (r_0^2 + f^2(t)(r_0^2 - r^2))dr^2 \tag{38}$$

$$-2(f(t)f'(t)(r_0^2 - r^2)dt - f^2(t)rdr)rdr)] + r^2 d\Omega^{d-2}.$$

It is not difficult to see that this expression can be simplified in the form

$$ds^2 \equiv -\frac{f'^2(t)}{(1+f^2)}(r_0^2 - r^2)dt^2 + \frac{r_0^2}{(r_0^2 - r^2)}dr^2 + r^2 d\Omega^{d-2}. \tag{39}$$

Now, writing $f(t)$ as

$$f(t) = \sinh(t/r_0), \tag{40}$$

one finally discovers that (39) can be written as

$$ds^2 \equiv -(1 - \frac{r^2}{r_0^2})dt^2 + \frac{dr^2}{(1 - \frac{r^2}{r_0^2})} + r^2 d\Omega^{d-2}. \tag{41}$$

This expression is, of course, very useful when one considers black-holes or cosmological models in the de Sitter (or anti-de Sitter) space.

CHAPTER 3

THE 1-SPHERE AND RELATIVITY

3.1. Special relativity from the Pythagorean theorem

Let us now derive time dilation and length contraction from a light path along a vertical and horizontal trajectories as described by an observer A at rest, which is moving with constant velocity v with respect a ground observer B. First, we assume that the light is moving vertically in the frame of the A-observer. But the light path for the B-observer will have two component: one vertical and one horizontal. The point is that considering together the result of the two observers A and B a rectangular triangle is formed. So one can apply the Pitagoras theorem in order to have the relation

$$c^2t^2 = v^2t^2 + c^2t_0^2. \tag{42}$$

From this expression one gets the time dilation formula

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{43}$$

Hidden in this derivation is the fact that light velocity c is the same for both A-frame and B-frame.

Similarly, if according to the A-observer the light path is moving forward and backward horizontally a distance L_0 the total time of the round light trip is

$$t_0 = 2L_0/c. \tag{44}$$

However, for the B-observer, in the forward case, we must have

$$ct_A = L + vt_A \tag{45}$$

and

$$ct_B = L - vt_B, \tag{46}$$

in the backwards case. So the total time of the round trip for the B-observer is given by

$$t = \frac{L}{c-v} + \frac{L}{c+v}. \tag{47}$$

Thus, one gets

$$t = \frac{2L/c}{1 - \frac{v^2}{c^2}}. \tag{48}$$

But, in order to derive the length contraction

$$L = \sqrt{1 - \frac{v^2}{c^2}} L_0, \tag{49}$$

traditionally one considers the vertical result (43) and the formula (44). The formulae (43) and (49) are, of course, two of the corner stone in the special theory of relativity.

3.2. Black Hole solution from a cycloid light path

Suppose now that A-observer, which is at rest, does not separate the light path as vertical or horizontal but instead it assumes that the light is moving in a circular path. A circle is in a sense a synchronized combination of both vertical and horizontal paths. The question arises; what will be the light trajectory for a B-observer which sees the frame A-observer moving at v velocity in the horizontal direction? One may anticipate, of course, that for the B-observer the light trajectory will be prolate cycloid (also called trochoid, in general). Our main goal here, however, is to establishing the connection of the light traveling in a circle with the light traveling horizontal or vertical direction of the previous section.

The first thing that one must mention is that the derivation of the length contraction of the previous section is in fact not completely correct because one is using a vertical result in the sense of time dilation in an experiment where the light path is moving in the horizontal direction with the respect the A-observer. The correct picture arises when one assumes that with respect the A-observer the light path corresponds to a circle instead of to separate horizontal and vertical trajectories. So the A-frame may use the equations

$$X_0 = -b_0 \cos\theta \tag{50}$$

and

$$Y_0 = b_0 sen\theta, \tag{51}$$

which lead to the light element

$$-c^2t_0^2 + X_0^2 + Y_0^2 = 0. \tag{52}$$

Substituting (50) and (51) into (52) one obtains the expression

$$-c^2t_0^2 + b_0^2 = 0, \tag{53}$$

which yields

$$ct_0 = b_0. \tag{54}$$

Now we would like to describe the trajectory described by the light with respect to a ground B-observer, when such trajectory corresponds to a circle with respect to the A-observer. The light path equations are in this case

$$X = vt - b\cos\theta \tag{55}$$

and

$$Y = b_0 sen\theta. \tag{56}$$

The light velocity can be obtained from

$$-c^2t^2 + X^2 + Y^2 = 0. \tag{57}$$

Substituting (55) and (56) into (57) one finds that

$$-c^2t^2 + (vt - b\cos\theta)^2 + (b_0 sen\theta)^2 = 0. \tag{58}$$

We shall now consider two particular cases: (1) when $\theta = \frac{\pi}{2}$ and (2) when $\theta = \pi$ and $\theta = 2\pi$. In the first case one sets $\theta = \frac{\pi}{2}$ in (58),

$$-c^2t^2 + (vt)^2 + (b_0)^2 = 0, \tag{59}$$

which in virtue of (54) leads to

$$-c^2t^2 + v^2t^2 + c^2t_0^2 = 0. \tag{60}$$

Hence one gets

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{61}$$

We recognize in this expression the time dilation (43).

In the second case, one sets, in (58), $\theta = \pi$ and $\theta = 2\pi$. From (58) one obtains that

$$-c^2t_1^2 + (vt_1 + b)^2 = 0 \tag{62}$$

and

$$-c^2t_2^2 + (-vt_2 + b)^2 = 0, \tag{63}$$

respectively. Considering the total time $t = t_1 + t_2$, these expressions will lead to the length contraction

$$b = \sqrt{1 - \frac{v^2}{c^2}} b_0. \tag{64}$$

Thus, we have shown that from (58) one can obtain the time dilation formula (61) and the length contraction expression (64) as a particular cases. In the infinitesimal case (61) becomes

$$dt = \frac{dt_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{65}$$

While (64) leads to

$$db = \sqrt{1 - \frac{v^2}{c^2}} db_0. \tag{66}$$

So, one may expect that in the most general case the key equations (65) and (66) can be combined in such a way that the general line element becomes

$$ds^2 = -(1 - \frac{v^2}{c^2})c^2 dt^2 + \frac{dr^2}{(1 - \frac{v^2}{c^2})} + r^2 d\Omega^{d-2}, \tag{67}$$

where we identify the radius of the circle b with r, and add the usual spherical element $r^2 d\Omega^{d-2}$. It is important to mention that $d\Omega^{d-2}$ refers to an expansion of terms in 2-form diferentials.

Now, considering the escape velocity

$$v^2 = \frac{2GM}{r}, \tag{68}$$

and substituting it into (67) one finds

$$ds^2 = -(1 - \frac{2GM}{c^2 r})c^2 dt^2 + \frac{dr^2}{(1 - \frac{2GM}{c^2 r})} + r^2 d\Omega^{d-2}. \tag{69}$$

which is of course the Schwarzschild line element for black-holes in higher dimensions. Here, however, one may follow another route. This is because the light path is a circle and therefore one has

$$v = \frac{a_0}{t_0} \tag{70}$$

and

$$c = \frac{b_0}{t_0}. \tag{71}$$

Substituting these equations into (67) one discovers that

$$ds^2 = -(1 - \frac{a_0^2}{b_0^2})c^2 dt^2 + \frac{dr^2}{(1 - \frac{a_0^2}{b_0^2})} + r^2 d\Omega^{d-2}, \tag{72}$$

which one recognizes as the line element of the De Sitter (or anti-De Sitter) metric (41).

CHAPTER 4

COSMOLOGY AND BLACK HOLES (TOWARDS A UNIFIED THEORY)

4.1 $(1+D+d)$-dimensional Cosmological Model

Consider a universe described by the line element

$$ds^2 = -N^2(t)dt^2 + a^2(t)d^D\Omega + b^2(t)d^d\Sigma, \tag{73}$$

Where the $d^D\Omega$ and $d^d\Sigma$ correspond to a D-Dimensional and d-dimensional homogeneous spaces with constant curvature $k_1 = 0, \pm 1$ and $k_2 = 0, \pm 1$ respectively.

Using (73) we find that the Einstein-Hilbert action in $\mathcal{D} = 1+s$ dimensions, with $s = D+d$ can be written as

$$S = -\frac{1}{V_s}\int d^{\mathcal{D}}x\sqrt{-g}(R-2\Lambda), \tag{74}$$

where V_s is an appropiate volume constant. Thus (74) becomes (See App. A)

$$\begin{aligned} S = -\textstyle\int dt N a^D b^d\{&2DN^{-2}a^{-1}\ddot{a} - 2DN^{-3}\dot{N}a^{-1}\dot{a} + D(D-1)N^{-2}a^{-2}\dot{a}^2 \\ &+D(D-1)k_1 a^{-2} + 2dN^{-2}b^{-1}\ddot{b} - 2dN^{-3}\dot{N}b^{-1}\dot{b} + d(d-1)N^{-2}b^{-2}\dot{b}^2 \\ &+d(d-1)k_2 b^{-2} + 2dDN^{-2}a^{-1}\dot{a}b^{-1}\dot{b} - 2\Lambda\}. \end{aligned} \tag{75}$$

Here, we have performed a volume integration over the space-like coordinates. The action (75) can be rewritten as

$$\begin{aligned} S = -\textstyle\int dt\{&[\tfrac{d}{dt}(2DN^{-1}a^{D-1}b^d\dot{a} + 2dN^{-1}a^D b^{d-1}\dot{b}) - D(D-1)N^{-1}a^{D-2}b^d\dot{a}^2 \\ &-d(d-1)N^{-1}a^D b^{d-2}\dot{b}^2 - 2dDN^{-1}a^{D-1}\dot{a}b^{d-1}\dot{b}] \\ &+D(D-1)k_1 N a^{D-2}b^d + d(d-1)k_2 N a^D b^{d-2} - 2\Lambda N a^D b^d\}. \end{aligned} \tag{76}$$

Since a total derivative does not contribute to the dynamics of the classical system we can drop the first term in (76). Thus (76) simplifies to

$$\begin{aligned} S = \textstyle\int dt\{&N^{-1}a^D b^d[D(D-1)a^{-2}\dot{a}^2 + d(d-1)b^{-2}\dot{b}^2 + 2dDa^{-1}\dot{a}b^{-1}\dot{b}] \\ &-D(D-1)k_1 N a^{D-2}b^d - d(d-1)k_2 N a^D b^{d-2} + 2\Lambda N a^D b^d\}. \end{aligned} \tag{77}$$

One can show that the field equations for the cosmology model in $1+D+d$ follows from (77) [56].

4.2 Schwarzschild $D-$ dimensional space-time

Let us start with

$$ds^2 = -e^{f(x^1)}(dx^0)^2 + e^{h(x^1)}(dx^1)^2 + \phi^2(x^1)\tilde{g}_{ij}(\xi^k)d\xi^i d\xi^j, \tag{78}$$

or more specifically

$$ds^2 = -e^{f(r)}(dt)^2 + e^{h(r)}(dr)^2 + \phi^2(r)\tilde{g}_{ij}(\xi^k)d\xi^i d\xi^j, \tag{79}$$

where ds^2 corresponds to the line element for a spherically symmetric static black hole solution in a $D-$ dimensional space-time M^D, and the metric $\tilde{g}_{ij}(\xi^k)$ refers to a $(D-2)$-dimensional maximally spherically symmetric space (homogeneous space) with curvilinear coordinates ξ^i that are independent of time t and r. Also, greek indices $\mu, \nu = 0, 1, 2, ..., D-1$ while latin indices $i, j = 2, 3, 4, ..., D-1$. Here, we have specified the functional form for f, h and ϕ in terms of r (or x^1) and considered the speed of light $c = 1$.

Using (78) it is easy to show that the resulting field equations are (see App. B)

$$\begin{aligned}
R_{00} &= \tfrac{1}{2}f'' + \tfrac{1}{4}f'^2 - \tfrac{1}{4}f'h' + \tfrac{(D-2)}{2}\tfrac{\phi'}{\phi}f' = 0,\\
R_{11} &= -\tfrac{1}{2}f'' - \tfrac{1}{4}f'^2 + \tfrac{1}{4}f'h' + (D-2)\left(\tfrac{1}{2}\tfrac{\phi'}{\phi}h' - \tfrac{\phi''}{\phi}\right) = 0,\\
R_{ij} &= \tfrac{1}{2}\phi\phi'\left(h' - f'\right) - \phi\phi'' - (D-3)\,\phi'^2 + k\,(D-3)\,e^h = 0.
\end{aligned} \tag{80}$$

(in relativistic units $G = c = 1$),where $k = \pm 1$, depending if $\tilde{g}_{ij}$ refers to a positive or negative curvature. Here, we used the notation $A' \equiv \frac{dA}{dr}$ and $A'' \equiv \frac{d^2A}{dr^2}$ for any function $A = A(r)$.

Thus, the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ is given by

$$\begin{aligned}
R &= e^{-h}(D-2)\left[\tfrac{\phi'}{\phi}\left(h' - f'\right) - 2\tfrac{\phi''}{\phi} - (D-3)\tfrac{\phi'^2}{\phi^2}\right] + e^{-h}\left(-f'' - \tfrac{1}{2}f'^2 + \tfrac{1}{2}f'h'\right)\\
&\quad + (D-2)(D-3)\tfrac{k}{\phi^2}
\end{aligned} \tag{81}$$

Consequently, up to total derivative, the higher dimensional Einstein-Hilbert action

$$S = \frac{1}{2} \int_{M^D} \sqrt{-g} R, \tag{82}$$

where g denote the determinant of $g_{\mu\nu}$. That is

$$\sqrt{-g} = e^{\frac{f+h}{2}} \phi^{(D-2)} \sqrt{\tilde{g}}, \tag{83}$$

Thus, considering (81) and (83), (82) becomes

$$\begin{aligned} S = \tfrac{(D-2)}{2} \textstyle\int_{M^D} \sqrt{\tilde{g}} [\Omega^{-1} \{ (\phi^{(D-2)} \mathcal{F}) ((D-3) \tfrac{\phi'^2}{\phi^2} + 2 \tfrac{\mathcal{F}'}{\mathcal{F}} \tfrac{\phi'}{\phi}) \} \\ + \Omega \{ k(D-3) \mathcal{F} \phi^{(D-4)} \}]. \end{aligned} \tag{84}$$

Here, we used the notation $\mathcal{F} \equiv e^{\frac{f}{2}}$ and $\Omega \equiv e^{\frac{h}{2}}$. Note that the case $D = 2$ is exceptional. Similar conclusion can be obtained in the case of $D = 3$. For our purpose it turns out convenient to assume that $D - 2 \neq 0$ and $D - 3 \neq 0$. Observe that in (84) Ω acts as auxiliary field.

4.3 $(n + D + d)$-dimensional Cosmological Model

Now we consider a more general line element

$$ds^2 = g_{AB}(x^C)dx^A dx^B + a^2(x^C)d^D\Omega + b^2(x^C)d^d\Sigma, \tag{85}$$

where the indices A, B run from 1 to n. Here, the expressions $d^D\Omega \equiv \tilde{g}_{ab}(x^d)dx^a dx^b$ and $d^d\Sigma \equiv \hat{g}_{ij}(x^k)dx^i dx^j$ correspond to a D-dimensional and d-dimensional homogenous spatial spaces, with constant curvature $k_1 = 0, \pm 1$ and $k_2 = 0, \pm 1$, respectively.

For the line element (85) we have the Hilbert-Einstein action in $(n+D+d)$-dimensions

$$S = -\frac{1}{V_{D+d}} \int d^{n+D+d}x\sqrt{-g}(R - 2\Lambda), \tag{86}$$

Using (85) the action (86) is reduced to (see App. C)

$$\begin{aligned} S = -\textstyle\int d^n x\sqrt{\bar{g}}a^D b^d\{&-2Da^{-1}\mathcal{D}_A\partial^A a - D(D-1)g^{AB}a^{-2}\partial_A a\partial_B a \\ &-2db^{-1}\mathcal{D}_A\partial^A b - d(d-1)g^{AB}b^{-2}\partial_A b\partial_B b \\ &-2Dd(b^{-1}a^{-1})g^{AB}\partial_A a\partial_B b + \bar{R} + a^{-2}\tilde{R} + b^{-2}\hat{R} - 2\Lambda\}. \end{aligned} \tag{87}$$

which can be rewritten as

$$\begin{aligned} S = -\textstyle\int d^n x\sqrt{\bar{g}}\{&\mathcal{D}_A(2Da^{D-1}\partial^A a b^d + 2db^{d-1}\partial^A b a^D) \\ &-D(D-1)g^{AB}a^{D-2}b^d\partial_A a\partial_B a - d(d-1)g^{AB}a^D b^{d-2}\partial_A b\partial_B b \\ &-2Dd(a^{D-1}b^{d-1})g^{AB}\partial_A a\partial_B b + a^D b^d(\bar{R} + a^{-2}\tilde{R} + b^{-2}\hat{R} - 2\Lambda)\}, \end{aligned} \tag{88}$$

where $\mathcal{D}_A$ is a covariant derivative associated with g_{AB}. Dropping the total derivative in (88) one obtains

$$\begin{aligned} S = \textstyle\int d^n x\sqrt{\bar{g}}a^D b^d\{&D(D-1)g^{AB}a^{-2}\partial_A a\partial_B a + d(d-1)g^{AB}b^{-2}\partial_A b\partial_B b \\ &+2Dd(a^{-1}b^{-1})g^{AB}\partial_A a\partial_B b - (\bar{R} + a^{-2}\tilde{R} + b^{-2}\hat{R} - 2\Lambda)\}. \end{aligned} \tag{89}$$

Here, $\bar{R}, \tilde{R}$ and $\hat{R}$ are the curvature scalars associated with $g_{AB}(x^C), \tilde{g}_{ab}(x^d)$ and $\hat{g}_{ij}(x^k)$, respectively.

Now, it is worth mentioning that (89) is invariant under the duality transformation

$$\begin{aligned} a &\longrightarrow \frac{1}{a}, \\ b &\longrightarrow \frac{1}{b}, \\ g_{AB} &\longrightarrow a^{\frac{4D}{n-2}} b^{\frac{4d}{n-2}} g_{AB}. \end{aligned} \tag{90}$$

provided that $n \neq 2$ and $\bar{R} + a^{-2}\tilde{R} + b^{-2}\hat{R} - 2\Lambda = 0$. What appears interesting from our analysis of the invariance of (89) under the duality transformation (90) is that the case $n = 2$ is distinguished among any other n value. In other words, from duality point of view two time physics turns out to be a singular case. In some sense duality symmetry is playing analogue role in several time cosmological physics as the Weyl invariance in p-brane physics (see [55] and references there in).

4.4 FRW-Cosmology/Black hole reduction

A particular case of the equation (85) is expressed by the line element

$$ds^2 = -N^2(t)dt^2 + a^2(t)d^D\Omega + b^2(t)d^d\Sigma, \tag{91}$$

Which as you noticed it was already analized in Section 4.1 and the obtained Einstein-Hilbert action was

$$\begin{aligned} S = \int dt\{&N^{-1}a^D b^d[D(D-1)a^{-2}\dot{a}^2 + d(d-1)b^{-2}\dot{b}^2 + 2dDa^{-1}\dot{a}b^{-1}\dot{b}] \\ &-D(D-1)k_1 N a^{D-2}b^d - d(d-1)k_2 N a^D b^{d-2} + 2\Lambda N a^D b^d\}. \end{aligned} \tag{92}$$

The problem with (91) is that it can be obtained from (85) by choosing $g_{11}(x^C) = -N^2$ and $g_{12}(x^C) = g_{21}(x^C) = 0$ and $g_{22}(x^C) = 0$. This means that the 2-dimensional metric $g_{AB}(x^C)$ in (85) is singular. So, the question arises, starting from the action (89) and assuming $detg_{AB}(x^C) \neq 0$ how can one obtains the action (92)? The answer to this question may be solved by performing a different kind of projection. Let us rewrite the ansatz (85) in the form

$$ds^2 = g_{AB}(x^1, x^2)dx^A dx^B + a^2(x^1, x^2)d^D\Omega + b^2(x^1, x^2)d^d\Sigma, \tag{93}$$

Let us assume that $g_{12}(x^1, x^2) = g_{21}(x^1, x^2) = 0$. This leads to

$$ds^2 = g_{11}(x^1, x^2)dx^1 dx^1 + g_{22}(x^1, x^2)dx^2 dx^2 + a^2(x^1, x^2)d^D\Omega + b^2(x^1, x^2)d^d\Sigma. \tag{94}$$

By performing the projections $g_{11}(x^1, x^2) \longrightarrow g_{11}(x^1) = -N^2(x^1)$ and $g_{22}(x^1, x^2) = 0$, $a^2(x^1, x^2) \longrightarrow a^2(x^1)$ and also $b^2(x^1, x^2) \longrightarrow b^2(x^1)$ one see that the line element (94) becomes

$$ds^2 = -N^2(x^1)dx^1 dx^1 + a^2(x^1)d^{D+1}\Omega + b^2(x^1)d^d\Sigma. \tag{95}$$

Here, we have defined $d^{D+1}\Omega = dx^2dx^2 + d^D\Omega$ and it refers to an expansion of terms in 2-form diferentials.Therefore, one has discovered that (95) has exactly the same form as (91) and therefore the action (92) follows, provided one makes the extension

$$g_{ab} \to \begin{pmatrix} 1 & 0 \\ 0 & g_{ab} \end{pmatrix}. \tag{96}$$

For the black hole case by making the reductions $g_{11}(x^1, x^2) \longrightarrow g_{11}(x^2) = -e^{f(x^2)}$ and $g_{22}(x^1, x^2) \longrightarrow g_{22}(x^2) = e^{h(x^2)}$, $a^2(x^1, x^2) \longrightarrow \varphi^2(x^2)$ and $b^2(x^1, x^2) = b^2(x^2)$, with $x^2 = r$, the line element (94) becomes

$$ds^2 = -e^{f(r)}(dx^1)^2 + e^{h(r)}dr^2 + \varphi^2(r)d^D\Omega + b^2(r)d^d\Sigma. \tag{97}$$

with its corresponding action

$$\begin{aligned} S = \tfrac{(D-2)}{2} \int_{M^D} \sqrt{\bar{g}}[\Omega^{-1}\{(\phi^{(D-2)}\mathcal{F})((D-3)\tfrac{\phi'^2}{\phi^2} + 2\tfrac{\mathcal{F}'}{\mathcal{F}}\tfrac{\phi'}{\phi})\} \\ +\Omega\{k(D-3)\mathcal{F}\phi^{(D-4)}\}]. \end{aligned} \tag{98}$$

already obtained in Section 4.2. Here, by convenience we choose $d = 0$. From this action one can obtain the field equations whose solution corresponds to the the well known higher dimensional Schwarchild black-hole solution, namely

$$ds^2 = -(1 - \frac{k}{\varphi^{D/2}(r)})(dx^1)^2 + \frac{dr^2}{(1 - \frac{k}{\varphi^{D/2}(r)})} + \varphi^2 d^D\Omega. \tag{99}$$

In this section we showed that the Einstein-Hilbert action in $n + D + d$-dimensions can be reduced to the action (89) which contains the dynamics of both cosmology and black holes.

CHAPTER 5

FINAL REMARKS

dimensional qubits (see Refs. [16]-[17] for details).

It is also known that the 1-qubit state is related to the Hopf map as follows (see Refs. [16]-[19] and references therein):

$$S^3 \xrightarrow{S^1} S^2. \tag{102}$$

So, the 1-sphere becomes in this case the fiber of such a map. This means that one may be able to link the relativity theory with (102) and therefore

It is known that some of the most interesting generalizations of spheres are the squashed spheres in supergravity and pseudo spheres in oriented matroid theory. The idea of squashed spheres arises in attempt to find a realistic Kaluza- Klein theory. In fact the typical spontaneous compactification in eleven dimensional supergravity is given by $M^4 \times S^7$. But S^7 is isomorphic to $SO(8)/SO(7)$ which implies that instead of the group $U(1) \times SU(2) \times SU(3)$ of the standard model, the transition group is $SO(8)$. Thus, one uses the concept of spontaneous symmetry braking in order to make the transition $SO(8) \rightarrow U(1) \times SU(2) \times SU(3)$. One possibility to achieve this goal is to assume that the symmetry braking induces the transition $S^7 \longrightarrow \mathcal{S}^7$, where $\mathcal{S}^7$ is the squashed seven sphere [22]. The simplest example of this process is provided by the squashed S^3 sphere [31]. In this case the original metric

$$ds^2 = \sigma_1 + \sigma_2 + \sigma_3, \tag{100}$$

with invariant group $SO(4)$ is broken to the form

$$ds^2 = \sigma_1 + \sigma_2 + \lambda\sigma_3, \tag{101}$$

where $\lambda \neq 1$ and σ_1, σ_2 and σ_3 have quadratic form. When $\lambda = 1$ (101) is reduced to the line element of S^3, that is to (100). The idea is to get the reduction $SO(4) \rightarrow U(1) \times SU(2)$ by the process of symmetry braking.

There are important topological aspects related with the present approach of higher dimensional spheres. Mathematically, it may be interesting to link our work with the Bott periodicity theorem (see Ref. [10] and references therein). Moreover, we would like also to describe an application of Division-algebra/Poincaré-conjecture correspondence in qubits theory. It has been mentioned in Ref. [16], and proved in Refs. [17] and [18], that for normalized qubits the complex 1-qubit, 2-qubit and 3-qubit are deeply related to division algebras via the Hopf maps, $S^3 \xrightarrow{S^1} S^2$, $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$, respectively. It seems that there is not a Hopf map for higher N-qubit states. Therefore, from the perspective of Hopf maps, and therefore of division algebras, one arrives to the conclusion that 1-qubit, 2-qubit and 3-qubit are more special than higher dimensional qubits (see Refs. [16]-[17] for details).

It is also known that the 1-qubit state is related to the Hopf map as follows (see Refs. [16]-[19] and references therein):

$$S^3 \xrightarrow{S^1} S^2. \tag{102}$$

So, the 1-sphere becomes in this case the fiber of such a map. This means that one may be able to link the relativity theory with (102) and therefore

RT 140470

with 1-qubits. This kind of program may eventually be important to study light paths in different topological contexts (see Refs. [39]-[41] and references therein). In the general case, one has the Hopf maps $S^3 \xrightarrow{S^1} S^2$, $S^7 \xrightarrow{S^3} S^4$ and $S^{15} \xrightarrow{S^7} S^8$ [4].

Now, considering the 2-qubit as a guide one notices that S^3 plays the role of fiber in the map $S^7 \xrightarrow{S^3} S^4$. Thus, in principle one may think in a more general map $\mathcal{M}^7 \xrightarrow{\mathcal{M}^3} \mathcal{M}^4$ leading to a more general 2-qubit system which one may call 2-Poinqubit (just to remember that this is a concept inspired by Poincaré conjecture.) At the end one may be able to obtain the transition 2-Poinqubit$\longrightarrow$ 2-qubit. Of course one may extend most of the arguments developed in this work to the other Hopf maps $S^3 \xrightarrow{S^1} S^2$ and $S^{15} \xrightarrow{S^7} S^8$.

It is interesting to mention that recently a correspondence between the division algebras, Hopf maps and the Poincaré conjecture has been established [15]. Let us briefly mention how such a correspondence arises. Geometric parallelizability of S^d means the "flattening" the space in the sense that

$$\mathcal{R}^i_{jkl}(\Omega^m_{rs}) = 0, \tag{103}$$

where

$$\mathcal{R}^i_{jkl} = \partial_k \Omega^i_{jl} - \partial_l \Omega^i_{jk} + \Omega^i_{mk}\Omega^m_{jl} - \Omega^i_{ml}\Omega^m_{jk} \tag{104}$$

is the Riemann curvature tensor, with

$$\Omega^i_{kl} = \Gamma^i_{kl} + T^i_{kl}. \tag{105}$$

Here T^i_{kl} denotes the torsion tensor.

It can be proved that, for homogeneous space, from the condition (103) can be obtained the first and the second Cartan-Shouten equations (see Ref. [30])

$$T^{kl}_i T_{jkl} = (d-1) r_0^{-2} g_{ij}, \tag{106}$$

and

$$T^l_{ei} T^f_{lj} T^e_{fk} = (d-4) r_0^{-2} T_{ijk}, \tag{107}$$

respectively. It turns out that (106) and (107) can be used eventually to prove that the only parallelizable spheres are S^1, S^3 and S^7 [5].

Now, we would like to generalize the key constraint (3) in the form

$$x^{d+1} = \varphi(x^i), \tag{108}$$

where φ is an arbitrary function of the coordinates x^i. In this case, the metric γ_{ij} becomes

$$\gamma_{ij} = \delta_{ij} + \partial_i\varphi\partial_j\varphi, \tag{109}$$

while the inverse γ^{ij} is given by

$$\gamma^{ij} = \delta^{ij} - \frac{\partial^i\varphi\partial^j\varphi}{1 + \partial^k\varphi\partial_k\varphi}. \tag{110}$$

The Christoffel symbols become

$$\Gamma^i_{kl} = \frac{\partial^i\varphi\partial_{kl}\varphi}{1 + \partial^m\varphi\partial_m\varphi}. \tag{111}$$

After lengthy but straightforward computation one discovers that the Riemann tensor R_{ijkl} obtained form (111) is

$$R_{ijkl} = \frac{1}{1 + \partial^m\varphi\partial_m\varphi}(\partial_{ik}\varphi\partial_{jl}\varphi - \partial_{il}\varphi\partial_{jk}\varphi). \tag{112}$$

One can verifies that in the particular case

$$\varphi = (r_0^2 - x^i x^j \delta_{ij})^{1/2}, \tag{113}$$

(19) follows from (112).

Let us now consider the Ricci flow evolution equation [14] (see also Refs. [11]-[13] and references therein)

$$\frac{\partial\gamma_{ij}}{\partial t} = -2R_{ij}. \tag{114}$$

In this case the metric $\gamma_{ij}(t)$ is understood as a family of Riemann metrics on M^3. It has been emphasized that the Ricci flow equation is the analogue of the heat equation for metrics γ_{ij}. The central idea is that a metric γ_{ij} associated with a closed simply connected manifold $\mathcal{M}^3$ evolves according to (114) towards a metric g_{ij} of S^3. Symbolically, this means that in virtue of (114) we have the metric evolution $\gamma_{ij} \longrightarrow g_{ij}$, which in turn must imply the homeomorphism $\mathcal{M}^3 \longrightarrow S^3$.

The question arises whether one can introduce the parallelizability concept into (114). Let us assume that $\mathcal{M}^3$ is a parallelizable manifold. We shall also assume that $\mathcal{M}^3$ is determined by the general constraint (108). It has been proved that (114) can be generalized to

$$\frac{\partial \gamma_{ij}}{\partial t} = -2T_i^{kl}T_{jkl}. \tag{115}$$

It is worth mentioning that by considering the transition $\mathcal{M}^3 \to S^3$ one finds that even in special relativity one may have that the evolution process $\varphi(v_x, v_y, v_z) \to \sqrt{c^2 - (v_x^2 + v_y^2 + v_z^2)}$ may be understood as the transition $\mathcal{C} \to c$ of the light velocity (see Ref. [15] for details). This means that the light velocity c is not a given constant but it is obtained as a result of evolution transition. It is also worth mentioning that the cycloid idea has been used in different context to develop an alternative hypothesis of special relativity [43].

It is also known that S^1 plays an important role in oriented matroid theory [32]-[33] (see Refs. [34]-[35] and references therein) it may be interesting for further research to explore the full connection of the present work and oriented matroid theory.

Let us now discuss some physical scenarios where the division-algebra/Poincaré-conjecture correspondence may be relevant. Let us start by recalling the Einstein field equations with cosmological constant Λ,

$$R_{ij} - \frac{1}{2}\gamma_{ij}R + \Lambda\gamma_{ij} = 0. \tag{116}$$

It is known that the lowest energy solution of (116) corresponds precisely to S^3 (or to S^d in general). In this case the cosmological constant Λ is given by $\Lambda = \frac{2}{r_0^2}$. From quantum mechanics perspective One may visualize $\mathcal{M}^3$ as an excited state which must decay (homeomorphically) to S^3, according to the Poincaré conjecture. Symbolically, one may write this as $\mathcal{M}^3 \to S^3$.

In Ref. [15] it is observed that the transition $\mathcal{M}^3 \to S^3$ may be applied in two important scenarios: special relativity and cosmology. In the first case the evolution process $\varphi(v_x, v_y, v_z) \to \sqrt{c^2 - (v_x^2 + v_y^2 + v_z^2)}$ may be understood as the transition $\mathcal{C} \to c$ of the light velocity (see Ref. [15] for details). While in the second case the standard Friedmann-Robertson-Walker universe corresponds to a time evolving radius of a S^3 space that can be modified in $\mathcal{M}^3$. Thus, at the end the acceleration may produce a phase transition changing $\mathcal{M}^3$ to a space of constant curvature which corresponds precisely to the de Sitter phase associated with S^3.

Moreover, in Ref. [15] it was proposed the complex generalization

$$i\frac{\partial \psi_{ij}}{\partial t} = -2\mathcal{R}_{ij}, \tag{117}$$

of (114). Here, the metric γ_{ij} and the Ricci tensor R_{ij} are may be complexified, $\gamma_{ij} \rightarrow \psi_{ij}$ and $R_{ij} \rightarrow \mathcal{R}_{ij}$, respectively. The idea is now to consider the evolving complex metric ψ_{ij}.

Eventually, one may be interested in a possible connection of the Poincaré conjecture with oriented matroid theory [32] (see also Refs. [33]-[38] and references therein). This is because for any sphere S^d one may associate a polyhedron which under stereographic projection corresponds to a graph in R^{d+1}. It turns out that matroid theory can be understood as a generalization of graph theory and therefore it may be interesting to see whether there is any connection between oriented matroid theory and Poincaré conjecture. In fact in oriented matroid theory there exist the concept of pseudo-spheres which generalizes the ordinary concept of spheres (see Ref. [32] for details). So one wonders if there exist the analogue of Poincaré conjecture for pseudo-spheres.

Finally in this work we have studied duality symmetries in Kaluza-Klein $n+D+d$ dimensional cosmological models. We first briefly reviewed the case $1+D+d$ cosmological model. We wrote the action of this model in such a way that the duality symmetry becomes manifest. In section 4.3, we studied, from the point of view of duality, the more general case of a $n+D+d$ cosmological model. We discovered that, except for the case $n=2$, the Einstein-Hilbert action in $n+D+d$ dimensions is invariant under the duality symmetry $a \rightarrow \frac{1}{a}$ and $b \rightarrow \frac{1}{b}$. We studied the $2+D+d$ cosmological model in some detail finding an explicit classical solution. One of the interesting features of the $2+D+d$ cosmological model is that, in spite of lacking a duality symmetry, it leads to a universe in which the second time can be considered as a compact time-like dimension, while the first usual time behaves as an open dimension. It turns out that this kind of solution was already anticipated by Bars and Kounnas [49].

It is clear from the present results that the traditional Friedmann-Robertson-Walker cosmological model is contained in the $2+D+d$ cosmological model. The question arises whether other traditional cosmological models such as the different Bianchi models are also contained in the $2+D+d$ model. The really interesting problem, however, is to find a mechanism to decide whether the $2+D+d$ model is the correct model of the universe. Experimentally, it is an interesting possibility because presumably the second time is shrinking to zero in the first stage of the evolution of the universe, leading after that to the usual evolving universe. Theoretically, one becomes intriguing why duality is broken in the case of two times cosmological model, distinguishing the $2+D+d$ model of other $n+D+d$ models. In this work we tried to understand classically this interesting feature of the $2+D+d$ cosmological model but beyond of finding a consistent solution with the present evolution of our universe there seems not to be a clear reason why the duality symmetry is broken.

An open problem for further research is to quantize the $n+D+d$ cosmological model. In this case it may be interesting to see what are the consequences of the duality symmetry in the corresponding Wheeler-de Witt equation and in the associated state of the universe.

APPENDIX A

Considering the $1+D+d$ dimensional metric $g_{\alpha\beta}$, with $\alpha, \beta = 0, 1..., 1+D+d$, the only nonvanishing elements are

$$\begin{aligned} g_{00} &= -N^2, \\ g_{ij} &= a^2(t)\tilde{g}_{ij}, \\ g_{ab} &= b^2(t)\hat{g}_{ab}, \end{aligned} \tag{A1}$$

where the metric $\tilde{g}_{ij}$ corresponds to the D-dimensional homogenous space, while $\hat{g}_{ab}$ is metric of the d-dimensional homogeneous space.

We find that the only non-vanishing Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \tag{A2}$$

are

$$\begin{aligned} \Gamma^{0}_{ij} &= N^{-2}a\dot{a}\tilde{g}_{ij}, \\ \Gamma^{i}_{j0} &= a^{-1}\dot{a}\delta^{i}_{j}, \\ \Gamma^{0}_{00} &= N^{-1}\dot{N}, \\ \Gamma^{i}_{jk} &= \tilde{\Gamma}^{i}_{jk}, \\ \Gamma^{0}_{ab} &= N^{-2}b\dot{b}\hat{g}_{ab}, \\ \Gamma^{a}_{b0} &= b^{-1}\dot{b}\delta^{a}_{b}, \\ \Gamma^{a}_{bc} &= \hat{\Gamma}^{a}_{bc}. \end{aligned} \tag{A3}$$

Using (A3) we discover that the only non-vanishing components of the Riemann tensor

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \tag{A4}$$

are

$$
\begin{aligned}
R^0_{i0j} &= (N^{-2}a\ddot{a} - N^{-3}\dot{N}a\dot{a})\tilde{g}_{ij},\\
R^i_{0j0} &= (-a^{-1}\ddot{a} + N^{-1}\dot{N}a^{-1}\dot{a})\delta^i_j,\\
R^i_{jkl} &= \tilde{R}^i_{jkl} + N^{-2}\dot{a}^2(\delta^i_k\tilde{g}_{jl} - \delta^i_l\tilde{g}_{jk}),\\
R^0_{a0b} &= (N^{-2}b\ddot{b} - N^{-3}\dot{N}b\dot{b})\hat{g}_{ab},\\
R^a_{0b0} &= (-b^{-1}\ddot{b} + N^{-1}\dot{N}b^{-1}\dot{b})\delta^a_b,\\
R^a_{bcd} &= \hat{R}^a_{bcd} + N^{-2}\dot{b}^2(\delta^a_c\hat{g}_{bd} - \delta^a_d\hat{g}_{bc}),\\
R^i_{ajb} &= N^{-2}a^{-1}\dot{a}b\dot{b}\delta^i_j\hat{g}_{ab},\\
R^a_{ibj} &= N^{-2}a\dot{a}b^{-1}\dot{b}\delta^a_b\tilde{g}_{ij}.
\end{aligned}
\tag{A5}
$$

From (A5) we get the non-vanishing components of the Ricci tensor $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$;

$$
\begin{aligned}
R_{00} &= -Da^{-1}\ddot{a} + DN^{-1}\dot{N}a^{-1}\dot{a} - db^{-1}\ddot{b} + dN^{-1}\dot{N}b^{-1}\dot{b},\\
R_{ij} &= (N^{-2}a\ddot{a} - N^{-3}\dot{N}a\dot{a} + (D-1)N^{-2}\dot{a}^2 + dN^{-2}a\dot{a}b^{-1}\dot{b})\tilde{g}_{ij} + \tilde{R}_{ij},\\
R_{ab} &= (N^{-2}b\ddot{b} - N^{-3}\dot{N}b\dot{b} + (d-1)N^{-2}\dot{b}^2 + DN^{-2}a^{-1}\dot{a}b\dot{b})\hat{g}_{ab} + \hat{R}_{ab}.
\end{aligned}
\tag{A6}
$$

Thus, the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ is given by

$$
\begin{aligned}
R &= 2DN^{-2}a^{-1}\ddot{a} - 2DN^{-3}\dot{N}a^{-1}\dot{a} + D(D-1)N^{-2}a^{-2}\dot{a}^2 + D(D-1)k_1a^{-2}\\
&\quad +2dN^{-2}b^{-1}\ddot{b} - 2dN^{-3}\dot{N}b^{-1}\dot{b} + d(d-1)N^{-2}b^{-2}\dot{b}^2 + d(d-1)k_2b^{-2}\\
&\quad +2dDN^{-2}a^{-1}\dot{a}b^{-1}\dot{b}.
\end{aligned}
\tag{A7}
$$

APPENDIX B

For a $D-$ dimensional static black hole the nonvanishing elementos for the metric $g_{\mu\nu}$ are

$$\begin{aligned} g_{00} &= -e^{f(r)}, \\ g_{11} &= e^{h(r)}, \\ g_{ij} &= \phi^2(r)\tilde{g}_{ij}\left(\xi^k\right), \end{aligned} \tag{B1}$$

where the metric $\tilde{g}_{ij}\left(\xi^k\right)$ determines a $(D-2)$-dimensional maximally spherically symmetric space and $i, j = 2, 3, ..., D-1$.

The non-vanishing Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \tag{B2}$$

are

$$\begin{aligned} \Gamma^0_{01} &= \tfrac{1}{2}f', \\ \Gamma^1_{00} &= \tfrac{1}{2}f'e^{f-h}, \\ \Gamma^1_{11} &= \tfrac{1}{2}h', \\ \Gamma^1_{ij} &= -e^{-h}\phi\phi'\tilde{g}_{ij}, \\ \Gamma^i_{1j} &= \tfrac{\phi'}{\phi}\delta^i_j, \\ \Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk}. \end{aligned} \tag{B3}$$

Using (B3) we discover that the only non-vanishing components of the Riemann tensor

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \tag{B4}$$

are

$$\begin{aligned}
R^0_{101} &= -\tfrac{1}{2}f'' - \tfrac{1}{4}f'^2 + \tfrac{1}{4}f'h', \\
R^1_{010} &= e^{f-h}\left(\tfrac{1}{2}f'' + \tfrac{1}{4}f'^2 - \tfrac{1}{4}f'h'\right), \\
R^0_{i0j} &= -\tfrac{1}{2}f'e^{-h}\phi\phi'\tilde{g}_{ij}, \\
R^i_{0j0} &= \tfrac{1}{2}f'e^{f-h}\tfrac{\phi'}{\phi}\delta^i_j, \\
R^1_{i1j} &= e^{-h}\left(\tfrac{1}{2}h'\phi\phi' - \phi\phi''\right)\tilde{g}_{ij}, \\
R^i_{1j1} &= \left(\tfrac{1}{2}h'\tfrac{\phi'}{\phi} - \tfrac{\phi''}{\phi}\right)\delta^i_j, \\
R^i_{jkl} &= \tilde{R}^i_{jkl} - \phi'^2 e^{-h}(\delta^i_k\tilde{g}_{jl} - \delta^i_l\tilde{g}_{jk}).
\end{aligned} \tag{B5}$$

From (B5) we get the non-vanishing components of the Ricci tensor $R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}$;

$$\begin{aligned}
R_{00} &= e^{f-h}\left(\tfrac{1}{2}f'' + \tfrac{1}{4}f'^2 - \tfrac{1}{4}f'h' + \tfrac{(D-2)}{2}\tfrac{\phi'}{\phi}f'\right), \\
R_{11} &= -\tfrac{1}{2}f'' - \tfrac{1}{4}f'^2 + \tfrac{1}{4}f'h' + (D-2)\left(\tfrac{1}{2}\tfrac{\phi'}{\phi}h' - \tfrac{\phi''}{\phi}\right), \\
R_{ij} &= e^{-h}\left[\tfrac{1}{2}\phi\phi'(h'-f') - \phi\phi'' - (D-3)\phi'^2 + k(D-3)e^h\right]\tilde{g}_{ij}.
\end{aligned} \tag{B6}$$

Considering

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{B7}$$

and if $T_{\mu\nu} = 0$ for the particular "no matter" scenario,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{B8}$$

Multiplying (B8) by $g^{\mu\nu}$ we obtain

$$R = 0. \tag{B9}$$

Thus

$$R_{\mu\nu} = 0, \tag{B10}$$

and the field equations

$$\begin{aligned}
R_{00} &= \tfrac{1}{2}f'' + \tfrac{1}{4}f'^2 - \tfrac{1}{4}f'h' + \tfrac{(D-2)}{2}\tfrac{\phi'}{\phi}f' = 0, \\
R_{11} &= -\tfrac{1}{2}f'' - \tfrac{1}{4}f'^2 + \tfrac{1}{4}f'h' + (D-2)\left(\tfrac{1}{2}\tfrac{\phi'}{\phi}h' - \tfrac{\phi''}{\phi}\right) = 0, \\
R_{ij} &= \tfrac{1}{2}\phi\phi'\left(h' - f'\right) - \phi\phi'' - (D-3)\,\phi'^2 + k\,(D-3)\,e^h = 0.
\end{aligned} \tag{B11}$$

Finally, from (B11) it is easy to prove the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ is given by

$$\begin{aligned}
R &= e^{-h}(D-2)\left[\tfrac{\phi'}{\phi}\left(h'-f'\right) - 2\tfrac{\phi''}{\phi} - (D-3)\tfrac{\phi'^2}{\phi^2}\right] + e^{-h}\left(-f'' - \tfrac{1}{2}f'^2 + \tfrac{1}{2}f'h'\right) \\
&\quad + (D-2)(D-3)\tfrac{k}{\phi^2}
\end{aligned} \tag{B12}$$

APPENDIX C

Considering the $n+D+d$ dimensional metric $g_{\alpha\beta}$, with $\alpha, \beta = 0, 1..., n, n+1, ..., n+D+d$, the nonvanishing elements for this case are

$$\begin{aligned} g_{AB} &= \bar{g}_{AB}(x^C), \\ g_{ij} &= a^2(x^C)\tilde{g}_{ij}, \\ g_{ab} &= b^2(x^C)\hat{g}_{ab}, \end{aligned} \tag{C1}$$

where the metric $\tilde{g}_{ij}$ corresponds to the D-dimensional homogenous space, while $\hat{g}_{ab}$ is metric of the d-dimensional homogeneous space. Furthermore, the indices $A, B...etc$ run from 1 to n, the indices $i, j...etc$ run from $n+1$ to $n+D$ and the indices $a, b...etc$ run from $n+D+1$ to $n+D+d$.

We find that the only non-vanishing Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}) \tag{C2}$$

are

$$\begin{aligned} \Gamma^{A}_{ij} &= -g^{AB}a\partial_B a\tilde{g}_{ij}, \\ \Gamma^{i}_{jA} &= a^{-1}\partial_A a\delta^{i}_{j}, \\ \Gamma^{A}_{BC} &= \bar{\Gamma}^{A}_{BC}, \\ \Gamma^{i}_{jk} &= \tilde{\Gamma}^{i}_{jk}, \\ \Gamma^{A}_{ab} &= -g^{AB}b\partial_B b\hat{g}_{ab}, \\ \Gamma^{a}_{bA} &= b^{-1}\partial_A b\delta^{a}_{b}, \\ \Gamma^{a}_{bc} &= \hat{\Gamma}^{a}_{bc}. \end{aligned} \tag{C3}$$

Using (C3) we discover that the only non-vanishing components of the Riemann tensor

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\nu\alpha} \tag{C4}$$

are

$$
\begin{aligned}
R^A_{BCD} &= \bar{R}^A_{BCD} \\
R^A_{iBj} &= (-aD_B\partial^A a)\tilde{g}_{ij}, \\
R^i_{AjB} &= (-a^{-1}D_B\partial_A a)\delta^i_j, \\
R^i_{jkl} &= \tilde{R}^i_{jkl} - g^{AB}\partial_A a\partial_B a(\delta^i_k\tilde{g}_{jl} - \delta^i_l\tilde{g}_{jk}), \\
R^A_{aBb} &= (-bD_B\partial^A b)\hat{g}_{ab}, \\
R^a_{AbB} &= (-b^{-1}D_B\partial_A b)\delta^a_b, \\
R^a_{bcd} &= \hat{R}^a_{bcd} - g^{AB}\partial_A b\partial_B b(\delta^a_c\hat{g}_{bd} - \delta^a_d\hat{g}_{bc}), \\
R^i_{ajb} &= -(a^{-1}b)g^{AB}\partial_A a\partial_B b\delta^i_j\hat{g}_{ab}, \\
R^a_{ibj} &= -(b^{-1}a)g^{AB}\partial_A a\partial_B b\delta^a_b\tilde{g}_{ij}.
\end{aligned}
\tag{C5}
$$

From (C5) we get the non-vanishing components of the Ricci tensor $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$;

$$
\begin{aligned}
R_{AB} &= \bar{R}_{AB} - Da^{-1}D_B\partial_A a - db^{-1}D_B\partial_A b, \\
R_{ij} &= -(aD_A\partial^A a + (D-1)g^{AB}\partial_A a\partial_B a + d(b^{-1}a)g^{AB}\partial_A a\partial_B b)\tilde{g}_{ij} + \tilde{R}_{ij}, \\
R_{ab} &= -(bD_A\partial^A b + (d-1)g^{AB}\partial_A b\partial_B b + D(a^{-1}b)g^{AB}\partial_A a\partial_B b)\hat{g}_{ab} + \hat{R}_{ab}.
\end{aligned}
\tag{C6}
$$

Thus, the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ is given by

$$
\begin{aligned}
R &= -2Da^{-1}D_A\partial^A a - D(D-1)g^{AB}a^{-2}\partial_A a\partial_B a \\
&\quad -2db^{-1}D_A\partial^A b - d(d-1)g^{AB}b^{-2}\partial_A b\partial_B b \\
&\quad -2Dd(b^{-1}a^{-1})g^{AB}\partial_A a\partial_B b + \bar{R} + a^{-2}\tilde{R} + b^{-2}\hat{R}.
\end{aligned}
\tag{C7}
$$

References

[1] R. Bott and J. Milnor, Bull. Amer. Math. Soc. **64**, 87 (1958).

[2] M. Kerbvair, Proc. Nat. Acad. Sci. **44**, 286 (1958).

[3] J. Milnor, Ann. of Math. (2) **68,** 444 (1958).

[4] J. F. Adams, Ann. of Math. **75**, 603 (1962).

[5] J. A. Nieto and L. N. Alejo-Armenta, Int. J. Mod. Phys. A **16**, 4207 (2001); hep-th/0005184.

[6] M. F. Atiyah, *K-theory* (Benjamin, Inc. New York, Amsterdam, 1967).

[7] C. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic Press, 1983).

[8] I. L. Kantor and A. S. Solodovnikov, *Hypercomplex Numbers - an Elementary Introduction to Algebras* (Springer, Berlin, 1989).

[9] F. Hirzebruch, *Numbers*, Ed. J. H. Ewing, (Spring-Verlag New York Inc., 1991).

[10] J. Baez , Bull. Amer. Math. Soc. **39**, 145 (2002).

[11] B. Kleiner and J. Lott, Notes on Perelman's Papers, arXiv: math. DG/0605667.

[12] J. W. Morgan and G. Tian, "Ricci Flow and the Poincare Conjecture", arXiv: math.DG/0607607.

[13] Huai-Dong Cao, Xi-Ping Zhu, Asian J. Math **10,** 165 (2006).

[14] R. Hamilton, J. Diff. Geom. **17**, 255 (1982).

[15] J. A. Nieto, J. Mod. Phys. **4**, 32 (2013); ArXiv:1303.2673 [physics.gen-ph].

[16] R. Mosseri and R. Dandoloff, J. Phys. A: Math. Gen. **34**, 10243 (2001).

[17] R. Mosseri, "Two and Three Qubits Geometry and Hopf Fibrations"; arXiv:quant-ph/0310053.

[18] B. A. Bernevig and H. D. Chen, J. Phys. A **36**, 8325 (2003).

[19] L. Borsten, M. J. Duff and P. Levay, Class. Quant. Grav. **29** (2012) 224008.

[20] M. Ryan and L. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, NJ, 1975).

[21] M. Green, J. Schwarz, E. Witten, *Superstring Theory* (Cambridge U. Press, Cambridge,UK, 1987).

[22] M. J. Duff, I. G. Koh and B. E. W. Nilsson, Phys. Lett. B **148**, 431 (1984).

[23] T. Kugo and P. Townsend, Nucl. Phys. B **221**, 357 (1983).

[24] J. M. Evans, Nuclear Physics B **298,** 92 (1988).

[25] M. D. Maia. Int. J. Mod. Phys. A **26,** 3813 (2011); Int. J. Mod. Phys. Conf. Ser. **03**, 195 (2011); e-Print: arXiv:1107.2910 [gr-qc].

[26] Donal O'Shea, *The Poincare Conjecture: In Search of the Shape of the Universe* (Walker & Company, New York, 2007).

[27] I. Bakas, D. Orlando and P. M. Petropoulos, JHEP **0701,** 040 (2007); e-Print: hep-th/0610281.

[28] J. Isenberg and M. Jackson, J. Differential Geometry **35,** 723 (1992).

[29] F. Bourliot, J. Estes, P. M. Petropoulos and P. Spindel, Phys. Rev. D **81,** 104001 (2010); e-Print: arXiv:0906.4558 [hep-th].

[30] R. Dundarer, F. Gursey and C. H. Tze, J. Math. Phys. **25**, 1496 (1984).

[31] M. J. Duff, H. Lu and C. N. Pope, Nucl. Phys. B **544**, 145 (1999).

[32] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented Matroids*, (Cambridge University Press, Cambridge, 1999).

[33] J. G. Oxley, *Matroid Theory* (Oxford University Press, New York, 2006).

[34] J. A. Nieto, Adv. Theor. Math. Phys. **8**, 177 (2004); hep-th/0310071.

[35] J. A. Nieto, Adv. Theor. Math. Phys. **10**, 747 (2006), hep-th/0506106.

[36] H. Whitney, Trans. Am. Math. Soc. **34**, 339 (1932).

[37] H. Whitney, Am. J. Math. **57**, 509 (1935).

[38] J. P. S. Kung, *A source book on matroid theory* (Birkhauser Boston, Inc., 1986).

[39] A. F. Rañada, "A topological theory of the electromagnetic field", Lett. Math. Phys. 18, 97-106 (1989).

[40] J. Leach et al., "Knotted threads of darkness", Nature **432**, 165 (2004).

[41] J. Swearngin, A. Thompson, A. Wickes, J. W. Dalhuisen, and D. Bouwmeester, "Gravitational Hopfions", e-Print: arXiv:1302.1431[gr-qc].

[42] G. Avila, S. J. Castillo, J. A. Nieto, "Geometric Structure of Higher-Dimensional Spheres", e-Print: arXiv:1311.2598 [physics.gen-ph].

[43] H. Eckardt, Prog. Phys. **2**, 56 (2009).

[44] M. Gasperini; *Elementary Introduction to Pre-Big-Bang Cosmology and to the Relic Graviton Background,* proceedings of SIGRAV Graduate School in Contemporary Relativity and Gravitational Physics, Villa Olmo, Como, Italy, 19-24 Apr (1999); hep-th/9907067.

[45] M. J. Duff, Int. J. Mod. Phys. **A11**, 5623 (1996); hep-th/9608117.

[46] T. Banks, *Tasi Lectures on Matrix Theory,* Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 99): Strings, Branes, and Gravity, Boulder, Colorado, 30 May - 25 Jun (1999); hep-th/9911068.

[47] E. Martinec, *Geometrical Structure of M Theory*; hep-th/9608017.

[48] E. Martinec,*Matrix Theory and N=(2,1) Strings*; hep-th/9706194.

[49] I. Bars and C. Kounnas, Phys. Lett. **B402**, 25 (1997); hep-th/9703060.

[50] I. Bars, *Two Time Physics with Gravitational and Gauge Field Backgrounds*; hep-th/0002140.

[51] C. M. Hull, JHEP **9811**, 017 (1998); hep-th/9807127.

[52] M. P. Blencowe and M. J. Duff, Nucl. Phys. **B310**, 387 (1988).

[53] V. I. Tkach, J. Socorro, J. J. Rosales and J. A. Nieto, Phys. Rev. **D60**, 067503 (1999); hep-th/9807129.

[54] R. A. Matzner and M. P. Ryan, J. Math. Phys. **25**, 2236 (1984).

[55] J. A. Nieto, Mod. Phys. Lett. **A16**, 2567 (2001); hep-th/0110227.

[56] J. A. Nieto, E. A. Leon (Sinaloa U.), V. M. Villanueva, Int. J. Mod. Phys. D **22** (2013) 1350047; arXiv:1302.1469 [gr-qc]